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Random walks on symmetric spaces and inequalities for matrix spectra

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To the memory of Natasha

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Abstract

Using harmonic analysis on symmetric spaces we reduce the singular spectral problem for products of matrices to the recently solved spectral problem for sums of Hermitian matrices. This proves R.C. Thompson's conjecture [Matrix Spectral Inequalities, Johns Hopkins University Press, Baltimore, MD, 1988]. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let a point with initial position x_0 in Euclidean space $\mathbb{E}^3$ make a sequence of jumps $x_0, x_1, \ldots, x_n$ of fixed lengths $a_i = |x_i - x_{i-1}|$ in random directions. What can one say about the distribution of the final point x_n?

This problem has a long history partially described in [13]. The first solution appears in the last published paper of Rayleigh [19]. He discovered that the probability density $p_n(x)$ is a piecewise polynomial function of the distance $d = d(x, x_0)$ from the initial point x_0 and calculated p_n explicitly for $n \leqslant 6$. Later on, Treloar [22] gave a closed form of the solution for arbitrary n.

In this work, we apply random walks on groups and symmetric spaces (see Section 3 for precise definitions) to matrix spectral problems. The main technical tool is a decomposition of the probability distribution by spherical functions (Theorems

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3.3.1 and 3.4.2). We include a number of examples, which cover some classical formulae, as well as new ones.

For application to the matrix spectral problems only three examples are essential, namely, the sphere $\mathbb{S}^3$, Euclidean space $\mathbb{E}^3$, and Lobachevskii space $\mathbb{L}^3$. They form a special case of a triple of symmetric spaces associated with any compact simply connected group G:

- the group G itself;
- its Lie algebra L_G;
- the dual symmetric space $H_G = G_{\mathbb{C}}/G$.

For the unitary group $G = \mathrm{SU}(n)$ the space L_G consists of (skew) Hermitian traceless matrices, while $H_G = \mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n) := \mathbb{H}_n$ may be identified with the space of positive Hermitian unimodular matrices H via polar decomposition $A = H \cdot U$, $A \in \mathrm{SL}(n, \mathbb{C})$, $U \in \mathrm{SU}(n)$. In the case $n = 2$, we recover the above triple $\mathbb{S}^3 \simeq \mathrm{SU}(2)$, $\mathbb{E}^3$ and $\mathbb{L}^3$.

The spaces G, L_G, and H_G have positive, zero, and negative curvature, and may be treated as members of one family depending on the scalar curvature $-\infty < K < \infty$. Let p_G, p_L and p_H be probability densities for random walks in G, L_G and H_G. For the unitary group $G = \mathrm{SU}(n)$ they have the following meaning:

- $p_L(H)$ gives the distribution of sums $H = H_1 + H_2 + \cdots + H_N$ of independent random Hermitian matrices H_k with given spectra
$$\lambda(H_k) = \left\{\lambda_1^{(k)} \geqslant \lambda_2^{(k)} \geqslant \cdots \geqslant \lambda_n^{(k)}\right\} := \lambda^{(k)}.$$
- $p_G(U)$ is the distribution of products $U = U_1 U_2 \cdots U_N$ of independent random unitary matrices $U_k \in \mathrm{SU}(n)$ with given spectra
$$\varepsilon(U_k) = \exp\left(\mathrm{i}\,\lambda^{(k)}\right).$$
- $p_H(A)$ is the distribution of products $A = A_1 A_2 \cdots A_N$ of random unimodular matrices $A_k \in \mathrm{SL}(n, \mathbb{C})$ with given *singular spectra*
$$\sigma(A_k) = \lambda\left(\sqrt{A_k A_k^*}\right) = \exp\left(\lambda^{(k)}\right).$$

In all three cases the densities $p_L(H) = p_L(\lambda)$, $p_G(U) = p_G(\varepsilon)$, and $p_H(A) = p_H(\sigma)$ depend only on the spectra $\lambda = \lambda(H)$, $\varepsilon = \varepsilon(U)$, and $\sigma = \sigma(A)$. The spectra in turn parametrize orbits of G in the corresponding symmetric spaces. The word "random" refers to the uniform distribution in the orbits.

In view of these interpretations, the classical spectral problems for

(i) sums of Hermitian matrices $H_1 + H_2 + \cdots + H_N$,
(ii) products of unitary matrices $U_1 U_2 \cdots U_N$,
(iii) singular spectrum of products $A_1 A_2 \cdots A_N$, $A_k \in \mathrm{SL}(n, \mathbb{C})$

are just questions about the *supports* of the densities $p_L(\lambda)$, $p_G(\varepsilon)$, and $p_H(\sigma)$. It turns out that the densities, and their supports, in cases (i) and (iii) are closely related.

Theorem A. *Let* exp : $\mathscr{T} \to T$ *be the exponential map for a maximal torus* $T \subset G$ *in a compact simply connected group* G, *and let the previous notations be in force. Then the following identity holds*:

$$p_L(\lambda)\prod_{k=0}^{N}\prod_{\alpha>0}\left(\lambda^{(k)},\alpha\right) = p_H(\exp \mathrm{i}\,\lambda)\prod_{k=0}^{N}\prod_{\alpha>0}\sinh\left(\lambda^{(k)},\alpha\right), \tag{1.1}$$

where the internal product is extended over all positive roots α *of* G.

Both sides of (1.1) are actually polynomials in $\lambda^{(0)} := -\lambda, \lambda^{(1)}, \ldots, \lambda^{(N)} \in \mathscr{T}$ in each chamber defined by the system of hyperplanes,

$$\left(w_0\lambda^{(0)} + w_1\lambda^{(1)} + \cdots + w_N\lambda^{(N)}, \omega_i\right) = 0,$$

where ω_i are fundamental weights, and $w_k \in W_G$ are elements of Weyl group W_G (Theorems 4.2.3 and 5.1.1). A similar formula holds for random walks in G, but only for sufficiently small λ (Theorem 4.2.3).

Since the exponential mapping for the hyperbolic space H_G is bijective, and the densities p_L and p_H differ only by nonvanishing factors $\sinh(\lambda^{(k)},\alpha)/(\lambda^{(k)},\alpha)$, the distributions have essentially the same support

$$\operatorname{supp}(p_H) = \exp(\operatorname{supp}(p_L)).$$

For the unitary group this may be stated as follows.

Theorem B. *The following conditions are equivalent*:

1. *There exist matrices* $A_i \in \mathrm{GL}(n,\mathbb{C})$ *with given singular spectra*

$$\sigma_i = \sigma(A_i) \quad \textit{and} \quad \sigma = \sigma(A_1A_2\cdots A_N).$$

2. *There exist Hermitian* $n \times n$ *matrices* H_i *with spectra*

$$\lambda(H_i) = \log\sigma_i \quad \textit{and} \quad \lambda(H_1 + H_2 + \cdots + H_N) = \log\sigma.$$

The theorem was conjectured by Thompson [20] (see also [21]), who was inspired by the striking similarity between known results for Hermitian and singular spectral problems. The Hermitian problem has recently been solved in my paper [16], see [2,9,10,18,24] for further improvements, including Horn's conjecture. There are analogues of Theorem B for orthogonal and simplectic groups.

The piecewise polynomial structure of the densities, which is given in explicit form in Section 5 of the paper, in principle shifts the spectral problems into the combinatorial domain. Nevertheless, currently this approach fails to produce a solution for the unitary spectral problem, comparable with an elegant one given by Agnihotri and Woodward [1].

Application of harmonic analysis on symmetric spaces to spectral problems of linear algebra was initiated by Berezin and Gelfand [3], see also [7]. The formulae for random walks in *finite groups* go back to Frobenius [8] (up to terminology), for more recent treatments see [4,14]. The main result (Theorem A) may be considered as a hyperbolic version of the so-called *wrapping theorem* for compact groups [6], which essentially is an extension of the identity (4.19) of Theorem 4.2.2 to arbitrary elements a_k of Lie algebra L_G. Unfortunately, this extension has no probabilistic interpretation, and hence no reduction of the unitary spectral problem to the Hermitian one beyond region (4.20).

2. Symmetric spaces

2.1.

Let us recall that a Riemann manifold X is said to be *symmetric* if the geodesic symmetry $\sigma : X \to X$ with center at any point x_0 is an isometry. By definition σ maps a point x on a geodesic through x_0 into a symmetric point x' on the same geodesic and at the same distance from x_0. It follows from the definition that a symmetric space X admits a connected transitive Lie group of isometries G and may be identified with the homogeneous space $X = G/K$ with *compact* isometry group K, which up to a finite index may be given by one of the formulae

$$K = \{g \in G \mid gx_0 = x_0\} = \{g \in G \mid g\sigma = \sigma g\}.$$

So in essence symmetric spaces are parametrized by *Cartan pairs* (G, σ) consisting of a Lie group G and an involution $\sigma : G \to G$ with compact centralizer K. Then there exists a unique, up to proportionality, G-invariant metric on $X = G/K$ and the geodesic symmetry with center at $x_0 = fK$

$$gK \mapsto ff^{-\sigma}g^{\sigma}K$$

is an isometry.

2.2. Examples

The following symmetric spaces are important either for motivation or for the main applications of our study.

2.2.1. Spaces of rank 1

The sphere $\mathbb{S}^n$, Euclidean space $\mathbb{E}^n$, and Lobachevskii space $\mathbb{L}^n$ have evident symmetric structures. For example, Euclidean space has Cartan presentation $\mathbb{E}^n =$

$M(n)/\mathrm{SO}(n)$ with group of rigid motions $M(n)$ as isometry group, and central symmetry as Cartan involution. These are typical examples of spaces of rank 1, for which double cosets $K\backslash G/K$ depend on one parameter.

2.2.2. The three spaces

A compact group G may be considered as a symmetric space with isometry group $G \times G$, acting by left and right multiplication $x \mapsto g_1 x g_2^{-1}$. The Cartan involution interchanges the factors in $G \times G$, and the isotropy group K is G itself diagonally embedded in $G \times G$.

The Lie algebra L_G of a group G is a symmetric space with noncompact isometry group generated by translations and the adjoint action of G.

Let $L_G \otimes \mathbb{C}$ be the complexification of L_G and $G_{\mathbb{C}}$ be the corresponding complex reductive group. Then $H_G = G_{\mathbb{C}}/G$ is a symmetric space with complex conjugation in $G_{\mathbb{C}}$ as Cartan involution. This space is called the *dual symmetric space* to G.

For the group SU(2) the three spaces are just the sphere $\mathbb{S}^3$, Euclidean space $\mathbb{E}^3$, and Lobachevskii space $\mathbb{L}^3$.

2.2.3. Positive Hermitian matrices

The dual space to the unitary group $\mathrm{SU}(n)$, that is, $\mathbb{H}_n := \mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)$, may be identified with the space of unimodular positive Hermitian matrices via the polar decomposition $A = H \cdot U$, with angular part $U \in \mathrm{SU}(n)$, and the positive Hermitian matrix $H = \sqrt{A \cdot A^*}$ as radial component. The eigenvalues of H are said to be the *singular values* of A. This is the central example for our study of the singular values spectral problem.

3. Random walks

3.1.

We begin with the classical example of random walk in Euclidean space $\mathbb{E}^n$, which may be defined as a sequence of random points in $\mathbb{E}^n$

$$0 = x_0, x_1, x_2, \ldots, x_N \tag{3.1}$$

such that the differences $\delta_i = x_i - x_{i-1}$ are *independent* and *uniformly distributed* in spheres of given radii a_i.

Treating $\mathbb{E}^n$ as the symmetric space $G/K = M(n)/\mathrm{SO}(n)$ we may identify the spheres with double cosets KgK. Then the random walk (3.1) is given by a sequence of elements

$$g_1, g_2, \ldots, g_N \in G, \tag{3.2}$$

which are independent and uniformly distributed in the double cosets $X_i = Kg_iK$. The original sequence of elements (3.2) may be reconstructed from these data as follows:

$$x_i = g_1 g_2 \cdots g_i K \in G/K = X.$$

So we arrived at the following:

Definition 3.1.1. A random walk in the symmetric space $X = G/K$ is a sequence of random elements

$$x_i = g_1 g_2 \cdots g_i K \in G/K, \tag{3.3}$$

where the g_i are independent and uniformly distributed in given double cosets $X_i = Kg_iK$.

Example 3.1.2 (*Random walk in space* $\mathbb{H}_n$). As we have seen in Section 2.2.3, the space of positive Hermitian matrices $\mathbb{H}_n$ is a symmetric space with Cartan representation $\mathbb{H}_n = \mathrm{GL}(n, \mathbb{C})/U(n)$. A double coset $U(n)gU(n) \subset \mathbb{H}_n$ in this case consists of matrices $A \in \mathrm{GL}(n, \mathbb{C})$ with fixed singular spectrum $\sigma(A)$.

The matrix A, considered as an operator in $\mathbb{C}^n$, transforms the unit sphere into an ellipsoid with semiaxis equal to the singular values of A. Hence, one may visualize a random walk in $\mathbb{H}_n$ as a sequence of ellipsoids in $\mathbb{C}^n$ obtained from the unit sphere by a succession of dilations with given coefficients $\sigma_1^{(k)}, \sigma_2^{(k)}, \ldots, \sigma_n^{(k)}$ along randomly chosen orthogonal directions $e_1^{(k)}, e_2^{(k)}, \ldots, e_n^{(k)}$.

Notation 3.1.3. For given double cosets $X_i = Kg_iK$ in the symmetric space $X = G/K$ let

$$P_X(x) = P(X_1, X_2, \ldots, X_N \mid x) \tag{3.4}$$

be the probability density for the distribution of the final element $x = x_N$ in the random walk (3.3).

In the following section, we evaluate the densities (3.4) in terms of spherical functions.

3.2. Spherical functions

To evaluate the densities we first need *spherical functions* on the symmetric space $X = G/K$.

Definition 3.2.1. A function $\varphi \in L^2(G/K)$ is said to be *spherical* if $\varphi(1) = 1$, and the following equation holds:

$$\int_K \varphi(xky)\,\mathrm{d}k = \varphi(x)\varphi(y) \quad \forall x, y \in G.$$

Note that the equation implies bi-invariance of spherical functions

$$\varphi(k_1 x k_2) = \varphi(x) \quad \forall k_1, k_2 \in K.$$

The importance of spherical functions for analysis on symmetric spaces may be seen from the following properties. Let $H_\varphi \subset L^2(G/K)$ be the G-invariant Hilbert subspace generated by the spherical function φ. Then

1. $G : H_\varphi$ is an irreducible representation (which is said to be *spherical*), and $\varphi \in H_\varphi$ is the unique, up to proportionality, bi-invariant function in H_φ.
2. Hence, in the compact case the space H_φ is finite-dimensional.
3. $H_\varphi \perp H_\psi$ for $\varphi \neq \psi$.
4. $L^2(G/K)$ is a direct sum (or integral for noncompact $X = G/K$) of the irreducible representations H_φ.

For all classical symmetric spaces the spherical functions are explicitly known [11,12].

Example 3.2.2. For Euclidean space $\mathbb{E}^n = M(n)/\mathrm{SO}(n)$, spherical functions depend only on the distance $d = |x|$ from the origin, and may be expressed via Bessel functions:

$$\varphi_\lambda(x) = 2^\nu \Gamma(\nu + 1) \cdot \frac{J_\nu(\lambda d)}{(\lambda d)^\nu}, \quad \nu = \frac{n-2}{2}.$$

Example 3.2.3. For a compact group G, considered as a symmetric space (Sectiion 2.2.2), the spherical functions are just normalized characters $\varphi(g) = \chi(g)/\chi(1)$ of irreducible representations $G : U_\chi$, and the corresponding spherical representation of $G \times G$ is $H_\varphi = U_\chi \otimes U_\chi$.

3.3. *Compact case*

Now we are in a position to evaluate the probability distribution for a random walk in a compact symmetric space.

Theorem 3.3.1. *The probability density of the random walk* (3.3) *in a compact symmetric space $X = G/K$ has the following decomposition into spherical functions*:

$$P(X_1, X_2, \ldots, X_N \mid x) = \sum_\varphi \dim H_\varphi \cdot \overline{\varphi}(x) \prod_{i=1}^N \varphi(X_i), \tag{3.5}$$

where the sum runs over all spherical functions.

Remark. Since spherical functions are bi-invariant, $\varphi(g_i)$ depends only on the double coset $X_i = K g_i K$. This explains the notation $\varphi(X_i) = \varphi(g_i)$.

Proof. To clarify the structure of the proof we split it into one-move steps.

Step 1. *For any spherical function φ and $x_i \in X$ the following identity holds*:

$$\int_{K\times K\times\cdots\times K}\varphi(k_1x_1k_2x_2\cdots k_Nx_N)\,\mathrm{d}k_1\,\mathrm{d}k_2\cdots\mathrm{d}k_N$$
$$=\varphi(x_1)\varphi(x_2)\cdots\varphi(x_N). \tag{3.6}$$

For $n=1$ the equation follows from the definition of spherical function

$$\int_K\varphi(kx)\,\mathrm{d}k=\varphi(1)\varphi(x)=\varphi(x),$$

and simple induction arguments prove it in general. □

Step 2. *The identity of Step* 1 *may be rewritten in the form*

$$\int_X\varphi(x)P(X_1,X_2,\ldots,X_N\mid x)\,\mathrm{d}x=\varphi(X_1)\varphi(X_2)\cdots\varphi(X_N) \tag{3.7}$$

where $X_i=Kx_i$.

Let us consider the mapping

$$\mu: K\times K\times\cdots\times K\to X$$
$$k_1\times k_2\times\cdots\times k_N\mapsto k_1x_1k_2x_2\cdots k_Nx_N.$$

The function $\varphi(k_1x_1k_2x_2\cdots k_Nx_N)$ is constant on the fibers of μ and

$$P(X_1,X_2,\ldots,X_N\mid x)\,\mathrm{d}x$$

is equal to the volume of the fiber $\mu^{-1}(\mathrm{d}x)$. Hence, by Fubini's theorem

$$\int_{K\times K\times\cdots\times K}\varphi(k_1x_1k_2x_2\cdots k_Nx_N)\,\mathrm{d}k_1\,\mathrm{d}k_2\cdots\mathrm{d}k_N$$
$$=\int_X\varphi(x)P(X_1,X_2,\ldots,X_N\mid x)\,\mathrm{d}x$$

and the result follows. □

Step 3. *The density has the following decomposition into series of spherical functions*

$$P(X_1,X_2,\ldots,X_N\mid x)=\sum_\varphi\frac{\overline{\varphi}(x)}{(\varphi,\varphi)}\varphi(X_1)\varphi(X_2)\cdots\varphi(X_N),$$

where $(f,g)=\int_X f(x)\overline{g}(x)\,\mathrm{d}x$.

As with any reasonable bi-invariant function, the density admits a decomposition into spherical harmonics

$$P(X_1,X_2,\ldots,X_N\mid x)=\sum_\varphi a_\varphi\varphi(x),$$

with coefficients

$$a_\varphi=\frac{1}{(\varphi,\varphi)}\int_X P(X_1,X_2,\ldots,X_N|x)\overline{\varphi}(x)\,\mathrm{d}x$$

$$\overset{(3.7)}{=} \frac{1}{(\varphi,\varphi)}\overline{\varphi}(X_1)\overline{\varphi}(X_2)\cdots\overline{\varphi}(X_N),$$

and the result follows. □

To get the final formula (3.5) we have to evaluate (φ,φ).

Step 4. *The following equality holds*:

$$(\varphi,\varphi) = \frac{1}{\dim H_\varphi}. \tag{3.8}$$

This step is equivalent to evaluation of the *Plancherel measure* for X (see the following). It may be proved as follows. Let us denote by $(g)_H : H_\varphi \to H_\varphi$ the linear operator of the spherical representation H_φ corresponding to the element $g \in G$. Then the operator

$$\int_{G\times K} (g^{-1}kg)_H \,\mathrm{d}g\,\mathrm{d}k$$

commutes with G and hence by Schur's lemma is a scalar

$$\int_{G\times K} \left(g^{-1}kg\right)_H \,\mathrm{d}g\,\mathrm{d}k = \lambda\cdot\mathrm{id}. \tag{3.9}$$

Applying this operator to the spherical function $\varphi(x)$ we get

$$\lambda\varphi(x) = \int\int_{K\times G} \varphi\left(g^{-1}kgx\right)\mathrm{d}k\,\mathrm{d}g = \int_G \varphi\left(g^{-1}\right)\varphi(gx)\,\mathrm{d}g,$$

where in the last equality we make use of the functional equation for spherical functions (stated as Definition 3.2.1 in our exposition). For $x = 1$ we get $\lambda = (\varphi,\varphi)$, and taking the trace of (3.9) we finally get

$$(\varphi,\varphi)\dim H_\varphi = \int\int_{G\times K} \chi\left(g^{-1}kg\right)\mathrm{d}g\,\mathrm{d}k = \int_K \chi(k)\,\mathrm{d}k = 1.$$

The last integral is equal to the multiplicity of the trivial component in $K : H_\varphi$, and hence is 1. □

Example 3.3.2 (*Random walks in* $\mathbb{S}^3$). We identify the sphere with the group $\mathrm{SU}(2)$. Then by Example 3.2.3, the normalized character $\varphi_k = \sin k\theta/(k\sin\theta)$ of the irreducible k-dimensional representation $G : U_k$ is a spherical function, and $H_k = U_k \otimes U_k$ is the corresponding spherical representation of $\mathrm{SU}(2)\times\mathrm{SU}(2)$. Applying Theorem 3.3.1, we arrive at the formula

$$P(\alpha_1,\alpha_2,\ldots,\alpha_N \mid x) = \sum_{k=1}^{\infty}\frac{1}{k^{N-1}}\frac{\sin k\theta}{\sin\theta}\prod_i \frac{\sin k\alpha_i}{\sin\alpha_i},$$

where the random walk is defined by a sequence of independent jumps by angles $\alpha_1,\alpha_2,\ldots,\alpha_N$, beginning at the North pole ($\theta = 0$), and $\theta = \theta(x)$ is the latitude of the final point $x \in \mathbb{S}^3$.

Rather unexpectedly we may sum up the series and get a finite answer (by God's will the wonder repeats itself in all compact groups). To proceed, we first express $\sin k\alpha$ and $\sin k\theta$ by exponentials

$$\frac{2^{-n-2}i^{-n-1}}{\sin\,\theta\,\sin\,\alpha_1\,\sin\,\alpha_2\cdots\sin\,\alpha_n} \times\sum_{\pm}\sum_{k\neq 0}(-1)^{\#(-)}\frac{\exp(\mathrm{i}\,k(\pm\theta\pm\alpha_1\pm\alpha_2\pm\cdots\pm\alpha_n))}{k^{n-1}},$$

where the first sum runs over all combinations of signs $\pm$. Then apply the Fourier expansion for Bernoulli polynomials $B_\nu(x)$

$$\sum_{k\neq 0}\frac{\exp(2\pi\,\mathrm{i}\,kx)}{k^\nu}=-\frac{(2\pi\,\mathrm{i})^\nu}{\nu!}\widetilde{B}_\nu(x),$$

where $\widetilde{B}_\nu(x+1)=\widetilde{B}_\nu(x)$ and $\widetilde{B}_\nu(x)=B_\nu(x)$ for $0<x<1$. As a result we finally get

$$\begin{aligned}&P^3_{\mathbb{S}}(\alpha_1,\alpha_2,\ldots,\alpha_N\mid x)\\&\quad=\frac{\pi^{n-1}}{(n-1)!4\,\sin\,\theta\prod_1^n\,\sin\,\alpha_i}\\&\qquad\times\sum_{\pm}(-1)^{\#(-)}\widetilde{B}_{n-1}\left(\frac{\theta\pm\alpha_1\pm\cdots\pm\alpha_n}{2\pi}\right),\end{aligned}\tag{3.10}$$

where we exclude the first $\pm$ sign using the symmetry $\widetilde{B}_\nu(-x)=(-1)^\nu\widetilde{B}_\nu(x)$.

Example 3.3.3 (*Random walks in* $\mathbb{E}^3$). Let us now suppose that the jumps $\alpha_i\geqslant 0$ are so small that the final point x never reaches the South pole, that is,

$$\alpha_1+\alpha_2+\cdots+\alpha_n<\pi.\tag{3.11}$$

Then formula (3.10) may be simplified as follows:

$$\begin{aligned}&P_{\mathbb{S}^3}(\alpha_1,\alpha_2,\ldots,\alpha_N\mid x)\\&\quad=\frac{\pi}{(n-2)!2^n\,\sin\,\theta\prod_1^n\,\sin\,\alpha_i}\\&\qquad\times\sum_{\theta\pm\alpha_1\pm\alpha_2\pm\cdots\pm\alpha_n<0}(-1)^{\#(-)}(\theta\pm\alpha_1\pm\cdots\pm\alpha_n)^{n-2}.\end{aligned}\tag{3.12}$$

For the proof, let us note that the sum over signs $\pm$ in (3.10) is nothing but the nth difference. Hence, for any *polynomial* $B_{n-1}(x)$ of degree $n-1$ the sum vanishes

$$\sum_{\pm}(-1)^{\#(-)}B_{n-1}\left(\frac{\theta\pm\alpha_1\pm\cdots\pm\alpha_n}{2\pi}\right)=0.$$

The function $\widetilde{B}_{n-1}$ in (3.10) is not polynomial, but under condition (3.11) its argument spreads over two intervals of polynomiality $(-1,0)$ and $(0,1)$. Splitting the sum into two polynomial parts

$$\sum_{\theta\pm\alpha_1\pm\cdots\pm\alpha_n>0}(-1)^{\#(-)}B_{n-1}\left(\frac{\theta\pm\alpha_1\pm\cdots\pm\alpha_n}{2\pi}\right)$$
$$+\sum_{\theta\pm\alpha_1\pm\cdots\pm\alpha_n<0}(-1)^{\#(-)}B_{n-1}\left(1+\frac{\theta\pm\alpha_1\pm\cdots\pm\alpha_n}{2\pi}\right),$$

and using the functional equation $B_\nu(x+1)-B_\nu(x)=\nu x^{\nu-1}$ we get the result.

Let us now suppose that the radius of the sphere $\mathbb{S}^3$ tends to infinity in such a way that $R\theta\to d$ and $R\alpha_i\to a_i$. Then taking limits in (3.12) we get the Treloar formula [22] for random walks in $\mathbb{E}^3$:

$$\begin{aligned}
&P_{\mathbb{E}^3}(a_1,a_2,\ldots,a_n\mid d)\\
&\quad=\lim_{R\to\infty}\frac{1}{2\pi^2R^3}P^3_{\mathbb{S}}(\alpha_1,\alpha_2,\ldots,\alpha_n\mid\theta)\\
&\quad=\frac{1}{\pi(n-2)!2^{n+1}da_1a_2\cdots a_n}\\
&\qquad\times\sum_{d\pm a_1\pm a_2\pm\cdots\pm a_n<0}(-1)^{\#(-)}(d\pm a_1\pm a_2\pm\cdots\pm a_n)^{n-2},
\end{aligned}\tag{3.13}$$

where $2\pi^2R^3=\operatorname{vol}\mathbb{S}^3$.

3.4. Plancherel measure and noncompact case

For a noncompact symmetric space $X=G/K$, the spherical representations H_φ are usually infinite-dimensional, and formula (3.5) makes no sense. Nevertheless, on the space of spherical functions (denote it by Λ) there exists the so-called *Plancherel measure* $\mathrm{d}\mu(\lambda)$, which may be characterized by the equation

$$\int_G|f(g)|^2\,\mathrm{d}g=\int_\Lambda|\widehat{f}(\lambda)|^2\,\mathrm{d}\mu(\lambda)\tag{3.14}$$

for any bi-invariant function $f\in L^2(K\backslash G/K)$. Here

$$\widehat{f}(\lambda)=\int_G f(g)\overline{\varphi}_\lambda(g)\,\mathrm{d}g\tag{3.15}$$

is the *spherical transform* of *f*.

Example 3.4.1. For a compact group G, the Plancherel measure is discrete. To evaluate the measure of a spherical function $f=\varphi_\lambda$ we begin with its spherical transform

$$\widehat{f}(\gamma) = \int_G \varphi_\lambda(g)\overline{\varphi}_\gamma(g)\,\mathrm{d}g = (\varphi_\lambda, \varphi_\gamma)\delta_{\lambda\gamma},$$

and substitute this value in (3.14)

$$(\varphi_\lambda, \varphi_\lambda)^2\mu(\lambda) = (\varphi_\lambda, \varphi_\lambda).$$

Then by (3.8)

$$\mu(\lambda) = \frac{1}{(\varphi_\lambda, \varphi_\lambda)} = \dim H(\varphi_\lambda).$$

The last step in the proof of Theorem 3.3.1 is nothing but a computation of the Plancherel measure. In a sense the Plancherel measure is an analogue of dimension for infinite-dimensional spherical representations. The Plancherel measure is known for all Riemannian symmetric spaces [11,12].

Theorem 3.4.2. *The density of a random walk in an arbitrary symmetric space $X = G/K$ is given by the formula*

$$P(X_1, X_2, \ldots, X_N \mid x) = \int_\Lambda \varphi_\lambda(x') \prod_i \varphi_\lambda(X_i)\,\mathrm{d}\mu(\lambda), \tag{3.16}$$

where x' is the symmetric element to x with respect to (*the image of*) *the unit element* $1 \in G$, *from which the random walk begins.*

Proof. The proof has the same logical structure as in the compact case, except that instead of series one has to use integrals. In a sense it is even simpler, since we do not need step 4, which is hidden in the inversion formula

$$f(x) = \int_\Lambda \varphi_\lambda(x)\widehat{f}(\lambda)\,\mathrm{d}\mu(\lambda) \tag{3.17}$$

for spherical transform (2.15). □

Remark 3.4.3. Theorems 3.3.1 and 3.4.2 are actually based on two properties of the spherical transform (3.15): multiplicativity with respect to the convolution $f * h(x) = \int_G f(xg)h(g^{-1}x)\,\mathrm{d}g$ of bi-invariant functions

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g},$$

and inversion formula (3.17). Both of these properties hold for any commutative *hypergroup* [4,14]. This provides a general template for such kind of results.[1]

Example 3.4.4. For Euclidean space $\mathbb{E}^n$ the spherical functions and the Plancherel measure are given by the formulae:

[1] The author is grateful to the referee for this remark.

$$\varphi_\lambda(x) = 2^\nu \Gamma(\nu+1)\frac{J_\nu(\lambda r)}{(\lambda r)^\nu}, \quad r = |x|,\ \nu = \frac{n-2}{2},$$

$$\mathrm{d}\mu(\lambda) = \frac{2}{(4\pi)^{\nu+1}\Gamma(\nu+1)}\lambda^{n-1}\,\mathrm{d}\lambda.$$

So a random walk in $\mathbb{E}^n$ with independent steps of length $a_1, a_2, \ldots, a_N$ has the density

$$P(a_1, a_2, \ldots, a_N \mid x) = \text{const.} \int_0^\infty \lambda^{n-1}\frac{J_\nu(\lambda r)}{(\lambda r)^\nu}\prod_{i=1}^N \frac{J_\nu(\lambda a_i)}{(\lambda a_i)^\nu}\,\mathrm{d}\lambda.$$

For the plane $\mathbb{E}^2$ this amounts to Kluyver's formula [15]

$$P_{\mathbb{E}^2}(a_1, a_2, \ldots, a_N \mid x) = \frac{1}{2\pi}\int_0^\infty \lambda J_0(\lambda|x|)J_0(\lambda a_1)J_0(\lambda a_2)\cdots J_0(\lambda a_N)\,\mathrm{d}\lambda,$$

and for $n = 3$ to that of Rayleigh [19]

$$P_{\mathbb{E}^3}(a_1, a_2, \ldots, a_N \mid x) = \frac{1}{2\pi^2}\int_0^\infty \lambda^2\frac{\sin(\lambda r)}{\lambda r}\prod_{i=1}^N \frac{\sin(\lambda a_i)}{\lambda a_i}\,\mathrm{d}\lambda. \tag{3.18}$$

The general case is due to Watson [23].

4. The three symmetric domains

4.1. *Positive Hermitian matrices*

Let us begin with the symmetric space $\mathbb{H}_n$ of positive Hermitian $n \times n$ matrices. The action of $\mathrm{SL}(n, \mathbb{C})$

$$H \mapsto AH\bar{A}^{\mathrm{t}}, \quad H \in \mathbb{H}_n,\ A \in \mathrm{SL}(n, \mathbb{C})$$

gives rise to the Cartan presentation $\mathbb{H}_n = \mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)$. An orbit of the unitary group $\mathrm{SU}(n)$ on $\mathbb{H}_n$ consists of unimodular Hermitian matrices H with fixed positive spectrum $\lambda(H)$ which we write in exponential form $\lambda(H) = \mathrm{e}^S$, where

$$S\colon\ s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n, \qquad s_1 + s_2 + \cdots + s_n = 0. \tag{4.1}$$

The corresponding double coset

$$\mathscr{C}(S) \subset \mathrm{SL}(n, \mathbb{C})//\mathrm{SU}(n) := \mathrm{SU}(n)\backslash\mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)$$

consists of all matrices $A \in \mathrm{SL}_n(\mathbb{C})$ with given *singular spectrum* $\sigma(A) = \lambda(\sqrt{A\bar{A}^{\mathrm{t}}})$.

Theorem 3.4.2, when applied to $\mathbb{H}_n$, yields a distribution of the singular spectrum of products

$$A = A_1A_2\cdots A_N$$

of independent random factors A_i uniformly distributed in the space of matrices $\mathscr{C}(S_i)$ with given singular spectrum $\sigma(A_i) = \mathrm{e}^{S_i}$. To get an explicit formula we need the spherical functions and the Plancherel measure for $\mathbb{H}_n$. They were found by Gelfand and Naimark in 1950 (see [11, Chapter IV, Theorem 5.7] for Harish–Chandra's extension on arbitrary complex semisimple groups). The spherical functions on $\mathbb{H}_n$ are SU-invariant and hence depend only on the spectrum e^S (4.1) of a matrix $H \in \mathbb{H}_n$. They may be written in the form

$$\varphi_\lambda(S) = \left(\frac{2}{\mathrm{i}}\right)^{n(n-1)/2} \frac{1!2!\cdots(n-1)!\,\det\left\|\mathrm{e}^{\mathrm{i}\,\lambda_p s_q}\right\|}{\prod_{p<q}(\lambda_q-\lambda_p)\prod_{p<q}\left(\mathrm{e}^{2s_q}-\mathrm{e}^{2s_p}\right)}, \tag{4.2}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$. One can easily see that φ_λ is invariant with respect to translations $\lambda_p \mapsto \lambda_p + \alpha$ and permutations of the components λ_p. So the spherical functions are parametrized by the cone

$$\varLambda = \begin{cases} \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n, \\ \lambda_1 + \lambda_2 + \cdots + \lambda_n = 0. \end{cases}$$

The Plancherel measure on $\varLambda$ is proportional to

$$\prod_{p<q}(\lambda_q - \lambda_p)^2 \,\mathrm{d}\lambda$$

where $\mathrm{d}\lambda$ is Lebesgue measure on $\varLambda \subset \mathbb{R}^{n-1}$.

Example 4.1.1 (*Random walk in Lobachevskii space* $\mathbb{L}^3$). Let us consider in detail the group $\mathrm{SL}(2,\mathbb{C})$, which is locally isomorphic to the Lorentz group $\mathrm{SO}(3,1)$. Hence, in this case the symmetric space of positive unimodular Hermitian matrices $\mathbb{H}_2$ is a model for the Lobachevskii space $\mathbb{L}^3 = \mathrm{SO}(3,1)/\mathrm{SO}(3)$. Theorem 3.4.2 yields the following formula for random walks in Lobachevskii space of curvature radius $-R$ with jumps of length a_i:

$$P_{\mathbb{L}^3}(a_1, a_2, \ldots, a_N \mid x) = \frac{1}{4\pi^2 R^3}\int_{-\infty}^{\infty} \lambda^2 \frac{\sin \mathrm{d}\lambda}{\lambda \sinh d} \prod_i \frac{\sin a_i\lambda}{\lambda\sinh a_i}\,\mathrm{d}\lambda, \tag{4.3}$$

where d is the distance of x from the initial point. Putting $a_0 = d$ and leaving aside the constants the integral reduces to the form

$$\int_{\mathbb{R}} \prod_{k=0}^{N} \sin a_k\lambda \frac{\mathrm{d}\lambda}{\lambda^{N-1}},$$

and may be evaluated as follows. First of all change the real line $\mathbb{R}$ to the contour $\mathbb{R}_\varepsilon$ passing around zero by a small semicircle in the upper halfplane, and then write down sines via exponentials:

$$\frac{1}{(2\,\mathrm{i})^{N+1}}\sum_{\pm}(-1)^{\#(-)}\int_{\mathbb{R}_\varepsilon}\exp(\mathrm{i}(\pm a_0 \pm a_1 \pm \cdots \pm a_N)\lambda)\frac{\mathrm{d}\lambda}{\lambda^{N-1}}.$$

If the sum $(\pm a_0 \pm a_1 \pm \cdots \pm a_N)$ is positive, then the contour may be closed by a big semicircle in the upper halfplane. Hence by the residue theorem the integral is zero. For the negative sum, one can close the contour in the lower halfplane, and in this case

$$\begin{aligned}
&\int_{\mathbb{R}_\varepsilon} \exp(\mathrm{i}(\pm a_0 \pm a_1 \pm \cdots \pm a_N)\lambda)\frac{\mathrm{d}\lambda}{\lambda^{N-1}} \\
&\quad = -2\pi\,\mathrm{i}\;\mathrm{Res}_0\;\frac{\exp(\mathrm{i}(\pm a_0 \pm a_1 \pm \cdots \pm a_N)\lambda)}{\lambda^{N-1}} \\
&\quad = -\frac{2\pi\,\mathrm{i}}{(N-2)!}[\mathrm{i}(\pm a_0 \pm a_1 \pm \cdots \pm a_N)]^{N-2}.
\end{aligned}$$

As a result we get closed formulae for the integral

$$\begin{aligned}
&\int_{\mathbb{R}} \prod_{k=0}^{N} \sin a_k\lambda \frac{\mathrm{d}\lambda}{\lambda^{N-1}} \\
&\quad = \frac{\pi}{2^{N-1}(N-2)!} \sum_{a_0\pm a_1\pm\cdots\pm a_N<0} (-1)^{\#(-)}[a_0 \pm a_1 \pm \cdots \pm a_N]^{N-2},
\end{aligned}$$

and for the density (4.3) of a random walk in Lobachevskii space of radius R

$$\begin{aligned}
&P_{\mathbb{L}^3}(a_1, \ldots, a_N \mid x) \\
&\quad = \frac{1}{\pi R^3 2^{N+1}(N-2)!\,\sinh d \prod_k \sinh a_k} \\
&\qquad \times \sum_{d\pm a_1\pm\cdots\pm a_N<0} (-1)^{\#(-)}[d \pm a_1 \pm \cdots \pm a_N]^{N-2}.
\end{aligned} \tag{4.4}$$

Remark 4.1.2. The last formula for Lobachevskii space $\mathbb{L}^3$ of radius $R = 1$ differs only by simple factors from those of Euclidean space (3.13) and the unit sphere (3.12):

$$\begin{aligned}
P_{\mathbb{E}^3}(a_1, a_2, \ldots, a_N \mid d) &= P_{\mathbb{L}^3}(a_1, a_2, \ldots, a_N \mid d)\frac{\sinh d}{d}\prod_{k=1}^{N}\frac{\sinh a_k}{a_k} \\
&= P_{\mathbb{S}^3}(a_1, a_2, \ldots, a_N \mid d)\frac{\sin d}{d}\prod_{k=1}^{N}\frac{\sin a_k}{a_k},
\end{aligned} \tag{4.5}$$

where the second equality holds only in the domain of injectivity of the exponential mapping for the sphere $a_1 + a_2 + \cdots + a_N < \pi$. The origin of this striking similarity lies in the identity

$$\sum_{m>0} m^2 \prod_{k=0}^{N} \frac{\sin a_k m}{m \sin a_k} = \int_0^\infty \lambda^2 \prod_{k=0}^{N} \frac{\sin a_k\lambda}{\lambda \sin a_k}\,\mathrm{d}\lambda \tag{4.6}$$

valid for $a_k \geqslant 0$ such that $a_1 + a_2 + \cdots + a_N < \pi$. In the following section, we extend both (4.5) and (4.6) to an arbitrary compact simply connected group.

4.2. Some identities

Let Σ be the root system of a simply connected compact group G with simple roots $\alpha_1, \alpha_2, \dots, \alpha_n$ and fundamental weights $\omega_1, \omega_2, \dots, \omega_n$. We will use the standard notation for the halfsum of the positive roots

$$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \omega_1 + \omega_2 + \cdots + \omega_n,$$

and write Weyl's character formula in the form

$$\chi_\omega = \frac{\Delta_\omega}{\Delta_\rho}, \quad \Delta_\omega = \sum_{w\in W} \operatorname{sign}(w)\, \mathrm{e}^{w\omega}, \tag{4.7}$$

where $\omega = \widetilde{\omega} + \rho$ is *strictly* inside the Weyl chamber ($\widetilde{\omega}$ is a dominant weight). The summation is over the Weyl group $W = W_G$.

We will represent the dimension of the character in a similar form

$$\dim \chi_\omega = \frac{d(\omega)}{d(\rho)}, \quad d(\omega) = \prod_{\alpha>0} \left(\omega, \alpha^v\right). \tag{4.8}$$

The advantage of these not quite standard notations is that the character χ_ω and its dimension may by extended to a skew-symmetric function of arbitrary weight $\lambda \in \Lambda \otimes \mathbb{R}$ in the space spanned by the weight lattice Λ:

$$\chi_{w\lambda} = \operatorname{sign}(w)\chi_\lambda, \quad d(w\lambda) = \operatorname{sign}(w) d(\lambda),$$

and in addition $d(\lambda)$ is a product of linear forms in λ.

Let now $\exp : \mathscr{T} \to T$ be the exponential mapping for a maximal torus $T \subset G$, normalized by the condition $\ker(\mathscr{T} \overset{\exp}{\to} T) = \{a \in \mathscr{T} \mid (\omega, a) \in \mathbb{Z}\ \forall \omega \in \Lambda\}$. Then

$$\chi_\omega(\exp a) = \frac{\Delta_\omega(\exp a)}{\Delta_\rho(\exp a)} = \frac{\sum_{w\in W} \mathrm{e}^{2\pi \mathrm{i}(w\omega, a)}}{\prod_{\alpha>0}\left(\mathrm{e}^{\pi \mathrm{i}(\alpha,a)} - \mathrm{e}^{-\pi i(\alpha,a)}\right)}, \quad a \in \mathscr{T}.$$

Since the spherical functions on G are normalized characters

$$\varphi_\omega(\exp a) = \frac{d(\rho)\Delta_\omega(\exp a)}{d(\omega)\Delta_\rho(\exp a)},$$

by Theorem 3.3.1 and Example 3.2.3 the random walk in G with jumps $\exp a_k$ has the density

$$P_G(\exp a) = \frac{\text{const.}}{\prod_k \prod_{\alpha>0} \sin \pi(\alpha, a_k)} \sum_{(\omega,\alpha_i^v)>0} d(\omega)^2 \prod_{k=0}^{N} \frac{\Delta_\omega(\exp a_k)}{d(\omega)}, \tag{4.9}$$

where the constant depends only on N, and to simplify the notations we put $a_0 = -a$.

According to Gelfand–Naimark and Harish–Chandra [11, Chapter IV, Theorem 5.7] spherical functions on the dual symmetric space $H_G = G_{\mathbb{C}}/G$ are obtained from those of G by the formal substitution $\rho \mapsto \mathrm{i}\,\rho$, and taking the element $\lambda \in \Lambda \otimes \mathbb{R}$ in the positive Weyl chamber instead of the integer weight $\omega \in \Lambda$:

$$\varphi_\lambda(\exp \mathrm{i}\, a) = \frac{d(\mathrm{i}\,\rho)\Delta_\lambda(\exp a)}{d(\lambda)\Delta_{\mathrm{i}\,\rho}(\exp a)} = \frac{d(\mathrm{i}\,\rho)}{d(\lambda)} \frac{\sum_{w\in W} \mathrm{e}^{2\pi\,\mathrm{i}(w\lambda,a)}}{\prod_{\alpha>0}\left(\mathrm{e}^{-\pi(\alpha,a)} - \mathrm{e}^{\pi(\alpha,a)}\right)}.$$

Since the Plancherel measure in this case is known $\mathrm{d}\mu(\lambda) \propto d(\lambda)^2 \mathrm{d}\lambda$, by Theorem 3.4.2 we get the density of the random walk in $H = H_G$ with steps $\exp \mathrm{i}\, a_k$:

$$\begin{aligned} P_H(\exp \mathrm{i}\, a) &= \frac{\text{const.}}{\prod_{k=0}^N \prod_{\alpha>0} \sinh \pi(\alpha, a_k)} \\ &\quad \times \int_{(\lambda,\alpha_i^v)>0} d(\lambda)^2 \prod_{k=0}^N \frac{\Delta_\lambda(\exp a_k)}{d(\lambda)}\, \mathrm{d}\lambda, \end{aligned} \tag{4.10}$$

where as before we put $a_0 = -a$.

We are now ready to prove the analogue of identity (4.6).

Theorem 4.2.1. *Let a_k satisfy the inequalities*

$$|(\omega_i, w_0 a_0 + w_1 a_1 + \cdots + w_N a_N)| < 1 \tag{4.11}$$

for all fundamental weights ω_i and $w_k \in W_G$. Then the following identity holds:

$$\sum_{(\omega,\alpha_i^v)>0} d(\omega)^2 \prod_{k=0}^N \frac{\Delta_\omega(\exp a_k)}{d(\omega)} = \int_{(\lambda,\alpha_i^v)>0} d(\lambda)^2 \prod_{k=0}^N \frac{\Delta_\lambda(\exp a_k)}{d(\lambda)}\, \mathrm{d}\lambda. \tag{4.12}$$

The sum in (4.12) *runs over integral weights inside the positive Weyl chamber, while the integral is taken over the chamber itself.*

Remark 4.2.2. The left-hand side of (4.12) is a periodic function of a_k with simple roots α_i^v as periods, while the right-hand side is manifestly a homogeneous function. Hence equality (4.12) cannot be valid for all a_k. We will see in the following section that the sum in (4.12) is a *polynomial* function of $a_0, a_1, \dots, a_N$ in each chamber defined by affine hyperplanes

$$(\omega, w_0 a_0 + w_1 a_1 + \cdots + w_N a_N) = p \in \mathbb{Z} \tag{4.13}$$

for $\omega \in \Lambda$ and $w_k \in W$. The theorem implies that the integral in (4.12) is polynomial in each cone defined by hyperplanes (4.13) passing through zero.

Proof of Theorem 4.2.1. We start with the Poisson summation formula

$$\sum_{\omega\in\Lambda} f(\omega) = \sum_{\ell\in L} \widehat{f}(\ell) \tag{4.14}$$

valid for any reasonable function f in the space $\Lambda \otimes \mathbb{R}$ spanned by the weight lattice Λ. Here $\widehat{f}$ is the Fourier transform

$$\widehat{f}(q) = \int_{\Lambda\otimes\mathbb{R}} f(p) \exp(-2\pi\,\mathrm{i}(p,q))\, \mathrm{d}p,$$

and $L = \ker(\mathscr{T} \xrightarrow{\exp} T)$ is the dual lattice to Λ. We apply (4.14) to the W-invariant function

$$f(\lambda) = d(\lambda)^2 \prod_{k=0}^{N} \frac{\Delta_\lambda(\exp a_k)}{d(\lambda)}$$

vanishing on the mirrors $(\lambda, \alpha_i^v) = 0$ to get

$$\begin{aligned}&\sum_{(\omega,\alpha_i^v)\neq 0} d(\omega)^2 \prod_{k=0}^{N} \frac{\Delta_\omega(\exp a_k)}{d(\omega)} \\ &\quad = \sum_{\ell\in L} \int_{\Lambda_{\mathbb{R}}} \exp(-2\pi\, \mathrm{i}(\lambda, \ell)) d(\lambda)^2 \prod_{k=0}^{N} \frac{\Delta_\lambda(\exp a_k)}{d(\lambda)}\, \mathrm{d}\lambda.\end{aligned} \tag{4.15}$$

Theorem 4.2.1 just says that the sum on the right-hand side of (4.15) reduces to the first term $\ell = 0$. For the proof let us begin with a slightly different integral

$$\int_{\Lambda_{\mathbb{R}}} d(\lambda)^2 \frac{\exp(2\pi i(\lambda, \ell))}{d(\lambda)} \prod_{k=0}^{N} \frac{\Delta_\lambda(\exp a_k)}{d(\lambda)}\, \mathrm{d}\lambda, \tag{4.16}$$

which by W-symmetrization may be written in the form

$$\frac{1}{|W|} \int_{\Lambda_{\mathbb{R}}} d(\lambda)^2 \frac{\Delta_\lambda(\exp \ell)}{d(\lambda)} \prod_{k=0}^{N} \frac{\Delta_\lambda(\exp a_k)}{d(\lambda)}\, \mathrm{d}\lambda. \tag{4.17}$$

The last integral enters into formula (4.10) for the density $P_H(\exp(-\mathrm{i}\,\ell))$ of the random walk in the hyperbolic space H_G. Since the set $\exp(\mathrm{i}L)$ is *discrete* in H_G, the density $P_H(\exp(-\mathrm{i}\,\ell))$, and integrals (4.16) and (4.17) vanish identically for $\ell \neq 0$ and sufficiently small steps a_k. Taking derivatives of integral (4.16) in the directions of all positive roots $\alpha^v > 0$, we kill the extra factor $d(\lambda) = \prod_{\alpha^v>0}(\lambda, \alpha^v)$ in the denominator, and arrive to the vanishing of all terms in the right-hand side of (4.15) with $\ell \neq 0$. This proves identity (4.12) for small a_k.

The precise form (4.11) of the domain, in which the identity holds, follows from piecewise polynomiality of its left-hand side, which will be proved in the following section, and homogeneity of the right-hand side. □

Now we are in a position to establish relations between the densities P_G, P_L and P_H of random walks in the compact group G, its Lie algebra L_G, and the dual symmetric space $H_G = G_{\mathbb{C}}/G$ with steps $\exp a_k$, a_k, $\exp \mathrm{i}\, a_k$.

Theorem 4.2.3. *The densities P_G, P_L, P_H are related by the formulae*

$$P_L(a) = P_H(\exp \mathrm{i}\, a) \prod_{k=0}^{N} \prod_{\alpha>0} \frac{\sinh \pi(\alpha, a_k)}{\pi(\alpha, a_k)} \tag{4.18}$$

$$= P_G(\exp\, a) \prod_{k=0}^{N} \prod_{\alpha>0} \frac{\sin\, \pi(\alpha, a_k)}{\pi(\alpha, a_k)}, \tag{4.19}$$

where $a_0 = -a$ *and the last equality is valid under the restriction*

$$|(\omega_i, w_0 a_0 + w_1 a_1 + \cdots + w_N a_N)| < 1 \tag{4.20}$$

for all fundamental weights ω_i *and* $w_k \in W$.

Proof. We have to prove only the first identity (4.18), since the second one follows from Theorem 4.2.1 and formulae (4.9) and (4.10) for the densities P_G and P_H.

To proceed we need a formula for the density P_L. We can readily get it by treating a random walk in the Lie algebra L with steps a_k as a properly rescaled walk in H_G with very small steps $\exp(\mathrm{i}\,\varepsilon a_k)$. This leads to the following calculation:

$$\begin{aligned}
&P_L(a_1, a_2, \ldots, a_N \mid a)\\
&\quad = \lim_{\varepsilon\to 0} \varepsilon^{\dim L} P_H(\exp\, \mathrm{i}\,\varepsilon a_1, \exp\, \mathrm{i}\,\varepsilon a_2, \ldots, \exp\, \mathrm{i}\,\varepsilon a_N \mid \exp\, \mathrm{i}\,\varepsilon a)\\
&\quad \overset{4.10}{=} \lim_{\varepsilon\to 0} \frac{C\varepsilon^{\dim L}}{\prod_{k=0}^{N}\prod_{\alpha>0} \sinh\, \pi(\alpha, \varepsilon a_k)} \int_{(\lambda,\alpha_i f^v)>0} d(\lambda)^2 \prod_{k=0}^{N} \frac{\Delta_\lambda(\exp\, \varepsilon a_k)}{d(\lambda)}\, \mathrm{d}\lambda\\
&\quad \overset{\lambda\mapsto\lambda/\varepsilon}{=} C \int_{(\lambda,\alpha_i^v)>0} d(\lambda)^2 \prod_{k=0}^{N} \frac{\Delta_\lambda(\exp\, a_k)}{d(\lambda)}\, \mathrm{d}\lambda \lim_{\varepsilon\to 0} \prod_{k=0}^{N}\prod_{\alpha>0} \frac{\varepsilon}{\sinh \pi(\alpha, \varepsilon a_k)}\\
&\quad \overset{4.10}{=} P_H(\exp\, \mathrm{i}\, a_1, \exp\, \mathrm{i}\, a_2, \ldots, \exp\, \mathrm{i}\, a_N \mid \exp\, \mathrm{i}\, a)\\
&\qquad \times \prod_{k=0}^{N}\prod_{\alpha>0} \frac{\sinh\, \pi(\alpha, a_k)}{\pi(\alpha, a_k)}. \qquad \square
\end{aligned}$$

Corollary 4.2.4. *The supports of the probability measures* P_L *and* P_H *for random walks in* L_G *and* H_G *with steps* a_k *and* $\exp\, \mathrm{i}\, a_k$ *are related by the equation*

$$\operatorname{supp}\, P_H = \exp(\mathrm{i} \operatorname{supp}\, P_L).$$

Proof. By (4.18) the measures differ only by nonvanishing factors $(\sinh\, \pi(\alpha, a_k))/(\pi(\alpha, a_k))$. $\square$

For the unitary group $\mathrm{SU}(n)$ this solves Thompson's conjecture [20].

Theorem 4.2.5. *Let* σ_i, $i = 1, 2, \ldots, N$, *and* σ *be positive spectra. Then the following statements are equivalent*:

1. *There exist matrices* $A_i \in \mathrm{GL}(n, \mathbb{C})$ *with singular spectra* $\sigma_i = \sigma(A_i)$ *and* $\sigma = \sigma(A_1 A_2 \cdots A_N)$.
2. *There exist Hermitian* $n \times n$ *matrices* H_i *with spectra* $\lambda(H_i) = \log \sigma_i$ *and* $\lambda(H_1 + H_2 + \cdots + H_N) = \log \sigma$.

Proof. Solvability of the equations $\lambda(H_1 + H_2 + \cdots + H_N) = \log \sigma$ and $\sigma = \sigma(A_1 A_2 \cdots A_N)$ in (Hermitian) matrices with given (singular) spectra means that σ and $\log \sigma$ are in the supports of the corresponding measures P_H and P_L. Hence the claim follows from the previous corollary. □

Remark 4.3.6. A similar result holds for other classical groups, say for the singular spectrum of a product of complex orthogonal matrices $A_i \in \mathrm{SO}(n, \mathbb{C})$ and the spectrum of a sum of real symmetric $n \times n$ matrices H_i.

5. Piecewise polynomiality

In this section, we prove piecewise polynomiality of sums like

$$\sum_{(\omega,\alpha_i^v)>0} d(\omega)^2 \prod_{k=0}^{N} \frac{\Delta_\omega(\exp a_k)}{d(\omega)}, \tag{5.1}$$

which enter in the density formula (4.9) for random walks in a compact group G. Our exposition follows [17]. The summands are W-invariant functions, hence we may extend the sum over all nonsingular weights $d(\omega) \neq 0$. Since $\Delta_\omega = \sum_{w\in W} \mathrm{sgn}(w)\, \mathrm{e}^{w\omega}$ the problem reduces to the sums of the form

$$\sum_{d(\omega)\neq 0} \frac{\mathrm{e}^{2\pi \mathrm{i}(\omega,a)}}{d(\omega)^{N-1}}$$

for $a = w_0 a_1 + w_1 a_2 + \cdots + w_N a_N$, $w_k \in W$. In addition, $d(\omega) = \prod_{\alpha^v>0}(\omega, \alpha^v)$ is a product of linear forms, hence we finally arrive at the series

$$f_L(x|\alpha_1, \alpha_2, \ldots, \alpha_N) = \sum_{\omega\in 2\pi \mathrm{i}\,\Lambda} \frac{\mathrm{e}^{(\omega,x)}}{(\omega, \alpha_1)(\omega, \alpha_2)\cdots(\omega, \alpha_N)}, \tag{5.2}$$

where the sum runs over those $\omega \in 2\pi \mathrm{i}\,\Lambda$ for which $(\omega, \alpha_k) \neq 0$. Here $\alpha_i \in L$ are arbitrary elements in a lattice L, Λ is the dual lattice, and $x \in L \otimes \mathbb{R}$.

Let us consider affine hyperplanes in $L_{\mathbb{R}}$ of the form $H + a$, $a \in L$, where the subspace $H \subset L \otimes \mathbb{R}$ is spanned by some vectors α_i. They divide $L \otimes \mathbb{R}$ into connected pieces called *chambers* of the system α_k.

Theorem 5.1.1. *Function* (5.2) *is polynomial of degree N on each chamber, and its highest form does not depend on the chamber.*

Remark 5.1.2. Function (5.2) is well defined as a *distribution* even if the system α_k does not span $L_{\mathbb{R}}$. For example, an empty system of vectors gives the δ-function of lattice L (it is just another way to write the Poisson summation formula (4.14)).

Example 5.1.3 (*Root systems*). In the case of the density function (5.1) we deal with the system of positive roots α^v, each taken with multiplicity $N-1$. It is well known that any subspace spanned by a set of roots is parabolic, i.e., spanned by a part of a basis [5, VI.1.7, Proposition 24]. Such a subspace of codimension 1 $\langle\alpha_1, \alpha_2, \ldots, \widehat{\alpha}_i, \ldots, \alpha_n\rangle$ is orthogonal to the fundamental weight ω_i. Hence the chambers of function (5.1) are defined by affine hyperplanes $(\omega, a) = p \in \mathbb{Z}$, with ω conjugate to a fundamental weight, and $a = w_0 a_0 + w_1 a_1 + \cdots + a_N w_N$. The system of hyperplanes $(\omega, x) = p$, as opposed to the mirrors $(\alpha, x) = p$, behaves highly irregularly. Apparently neither the combinatorial structure of the chambers nor even the number of the chambers modulo translations are known.

Both assertions of Theorem 5.1.1 become evident from the following combinatorial description of function (5.2).

Proposition 5.1.4. *Let us define* $\varphi : \mathbb{R}^N \to L \otimes \mathbb{R}$ *by*

$$\varphi : (t_1, t_2, \ldots, t_N) \mapsto t_1\alpha_1 + t_2\alpha_2 + \cdots + t_N\alpha_N. \tag{5.3}$$

Then

$$f_L(x \mid \alpha_1, \alpha_2, \ldots, \alpha_N) = \begin{pmatrix} \textit{mean value of } \langle t_1\rangle\langle t_2\rangle\cdots\langle t_N\rangle \\ \textit{on the fiber } \varphi^{-1}(L - x) \end{pmatrix}, \tag{5.4}$$

where $\langle t\rangle = [t] - (1/2) = \widetilde{B}_1(t)$ *is the periodic extension of the first Bernoulli polynomial.*

Remark 5.1.5. The right-hand side of (5.4) should be understood in the following way. Since the product $\langle t_1\rangle\langle t_2\rangle\cdots\langle t_N\rangle$ is periodic, the mean value may be taken over sections of the unit cube $0 \leqslant t_i \leqslant 1$ by the affine subspaces $\varphi^{-1}(a - x)$, $a \in L$. Eq. (5.4) implies polynomiality of $f_L(x)$ near those x for which the affine subspaces are in general position to the unit cube, i.e., do not intersect its faces of dimension $m < n = \dim L_{\mathbb{R}}$. In other words the polynomiality fails only for $x \equiv t_{i_1}\alpha_{i_1} + t_{i_2}\alpha_{i_2} + \cdots + t_{i_m}\alpha_{i_m} \mod L$, $m < n$, i.e., on the walls of the chambers.

Proof of Proposition 5.1.4. In the following, we will understand the right-hand side of formula (5.2) as the Fourier expansion of a *generalised* function. In particular, $f_L(x|\emptyset)$ is the Fourier expansion of δ-function of the lattice L. With this understanding we have the recurrence relation

$$f_L(x \mid \alpha_1, \alpha_2, \ldots, \alpha_N) = \int_0^1 \left(t - \frac{1}{2}\right) f_L(x + t\alpha_1 | \alpha_2, \alpha_3, \ldots, \alpha_N)\, \mathrm{d}t, \tag{5.5}$$

which may be proved as follows:

$$\int_0^1 \left(1 - \frac{1}{2}\right) f_L(x + t\alpha_1 \mid \alpha_2, \alpha_3, \ldots, \alpha_N)\, \mathrm{d}t$$

$$
\begin{aligned}
&= \sum_{\omega\in 2\pi\mathrm{i}\,\Lambda} \frac{\mathrm{e}^{(x,\omega)}}{(\alpha_2,\omega)(\alpha_3,\omega)\cdots(\alpha_N,\omega)} \int_0^1 \left(t-\frac{1}{2}\right) \mathrm{e}^{(\omega,\alpha_1)t}\,\mathrm{d}t \\
&= \sum_{\omega\in 2\pi\mathrm{i}\,\Lambda} \frac{\mathrm{e}^{(x,\omega)}}{(\alpha_1,\omega)(\alpha_2,\omega)\cdots(\alpha_N,\omega)} \\
&= f_L(x \mid \alpha_1,\alpha_2,\ldots,\alpha_N).
\end{aligned}
$$

In this calculation we use

$$
\int_0^1 \left(t-\frac{1}{2}\right) \exp((\omega,\alpha_1)t)\,\mathrm{d}t = \begin{cases} 0 & \text{if } (\omega,\alpha_1)=0, \\ 1/(\omega,\alpha_1) & \text{if } (\omega,\alpha_1)\neq 0. \end{cases} \tag{5.6}
$$

Applying (5.5) N times we get

$$
\begin{aligned}
&f_L(x \mid \alpha_1,\alpha_2,\ldots,\alpha_N) \\
&\quad= \int_{[0,1]^N} \left(t_1-\frac{1}{2}\right)\cdots\left(t_N-\frac{1}{2}\right) \\
&\qquad\times f_L(x+t_1\alpha_1+\cdots+t_N\alpha_N)\,\mathrm{d}t_1\,\mathrm{d}t_2\cdots\mathrm{d}t_N \\
&\quad= \begin{pmatrix} \text{mean value of } \langle t_1\rangle\langle t_2\rangle\cdots\langle t_N\rangle \\ \text{on the fiber } \varphi^{-1}(L-x) \end{pmatrix}.
\end{aligned}
$$

In the second line $f_L(x) = f_L(x \mid \emptyset)$ is the δ-function of the lattice L. $\square$

In the density function (5.1) we deal with a system of positive roots $\alpha > 0$, each taken with multiplicity $N-1$. In this case, the following version of the proposition may be more relevant.

Corollary 5.1.6. *The function*

$$
f_L\left(x \mid \alpha_1^{m_1},\alpha_2^{m_2},\ldots,\alpha_N^{m_N}\right) = \sum_{\omega\in 2\pi\mathrm{i}\,\Lambda} \frac{\mathrm{e}^{(\omega,x)}}{(\omega,\alpha_1)^{m_1}(\omega,\alpha_2)^{m_2}\cdots(\omega,\alpha_N)^{m_N}}
$$

is equal to the mean value of the product $\prod_{i=1}^{N}(-1)^{m_i+1}(\widetilde{B}_{m_i}(t_i))/m_i!$ *on* $\varphi^{-1}(L-x)$*. Here* $\widetilde{B}_m$ *is the periodic extension of* mth *Bernoulli polynomial on* $(0,1)$.

Proof. To get the result one has to modify the proof of the proposition, using instead of (5.6) the formula

$$
\frac{(-1)^{\nu+1}}{\nu!} \int_0^1 B_\nu(t)\,\mathrm{e}^{(\omega,\alpha_1)t}\,\mathrm{d}t = \begin{cases} 0 & \text{if } (\omega,\alpha_1)=0, \\ 1/(\omega,\alpha_1)^{\nu} & \text{if } (\omega,\alpha_1)\neq 0, \end{cases}
$$

which follows from the Fourier expansion of Bernoulli polynomials (see Example 3.3.2). □

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