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Note

A note on a polynomial time solvable case of the quadratic assignment problem

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Abstract

We identify a class of instances of the Koopmans–Beckmann form of the Quadratic Assignment Problem that are solvable in polynomial time. This class is characterized by a path structure in the flow data and a grid structure in the distance data. Chr18b, one of the test problems in the QAPLIB, is in this class even though this feature of it has not been noticed until now.

Keywords: Quadratic assignment problem; Solvability

In this note, we identify a class of instances of the Quadratic Assignment Problem (QAP) that is solvable in polynomial time. Burkard et al. [1] and Çela [2] give comprehensive surveys of known polynomially solvable cases. Especially we refer to Burkard et al. [3], Deineko and Woeginger [4], and Erdoğan and Tansel [5]. The class we propose is new and is prompted by a closer examination of the test instance chr18b, available in QAPLIB [6], when we observed that this test instance has attained a naïve lower bound in all computational tests we have performed in a recent study of ours [7]. Further investigation of chr18b has revealed that its "flow" data can be characterized by a path structure while its "distance" data can be characterized by that of a "grid" graph. While the structure of the flow data for chr18b can be extracted quite directly, it is not obvious that its distance data comes from a grid structure. This may explain why the polynomial time solvability of chr18b has gone unnoticed for many years until now.

Consider the Koopmans–Beckmann form of the QAP [8] where there are n facilities that must be assigned to n locations. If facilities i and k are assigned to locations j and l, respectively, cost $f_{ik}d_{jl}$ is incurred. We assume f_{ik} are nonnegative, d_{jl} are positive for $j \neq l$ and $d_{jl} = 0$ for $j = l$. Let $I = \{1, \ldots, n\}$ and $\boldsymbol{a} = (a(1), \ldots, a(n))$ be any permutation of the integers $1, \ldots, n$. The Koopmans–Beckmann form of the QAP is to find a permutation (from among $n!$ possible permutations) that minimizes the objective function $\sum_{i,k \in I} f_{ik} d_{a(i)a(k)}$.

Let $F = [f_{ik}]$ and $D = [d_{jl}]$ be the n by n matrices specifying the problem data. Let $G_F = (I, A_F)$ be the undirected graph with node set I and arc set $A_F = \{(i, j) : f_{ij} > 0 \text{ or } f_{ji} > 0\}$. We refer to G_F as the *flow graph* induced by F. We say the flow graph has a *path structure* if it has no cycles and every node has a degree of 0, 1, or 2. A path structure implies each component of the graph is either a path or an isolated node. If the graph is connected, then a path structure is equivalent to a Hamiltonian path.

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We now associate a graph with the distance data D. Given two positive integers a and b, we define an a by b *grid graph* G_{ab} to be an undirected graph with ab nodes such that the nodes are arranged in a rows and b columns and the node in row i and column j is labeled ij ($i = 1, \dots, a$; $j = 1, \dots, b$). The arc set consists of arcs that connect nodes ij and kl if and only if either $i = k$ and $l \in \{j-1, j+1\}$ or $j = l$ and $k \in \{i-1, i+1\}$. Assign the length 1 to each arc of a grid graph. We say an n by n matrix $D = [d_{jl}]$ is induced by a grid graph if there exists two positive integers a and b such that $n = ab$ and that the n by n distance matrix (defined by shortest path lengths) D_{ab} of the grid graph is identical to D up to a positive multiplier; that is, $D = hD_{ab}$ for some positive constant h.

From our study of the test problem chr18b, we have found that its flow graph is a Hamiltonian path and its distance matrix is induced by a grid graph (with $a = 6$ and $b = 3$).

Theorem. *A size n instance of the QAP defined by flow and distance matrices F and D, respectively, is solvable in $O(n)$ time if the flow graph G_F has a path structure and D is induced by an a by b grid graph with $ab = n$.*

This result is a special case of a more general result that we give next. Let d^* be the smallest positive element of D and G^* be the undirected graph with node set I and arc set A^* consisting of arcs (j, l) for which $d_{jl} = d^*$. Observe that if the flow graph G_F is isomorphic to a subgraph of G^*, then an assignment is defined by this isomorphism that produces the objective value $d^* \sum_{(i,k)\in A_F} f_{ik}$ which is the smallest objective value that can be. This implies that whenever G_F has a path structure and G^* is Hamiltonian (a graph in which a Hamiltonian path can be identified in polynomial time), G_F is a subgraph of such a Hamiltonian path in G^* so that the QAP instance is solvable in polynomial time. A special case occurs when D is induced by a grid graph G_{ab} since G^* in this case is G_{ab} itself. Finding a Hamiltonian path in G_{ab} is done in constant time and the evaluation of the objective value takes $O(n)$ time that completes the proof.

If G_F has more than $n - 1$ arcs, it is not a forest and cannot have a path structure. In the remaining case, a breadth-first search [9] identifies a path structure in $O(n)$ time whenever such a structure exists. Determining if G^* has a grid structure or not is relatively more complicated but is still done in $O(n)$ time by a procedure that we outline next. If G^* is a path, it has a grid structure with $a = 1$ and $b = n$. If not, there must be four nodes of degree 2 and all remaining nodes must have degrees of 3 or 4. Nodes of degree 2 and 3 make up the border while the remaining nodes make up the inner nodes. Initially, mark all nodes of degree 4 and their incident arcs as *colored*. If the uncolored subgraph is a Hamiltonian cycle, then it uniquely qualifies as the border. A one-pass traversal along this cycle beginning at a node of degree 2 determines in linear time both the labels of the nodes on the border and the dimensions a and b. Begin now uncoloring the colored subgraph by first uncoloring the colored arcs that are incident to border nodes and then uncoloring their colored end nodes. Next, uncolor the colored arcs whose both ends are incident to already uncolored nodes. This last step defines a new border that consists of the most recently uncolored arcs and nodes. Repeat this process relative to the new border until all colored arcs and nodes are uncolored. In this process, every arc is processed once. Since the number of arcs in a grid graph is bounded above by $2n$, the whole process is done in $O(n)$ time.

It follows that determining whether or not a given QAP instance qualifies as a polynomial time solvable case is done in $O(n)$ time whenever G_F and G^* (equivalently, the positions of the positive entries in F and of the minimal positive elements in D) are available as part of the input. If this is not the case, constructing G_F and G^* directly from F and D is done in $O(n^2)$ time, thereby dominating the time bound of the subsequent steps.

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