On the Design of AQM Supporting TCP Flows Using Robust Control Theory

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Abstract—Recently it has been shown that the active queue management schemes implemented in the routers of communication networks supporting transmission control protocol (TCP) flows can be modeled as a feedback control system. Based on a delay differential equations model of TCPs congestion-avoidance mode different control schemes have been proposed. Here a robust controller is designed based on the known techniques for $\mathcal{H}^\infty$ control of systems with time delays.

Index Terms—Active queue management (AQM), communication networks, control of uncertain systems with time delays, $\mathcal{H}^\infty$ control.

I. INTRODUCTION

Recently several mathematical models of active queue management (AQM) schemes supporting transmission control protocol (TCP) flows in communication networks have been proposed [1]–[3]. From these models a control theory-based approach can be used to analyze or to design AQM schemes. The authors of [2] have derived a delay differential equations model of TCPs congestion avoidance mode and further simplified this model focusing the design of a proportional–integral controller on the low-frequency dynamics, considering the high-frequency dynamics as parasitic. Their controller could guarantee some robustness with respect to the network parameters uncertainties. However, if the uncertainties to be tolerated for stability are "relatively" large, the system's response becomes sluggish. Motivated by their work, we design in this note an $\mathcal{H}^\infty$ controller for their original linear system, without neglecting high-frequency dynamics, that ensures robust stability and good performance for a wider range of network parameters uncertainties. As in [2], we assume that the AQM mechanism brings the system to the neighborhood of an equilibrium (operating point), so that we take the same linear model. Large deviations from this operating point (e.g., in the form of TCP time-out and slow-start phases, buffer overflow, empty queue) are ignored.

Other control theoretic based design of AQM include [4]–[6] while the importance of considering time delays is pointed out in [7]–[9] and a general overview of Internet congestion control literature can be found in [10].

II. MATHEMATICAL MODEL OF AN AQM SCHEME SUPPORTING TCP FLOWS

We consider in this note the network configuration consisting of a single router receiving N TCP flows, we assume that the AQM scheme implemented at the router marks packets using explicit congestion notification (ECN) [11] to inform the TCP sources of impending congestion. In the following, we ignore the TCP slow start and time out
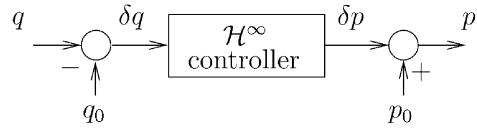
Fig. 1. AQM implementation.

mechanisms, thus providing a model and analysis during the congestion avoidance mode only. In TCP, the congestion window size ($W(t)$) is increased by one every round trip time if no congestion is detected, and is halved upon a congestion detection. This additive-increase multiplicative-decrease behavior of TCP has been modeled in [1] by the following difference equation (case of one TCP flow interacting with a single router):

$$dW(t) = \frac{dt}{R(t)} - \frac{W(t)}{2} \, dN(t) \tag{1}$$

with $R(t) = q(t)/C + T_p$ where T_p is the propagation delay, $q(t)$ is the queue length at the router, C is the router's transmission capacity, thus, $q(t)/C$ is the queuing delay and $R(t)$ is the round trip time delay, and $dN(t)$ is the number of marks the flow suffers. In a network topology of N homogeneous TCP sources and one router, a model relating the average value of these variables and the router's queue dynamics becomes [2]

$$\dot{W}(t) = \frac{1}{R(t)} - \frac{W(t)}{2} \frac{W(t - R(t))}{R(t - R(t))} p(t - R(t)) \tag{2}$$

$$\dot{q}(t) = \left[\frac{N(t)}{R(t)} W(t) - C\right]^+ \tag{3}$$

where $p(t)$ is the probability of packet mark due to the AQM mechanism at the router. Here we use the notation $[x]^+ = x$ if $x \geq 0$, and $[x]^+ = 0$ if $x < 0$.

The linearization of (2) and (3) about the operating point is carried out in [2] and the perturbed variables about the operating point satisfy

$$\delta\dot{W}(t) = -\frac{N}{R_0^2 C}(\delta W(t) + \delta W(t - R_0))$$
$$- \frac{1}{R_0^2 C}(\delta q(t) - \delta q(t - R_0)) - \frac{R_0 C^2}{2N^2}\delta p(t - R_0) \tag{4}$$

$$\delta\dot{q}(t) = \frac{N}{R_0}\delta W(t) - \frac{1}{R_0}\delta q(t) \tag{5}$$

where the operating point is defined by the solution (R_0, W_0, p_0) of the following set of equations:

$$R_0 = \frac{q_0}{C} + T_p \tag{6}$$

$$W_0 = \frac{R_0 C}{N} \tag{7}$$

$$p_0 = \frac{2}{W_0^2} \tag{8}$$

for a desired equilibrium queue level q_0. Then, $W(t) = W_0 + \delta W(t)$, and similarly for $R(t), p(t), q(t)$. Clearly, implementation of the controller depends on q_0, see Fig. 1. In the random early detection (RED) algorithm, q_0 is adjusted by setting appropriate parameters to satisfy

$$p_0 = L_{\text{RED}}(q_0 - \min_{\text{th}}) \tag{9}$$

where L_{RED} and $\min_{\text{th}}$ are the AQM-RED parameters (L_{RED} is the ratio of a small change in packet mark probability to a small change in queue length, and $\min_{\text{th}}$ is the minimum queue length beyond which

packet marking is applied linearly), see [1], [2]. Thus, (6)–(9) define the operating point that is adjusted by the RED parameters. Then, around the operating point, RED can be seen as a linear proportional controller with gain L_{RED}.

Here, we consider the same linear plant derived in [2]. Note that for the linearization the time-varying nature of the roundtrip time delay in the terms "$t - R(t)$" is ignored and these terms are approximated by "$t - R_0$." However, the queue length still depends on the round-trip time in the dynamical equation (3).

From (4) and (5) we derive the transfer function from δp to δq

$$\frac{\delta q(s)}{\delta p(s)} = -\frac{NW_0^3}{2} \frac{A(s)\mathrm{e}^{-R_0 s}}{1 + A(s)R_0 s \mathrm{e}^{-R_0 s}} \qquad (10)$$

where

$$A(s) = \frac{1}{W_0(R_0 s)^2 + (W_0 + 1)R_0 s + 2}.$$

Considering a negative feedback control system with the AQM being the controller, the system to be controlled is given by

$$P(s) = \frac{NW_0^3}{2} \frac{A(s)\mathrm{e}^{-R_0 s}}{1 + A(s)R_0 s \mathrm{e}^{-R_0 s}} \qquad (11)$$

$$= \frac{NW_0^3}{2} \frac{\mathrm{e}^{-R_0 s}}{\frac{R_0 C}{N} R_0^2 s^2 + \left(\frac{R_0 C}{N} + 1\right)R_0 s + 2 + R_0 s \mathrm{e}^{-R_0 s}}. \qquad (12)$$

Lemma 2.1: The plant P defined in (11) is stable for all positive values of R_0, C, and N.

Proof: The poles of the transfer function A are in the left-half part of the complex plane for all values of the parameters W_0 and R_0 positives, thus A is always stable. We also have

$$\left| A(s) R_0 s \mathrm{e}^{-R_0 s} \right|_{s=j\omega}$$
$$= \frac{R_0 \omega}{\sqrt{(2 - W_0 R_0^2 \omega^2)^2 + (W_0 + 1)^2 (R_0 \omega)^2}}$$
$$< 1$$

for all positive values of the parameters W_0 and R_0. So according to the Nyquist stability test the transfer function

$$\frac{1}{1 + A(s) R_0 s \mathrm{e}^{-R_0 s}}$$

is stable for all positive W_0 and R_0. We can thus conclude that the plant P defined in (11) is stable for all positive values of R_0, C, and N. ∎

In the following, we design an AQM scheme based on $\mathcal{H}^\infty$ control techniques that improves the system's transient while stabilizing the plant, and ensures robustness with respect to uncertainties in the values of the system's parameters.

III. MODELING OF THE PARAMETRIC UNCERTAINTIES OF THE PLANT

Note that P in (11) can be written as

$$P(s) = P_2(s) \frac{P_1(s)}{1 + P_1(s)} \qquad (13)$$

with

$$P_1(s) = A(s) R_0 s \mathrm{e}^{-R_0 s} \qquad (14)$$

and

$$P_2(s) = \frac{NW_0^3}{2R_0 s}. \qquad (15)$$

We assume that the parameters have known nominal values $N_n, R_{0n}, C_n, W_{0n} = (R_{0n}C_n/N_n)$ and that we have the following bounds for their uncertainty:

$$|N - N_n| \le \Delta N^+$$
$$|R_0 - R_{0n}| \le \Delta R_0^+$$
$$|C - C_n| \le \Delta C^+.$$

Assuming that $(R_{0n} - \Delta R_0^+), (C_n - \Delta C^+), (N_n - \Delta N^+)$ are positives, we also have

$$|W_0 - W_{0n}| \le \Delta W_0^+$$

where

$$\Delta W_0^+ = \max \left\{ \begin{array}{l} \frac{(R_{0n}+\Delta R_0^+)(C_n+\Delta C^+)}{(N_n-\Delta N^+)} - W_{0n} \\ W_{0n} - \frac{(R_{0n}-\Delta R_0^+)(C_n-\Delta C^+)}{(N_n+\Delta N^+)} \end{array} \right\}.$$

The plant P can be written as

$$P(s) = P_n(s)(1 + \Delta P(s)) \qquad (16)$$

where $P_n(s)$ is the nominal plant ($P(s)$ defined from the nominal values of the parameters N, R_0, C) and $\Delta P(s)$ is the multiplicative plant uncertainty. We would like to design a controller for the nominal plant $P_n(s)$ that would also stabilize the actual plant $P(s)$ that would be obtained by the same linearization process around the actual equilibrium point. In the following our goal is to find a bound $W_2(s)$ for the multiplicative plant uncertainty $\Delta P(s)$. For clarity of presentation we omit in the following the argument of the transfer functions (i.e., we just write P instead of $P(s)$). Note that

$$\frac{P_{1n} + \Delta P_1}{1 + P_{1n} + \Delta P_1} (P_{2n} + \Delta P_2)$$
$$= \frac{P_{1n}P_{2n}}{1 + P_{1n}} \left(\frac{1}{1 + \frac{\Delta P_1}{1+P_{1n}}} + \frac{\frac{\Delta P_2}{P_{2n}}}{1 + \frac{\Delta P_1}{1+P_{1n}}} + \frac{\frac{\Delta P_1(P_{2n}+\Delta P_2)}{P_{1n}P_{2n}}}{1 + \frac{\Delta P_1}{1+P_{1n}}} \right)$$
$$= \frac{P_{1n}P_{2n}}{1 + P_{1n}} \left(1 + \frac{-\frac{\Delta P_1}{1+P_{1n}} + \frac{\Delta P_2}{P_{2n}} + \frac{\Delta P_1(P_{2n}+\Delta P_2)}{P_{1n}P_{2n}}}{1 + \frac{\Delta P_1}{1+P_{1n}}} \right)$$

so

$$\Delta P = \frac{-\frac{\Delta P_1}{1+P_{1n}} + \frac{\Delta P_2}{P_{2n}} + \frac{\Delta P_1(P_{2n}+\Delta P_2)}{P_{1n}P_{2n}}}{1 + \frac{\Delta P_1}{1+P_{1n}}}$$
$$= \frac{1}{1 + P_{1n} + \Delta P_1} \cdot \left(\frac{\Delta P_1}{P_{1n}} + \frac{\Delta P_2}{P_{2n}}(1 + P_{1n}) \right.$$
$$\left. + \frac{\Delta P_1}{P_{1n}} \frac{\Delta P_2}{P_{2n}}(1 + P_{1n}) \right). \qquad (17)$$

It can be shown that a bound W_2 of the multiplicative plant uncertainty, i.e., W_2 that satisfies

$$|\Delta P(s)|_{s=j\omega} \le |W_2(s)|_{s=j\omega} \quad \forall \omega \in \mathbb{R}^+ \qquad (18)$$

is (see details of the derivation in the Appendix)

$$W_2(s) = a + bs + cs^2 \qquad (19)$$

with a, b, and c defined in (38)–(40).

IV. $\mathcal{H}^\infty$ OPTIMIZATION PROBLEM

We design the controller to minimize the following $\mathcal{H}^\infty$ cost function:

$$\inf \left\| \begin{bmatrix} W_1(1 + P_n(s)C(s))^{-1} \\ W_2 P_n(s)C(s)(1 + P_n(s)C(s))^{-1} \end{bmatrix} \right\|_\infty =: \gamma_{\mathrm{opt}} \qquad (20)$$

where the infimum is taken over all C stabilizing P_n, and $W_1(s) = 1/s$ for good tracking of step-like reference inputs (desired queue size). Applying the formulas given in [12] and [13], the optimal solution to (20) is found as (a similar derivation can be found in [14])

$$C_{\text{opt}}(s) = \frac{2\gamma}{cN_n W_{0n}^3}$$
$$\times \frac{W_{0n} R_{0n}^2 s^2 + (W_{0n} + 1)R_{0n}s + 2 + R_{0n}se^{-R_{0n}s}}{s^2}$$
$$\cdot \frac{1}{1 + \frac{a_2 c_2 \gamma^2}{s} + F(s)} \qquad (21)$$

where F is a finite impulse response filter defined as

$$f(t) = \begin{cases} (b_2 + a_2 - a_2 c_2 \gamma^2)\cos\left(\frac{t}{\gamma}\right) \\ \quad + \gamma\left(c_2 + a_2 b_2 - \frac{1}{\gamma^2}\right)\sin\left(\frac{t}{\gamma}\right), & \text{for } t < R_{0n} \\ 0, & \text{otherwise} \end{cases} \qquad (22)$$

with

$$a_2 = \frac{1}{c\gamma}\sqrt{\frac{\gamma^2 - a^2}{x}} \qquad (23)$$

$$b_2 = \sqrt{\frac{(b^2 - 2ac)\gamma^2 - c^2}{c^2\gamma^2} + 2\sqrt{x} - \frac{\gamma^2 - a^2}{c^2\gamma^2 x}} \qquad (24)$$

$$c_2 = \sqrt{x} \qquad (25)$$

where x is the unique positive root of

$$x^3 + \frac{b^2 - 2ac - a^2\gamma^2}{c^2\gamma^2}x^2 - \frac{(\gamma^2 - a^2)[(2ac - b^2)\gamma^2 + c^2]}{c^4\gamma^4}x$$
$$- \frac{(\gamma^2 - a^2)^2}{c^4\gamma^4} = 0 \qquad (26)$$

and γ is determined as the largest root of

$$1 - \frac{\gamma}{c}e^{-R_{0n}s}\frac{s}{(s + a_2)(s^2 + b_2 s + c_2)}\bigg|_{s=\frac{i}{\gamma}} = 0. \qquad (27)$$

V. IMPLEMENTATION ISSUES

The packets are marked with probability $p(t)$ computed by the $\mathcal{H}^\infty$ controller (21) according to the block diagram shown in Fig. 1 where q_0 is the desired steady-state queue length and

$$p_0 = \frac{2N_n^2}{\left(\frac{q_0}{C_n} + T_{pn}\right)^2 C_n^2} \qquad (28)$$

is calculated from (6)–(8).

VI. SIMULATIONS

The simulations are carried out using `simulink`, the nonlinear model defined by (2) and (3) representing the dynamics of N TCP flows loading a router. The router implements the AQM scheme defined by (21). The following scenario is considered.

- Nominal values known to the controller: $N_n = 50$ TCP sessions, $C_n = 300$ packets/s, $T_p = 0.2$ s, so $R_{0n} = 0.533$ s and $W_{0n} = 3.2$ packets (we assume here a fluid model and, thus, we do not consider packetization issues).
- Real values of the plant: $N = 40$ TCP sessions, $C = 250$ packets/s, $T_p = 0.3$ s, so $R_0 = 0.7$ s and $W_0 = 4.375$ packets.
- The following controller design parameters are considered: $\Delta N^+ = 10$, $\Delta R_0^+ = 0.1$, $\Delta C^+ = 50$, which implies $\Delta W_0^+ = 2.3417$.
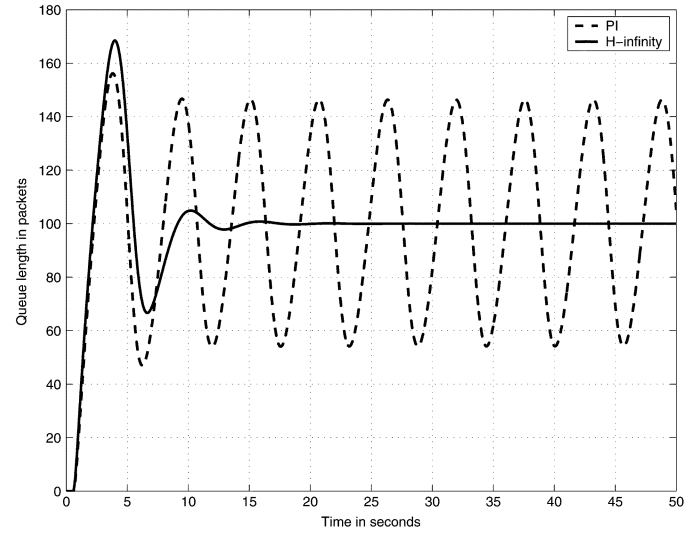

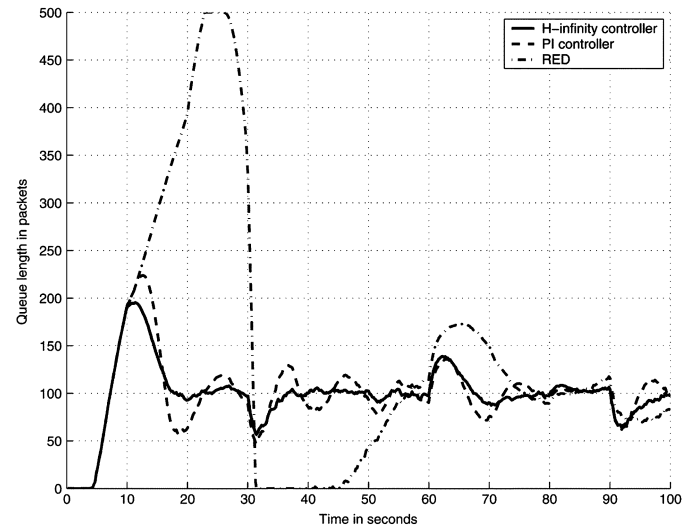Fig. 2. Comparison between $\mathcal{H}^\infty$ and PI controllers.

Fig. 3. Comparison between $\mathcal{H}^\infty$, PI controllers and RED.

For comparison we also simulate the PI AQM scheme proposed in [2]. The parameters of the PI controller are computed as suggested by the authors of [2] using the aforementioned known nominal values. We can see in Fig. 2 that the $\mathcal{H}^\infty$ controller performs significantly better than the PI controller for this set of parameters.

In Figs. 3 and 4, we analyze the robustness of the two schemes with respect to variations in the network parameters. The outgoing link capacity C is a normally distributed random signal with mean 250 packets/s and variance 50 added to a pulse of period 60 s, amplitude 60 packets/s. The number of TCP flows N is a normally distributed random signal with mean 45 and variance 30 added to a pulse of period 20 s and amplitude 10. The propagation delay T_p is a normally distributed random signal with mean 0.8 s and variance 0.05 s added to a pulse of period 20 s and amplitude 0.2 s. The controllers have the following value known to them: $C = 300$ packets/s, $N = 50$, $T_p = 0.7$ s and the desired queue length is $q_0 = 100$ packets. In addition the $\mathcal{H}^\infty$ controller uses the following design parameters: $\Delta N^+ = 10$, $\Delta R_0^+ = 0.1$, and $\Delta C^+ = 50$. RED [15] has the following parameters: $p_{\max} = 0.1$, $\min_{\text{th}} = 80$, $\max_{\text{th}} = 150$, (these determine the AQM-RED controller gain as $L_{\text{RED}} = (p_{\max})/(\max_{\text{th}} - \min_{\text{th}})$) and queue averaging weight $= 0.0001$. It can be seen that both the
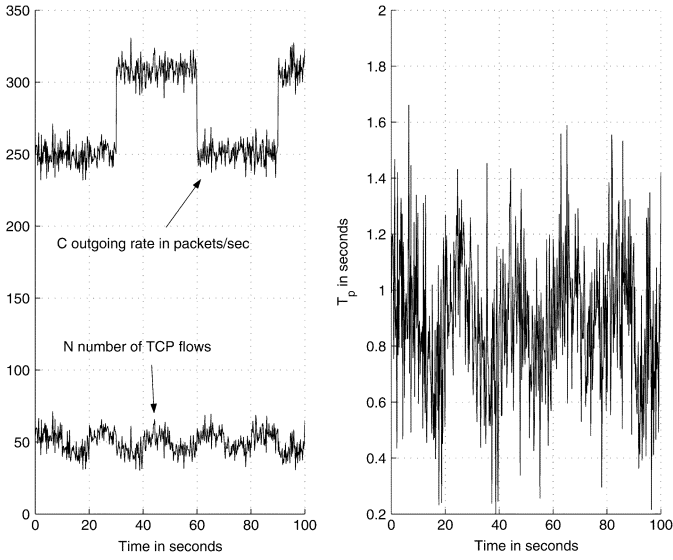
Fig. 4. Values of C, N, and T_p corresponding to Fig. 3.

PI and the $\mathcal{H}^\infty$ controllers perform better than RED as the initial over-shoot is reduced, and the system oscillates less under parameter variations. The wide oscillations of RED can cause the queue to become empty, thus decreasing the utilization of the outgoing link.

VII. CONCLUDING REMARKS

We developed in this note a robust AQM control scheme supporting ECN and TCP flows. The simulation experiments based on a non-linear fluid flow model, that includes delays, show that the proposed AQM scheme performs better than RED and the proportional-integral scheme by obtaining faster transients and less oscillatory responses, which translates into higher link utilization, low packet loss rate and small queue fluctuations. A challenging extension of this work would be to consider networks having multiple links and sources, but such an extension is not trivial due to the interaction between the control schemes of each router.

APPENDIX

From (17), we see that we can bound the multiplicative plant uncertainty in the following way:

$$
|\Delta P(s)|_{s=j\omega}
$$
$$
< \frac{1}{1 - \text{upper bound on} |P_{1n} + \Delta P_1|_{s=j\omega}}
$$
$$
\cdot \left(\left| \frac{\Delta P_1}{P_{1n}} \right|_{s=j\omega} + \left| \frac{\Delta P_2}{P_{2n}} (1 + P_{1n}) \right|_{s=j\omega} \right.
$$
$$
\left. + \left| \frac{\Delta P_1}{P_{1n}} \right|_{s=j\omega} \cdot \left| \frac{\Delta P_2}{P_{2n}} (1 + P_{1n}) \right|_{s=j\omega} \right). \qquad (29)
$$

Note that

$$
\frac{1}{W_{0n} R_{0n}^2 s^2 + (W_{0n} + 1) R_{0n} s + 2} = \frac{\frac{1}{W_{0n} R_{0n}^2}}{s^2 + 2\xi \omega_0 + \omega_0^2} \qquad (30)
$$

with

$$
\omega_0 = \frac{1}{R_{0n}} \sqrt{\frac{2}{W_{0n}}} \quad \xi = \frac{W_{0n} + 1}{2\sqrt{2 W_{0n}}}.
$$

We have

$$
W_{0n} + 1 \geq 2\sqrt{W_{0n}} \qquad \forall \, W_{0n}
$$

thus

$$
\xi \geq \frac{1}{\sqrt{2}} \qquad \forall W_{0n}
$$

and, consequently, the transfer function (30) do not exhibit any resonance phenomenon and

$$
\left| \frac{1}{W_{0n} R_{0n}^2 s^2 + (W_{0n} + 1) R_{0n} s + 2} \right|_{s=j\omega}
$$
$$
\leq \left| \frac{1}{W_{0n} R_{0n}^2 s^2 + (W_{0n} + 1) R_{0n} s + 2} \right|_{s=0}
$$
$$
\leq \frac{1}{2}.
$$

$$
\Delta P_2(s) = P_{2n}(s) + \Delta P_2(s) - P_{2n}(s)
$$
$$
= \frac{N_n W_{0n}^3 + N_n \left(3W_{0n}^2 \Delta W_0 + 3W_{0n} \Delta W_0^2 + \Delta W_0^3 \right) + \Delta N (W_{0n} + \Delta W_0)^3}{2R_{0n} s \left(1 + \frac{\Delta R_0}{R_{0n}} \right)} - \frac{N_n W_{0n}^3}{2R_{0n} s}
$$
$$
= \frac{N_n \left(3W_{0n}^2 \Delta W_0 + 3W_{0n} \Delta W_0^2 + \Delta W_0^3 \right) + \Delta N (W_{0n} + \Delta W_0)^3 - N_n W_{0n}^3 \frac{\Delta R_0}{R_{0n}}}{2R_{0n} s \left(1 + \frac{\Delta R_0}{R_{0n}} \right)}
$$

$$
\left| \frac{\Delta P_1}{P_{1n}} \right|_{s=j\omega}
$$
$$
= \left| 1 - \frac{P_{1n} + \Delta P_1}{P_{1n}} \right|_{s=j\omega}
$$
$$
= \left| 1 - \frac{R_{0n} + \Delta R_0}{R_{0n}} \cdot e^{-\Delta R_0 s} \right.
$$
$$
\left. \cdot \frac{W_{0n} R_{0n}^2 s^2 + (W_{0n} + 1) R_{0n} s + 2}{(W_{0n} + \Delta W_0)(R_{0n} + \Delta R_0)^2 s^2 + (W_{0n} + \Delta W_0 + 1)(R_{0n} + \Delta R_0) s + 2} \right|_{s=j\omega}
$$
$$
\leq 1 + \left(1 + \frac{\Delta R_0^+}{R_{0n}} \right) \frac{1}{2} \left| W_{0n} R_{0n}^2 s^2 + (W_{0n} + 1) R_{0n} s + 2 \right|_{s=j\omega}. \qquad (32)
$$
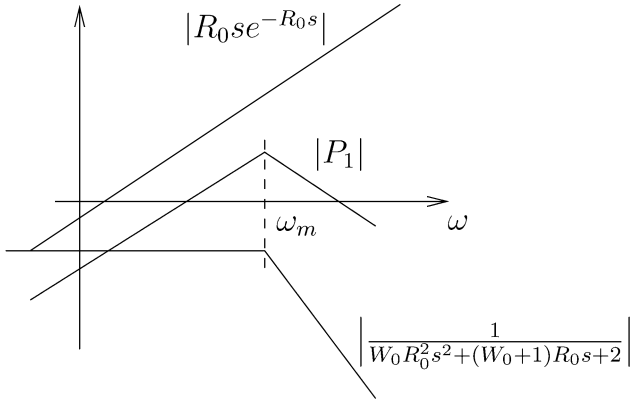
Fig. 5. Case $\Delta < 0$.
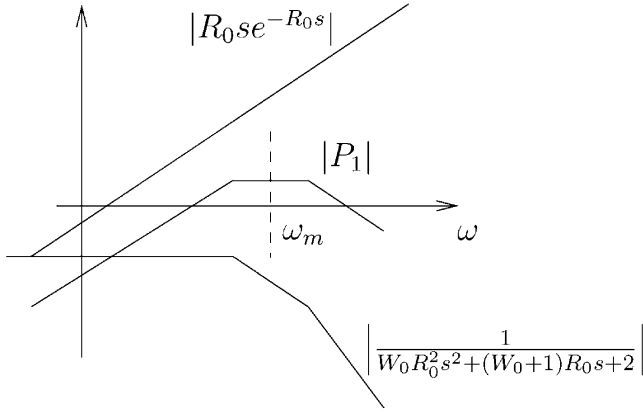
Fig. 6. Case $\Delta > 0$.

Similarly, we have

$$|(W_{0n} + \Delta W_0)(R_{0n} + \Delta R_0)^2 s^2$$
$$+ (W_{0n} + \Delta W_0 + 1)(R_{0n} + \Delta R_0)s + 2|_{s=j\omega} \geq 2.$$

The discriminant of the second order polynomial in s formed by the denominator of the transfer function $P_1(s)$ is

$$\Delta = (W_0 + 1)^2 R_0^2 - 8W_0 R_0^2$$
$$= R_0^2 (W_0^2 - 6W_0 + 1).$$

A sketch of the bode plot of P_1 is shown in Figs. 5, and 6. For the case of $\Delta < 0$ corresponding to Fig. 5 an upper bound for the magnitude of $P_1(s)$ evaluated on the imaginary axis is $|P_1(\omega_m)|$ where $\omega_m = (1/R_0)\sqrt{(2/W_0)}$. For the case of $\Delta > 0$ corresponding to Fig. 6 an upper bound for the magnitude of $P_1(s)$ evaluated on the imaginary axis is $|P_1(\omega_m)|$ where ω_m is the geometric mean between the roots of the second-order polynomial in s formed by the denominator of the transfer function $P_1(s)$, which is also $\omega_m = (1/R_0)\sqrt{(2/W_0)}$. Thus, we have

$$|P_{1n} + \Delta P_1|_{s=j\omega} < \max_{\Delta W_0, \Delta R_0} \left(\frac{R_{0n} + \Delta R_0}{2} \cdot \omega_m \right)$$
$$< \max_{\Delta W_0} \left(\frac{1}{2}\sqrt{\frac{2}{W_{0n} + \Delta W_0}} \right).$$

Assuming $W_{0n} + \Delta W_0 > 1$, we have

$$|P_{1n} + \Delta P_1|_{s=j\omega} < \frac{\sqrt{2}}{2}. \tag{31}$$

We also have the first equation shown at the bottom of the previous page, as well as (32), also shown at the bottom of the previous page, and a bound of the magnitude of $\Delta P_2(s)$ on the imaginary axis is

$$|\Delta P_2(s)|_{s=j\omega} < \left| \frac{Z_2}{s} \right|_{s=j\omega}$$

with

$$Z_2 = \frac{N_n \left(3W_{0n}^2 \Delta W_0^+ + 3W_{0n}(\Delta W_0^+)^2 + (\Delta W_0^+)^3 \right)}{2R_{0n} \left(1 - \frac{\Delta R_0^+}{R_{0n}} \right)}$$
$$+ \frac{\Delta N^+(W_{0n} + \Delta W_0^+)^3 + N_n W_{0n}^3 \frac{\Delta R_0^+}{R_{0n}}}{2R_{0n} \left(1 - \frac{\Delta R_0^+}{R_{0n}} \right)}$$

so we have

$$\left| \frac{\Delta P_2}{P_{2n}} \right|_{s=j\omega} < \frac{2R_{0n} Z_2}{N_n W_{0n}^3}. \tag{33}$$

We also have

$$|1 + P_{1n}|_{s=j\omega} \leq 1$$
$$+ \left| \frac{R_{0n} s}{W_{0n} R_{0n}^2 s^2 + (W_{0n} + 1)R_{0n} s + 2} \right|_{s=j\frac{1}{R_{0n}}\sqrt{\frac{2}{W_{0n}}}}$$
$$\leq 1 + \frac{1}{W_{0n} + 1}$$
$$\leq \frac{3}{2} \tag{34}$$

if we assume $W_{0n} > 1$.

With (31)–(34), (29) can be written as

$$|\Delta P(s)|_{s=j\omega} < \tilde{W}_2(\omega) \tag{35}$$

with

$$\tilde{W}_2(\omega) = \frac{1}{1 - \frac{\sqrt{2}}{2}} \left\{ \left[1 + \frac{1}{2} \left(1 + \frac{\Delta R_0^+}{R_{0n}} \right) \right. \right.$$
$$\left. \cdot \sqrt{(2 - W_{0n} R_{0n}^2 \omega^2)^2 + (W_{0n} + 1)^2 R_{0n}^2 \omega^2} \right]$$
$$\left. \cdot \left(1 + \frac{3R_{0n} Z_2}{N_n W_{0n}^3} \right) + \frac{3R_{0n} Z_2}{N_n W_{0n}^3} \right\}$$
$$= a_3 + \sqrt{b_3 + c_3 \omega^2 + d_3 \omega^4}$$

and

$$a_3 = \frac{1}{1 - \frac{\sqrt{2}}{2}} \left\{ 1 + \frac{6R_{0n} Z_2}{N_n W_{0n}^3} \right\}$$
$$e_3 = \frac{1}{4} \left(\frac{1}{1 - \frac{\sqrt{2}}{2}} \right)^2 \left(1 + \frac{\Delta R_0^+}{R_{0n}} \right)^2 \left(1 + \frac{3R_{0n} Z_2}{N_n W_{0n}^3} \right)^2$$
$$b_3 = 4e_3$$
$$c_3 = \left(W_{0n}^2 + R_{0n}^2 - 2W_{0n} R_{0n}^2 \right) e_3$$
$$d_3 = W_{0n}^2 R_{0n}^4 e_3.$$

Note that we have

$$\tilde{W}_2(\omega) < |W_2(s)|_{s=j\omega} \tag{36}$$

with

$$W_2(s) = a + bs + cs^2 \tag{37}$$

where

$$a = a_3 + \sqrt{b_3} \tag{38}$$

$$c = \sqrt{d_3} \tag{39}$$

$$b = \max \left\{ \sqrt{2 a_3 \sqrt{d_3} + 2ac + c_3}, \sqrt{\frac{a_3 c_3}{\sqrt{b_3}} + 2ac + c_3} \right\}. \tag{40}$$

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