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Integrality of Rational D-Series

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INTRODUCTION

Atiyah and Todd [1] determined the smallest positive integer M_k such that $((\log(1+x))/x)^{M_k} \in Z[\![x]\!] \pmod{x^{k+1}}$. Motivated by this we define two invariants d_k and e_k for rational power series with constant term ∓ 1 which are additive and multiplicative measures of their integrality. The result of [1] is then interpreted as $e_k((\log(1+x))/x) = M_k$.

We prove that d_k and e_k are divisible by the same set of primes. We define a power series to be stable mod x^{k+1} iff $d_i = e_i$ for $1 \leqslant i \leqslant k$ and give sufficient conditions in Lemma 2.2 for stability.

We then define a class of rational power series called rational D-series which are dominated by their first non-trivial coefficients with respect to different primes; i.e., the p-components of the denominators whose coefficients are bounded by an inequality involving the p-component of the first non-trivial coefficient with respect to p.

Theorem 3.5 which is our main result gives vital information about the invariant e_k of rational D-series and about sufficient conditions for the stability of strict D-series. The result of [1] follows as a corollary. Theorem 3.5 is also applied to obtain upper bounds for e_k of the series $x/(e^x - 1)$ and $x/\sin x$.

In Section 5 the finite quotient group $D(\alpha, \beta; k)$ of a weak D-series is defined and Theorem 3.5 is applied to compute the highest invariant of the p-summand of $D(\alpha, \beta; k)$. It would be an interesting problem to determine the groups $D(\alpha, \beta; k)$ completely.

In Section 6 analogues of our results are derived when the ring of rational power series is equipped with the starproduct defined by $x^i * x^j = x^{ij}$ instead of the usual product given by $x^i \cdot x^j = x^{i+j}$.

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1. Integrality of Rational Series

Let Z be the ring of integers and Q the field of rationals. Let $z[\![x]\!]$ and $Q[\![x]\!]$ be the rings of formal power series with coefficients in Z and Q, respectively. Let $Z^*[\![x]\!] = (f(x) \in Z[\![x]\!]/f(0) = \mp 1)$ be the group of units of $Z[\![x]\!]$. Similarly, let $Q^*[\![x]\!] = (f(x) \in Q[\![x]\!]/f(0) = \mp 1)$. For any $k \in Z^+$, $Z[\![x]\!]/(x^{k+1})$ is a subring of $Q[\![x]\!]/(x^{k+1})$. Let

$$A_k = \frac{Q[\![x]\!]/(x^{k+1})}{Z[\![x]\!]/(x^{k+1})}$$

be their additive quotient group.

The group $Z_{k,X}$ of units of $Z[\![x]\!]/(x^{k+1})$ is the image of $Z^*[\![x]\!]$ under the quotient map $Z[\![x]\!] \to Z[\![x]\!]/(x^{k+1})$.

Let $Q_{k,X}$ be the image of $Q^*[\![x]\!]$ under the quotient map $Q[\![x]\!] \to Q[\![x]\!]/(x^{k+1})$.

Let $M_k = Q_{k,X}/Z_{k,X}$ be their multiplicative quotient group.

Definition 1.1. For each $k \in Z^+$ we define d_k and e_k to be the algebraic order in the additive group A_k and the multiplicative group M_k, respectively; i.e.,

for $f(x) \in Q[\![x]\!]$, $d_k(f)$ is the smallest positive integer d_k such that $d_k f(x) \in Z[\![x]\!] \pmod{x^{k+1}}$.

For $f(x) \in Q^*[\![x]\!]$, $e_k(f)$ is the smallest positive integer e_k such that $f(x)^{e_k} \in Z[\![x]\!] \pmod{x^{k+1}}$.

For $f(x) \in Q^*[\![x]\!]$, both invariants $d_k(f)$ and $e_k(f)$ are defined.

Lemma 1.2. *For $f(x) \in Q^*[\![x]\!]$, $d_k(f)$ depends only on the equivalence class of $f(x)$ in M_k.*

Proof. Let $f(x) = \mp 1 + \sum_{i=1}^{k} a_i x^i \pmod{x^{k+1}}$, $h(x) = \mp 1 + \sum_{i=1}^{k} b_i x^i \pmod{x^{k+1}}$, where $a_i \in Q$, $b_i \in Z$ $(1 \leqslant i \leqslant k)$.

Let $F(x) = f(x)\,h(x) = \mp 1 + \sum_{i=1}^{k} c_i x^i \pmod{x^{k+1}}$.

We prove by induction on i that $d_i(F) = d_i(F) = d_i(f)$ $(0 \leqslant i \leqslant k)$. It is true for $i = 0$ since $d_0(F) = d_0(f) = 1$. Let $i \geqslant 1$ and assume it to be true for $(i-1)$. $c_i = \mp a_i + a_{i-1} b_1 + \cdots + a_1 b_{i-1} \mp b_i$.

$$d_{i-1}(f) = d_{i-1}(F)/d_i(F).$$

$$d_{i-1}(f)(a_{i-1} b_1 + a_{i-2} b_2 + \cdots + a_1 b_{i-1} \mp b_i) \in Z,$$

and hence,

$$d_i(F)(a_{i-1} b_1 + a_{i-2} b_2 + \cdots + a_1 b_{i-1} \mp b_i) \in Z.$$

Thus $d_i(F)\, c_i \in Z$ iff $d_i(F)\, a_i \in Z$ and hence $d_i(F) = d_i(f)$. Putting $i = k$ we obtain $d_k(F) = d_k(f)$. Q.E.D.

Notation. For a rational number q, $D(q)$ denotes its denominator in its lowest term.

LEMMA 1.3. *Let $f(x) \in Q^*[\![x]\!]$ and expand*

$$f(x)^{e_{k-1}(f)} = \mp 1 + \sum_{j=1}^{k-1} m_j x^j + b_k x^k \pmod{x^{k+1}},$$

where $m_j \in Z$ $(1 \leqslant j \leqslant k-1)$ and $b_k \in Q$. Then $e_k(f) = e_{k-1}(f)\, D(b_k)$.

Proof. $e_{k-1}(f)/e_k(f)$.

$$f(x)^{ne_{k-1}(f)} = \mp 1 + \sum_{j=1}^{k-1} n_j x^j + c_k x^k \pmod{x^{k+1}},$$

where $n_j \in Z$ $(1 \leqslant j \leqslant k-1)$ $c_k = nb_k + n_k$ for $n_k \in Z$. Thus $f(x)^{ne_{k-1}(f)} \in Z[\![x]\!]$ $(\operatorname{mod} x^{k+1})$ iff $nb_k \in Z$. If $e_k(f) = n_0 e_{k-1}(f)$ then n_0 is the smallest positive integer such that $n_0 b_k \in Z$; i.e., $n_0 = D(b_k)$.

COROLLARY 1.3. *If $f(x) \in Q^*[\![x]\!]$, then $e_k(f)$ is finite and is given by $e_k(f) = D(b_1)\, D(b_2) \cdots D(b_k)$.*

Proof. It follows from Lemma 1.3 by induction on k.

DEFINITION 1.4. Let $f(x) \in Q^*[\![x]\!]$ and let $f(x) = \mp 1 + \sum_{j=1}^{k} a_j x^j$ $(\operatorname{mod} x^{k+1})$. We define $S_k(f)$ to be the set of primes dividing the denominators of the coefficients a_j $(1 \leqslant j \leqslant k)$.

LEMMA 1.5. *Let $f(x) \in Q^*[\![x]\!]$. Then $d_k(f)$ and $e_k(f)$ are divisible by the same set $S_k(f)$ of primes.*

Proof. By definition $d_k(f) = \operatorname{LCD}(a_1, a_2, ..., a_k)$ and hence a prime $p/d_k(f)$ iff $p \in S_k(f)$. Let $S'_k(f)$ be the set of primes dividing the denominators of $b_1, b_2, ..., b_k$ as they are defined in Lemma 1.3. It follows from Corollary 1.3 that a prime $p/e_k(f)$ iff $p \in S'_k(f)$. We thus prove by induction on k that $S_k(f) = S'_k(f)$. For $k = 1$, $a_1 = b_1$ and hence $S_1(f) = S'_1(f)$.

Let $k > 1$ and assume it to be true for $(k-1)$. Raising $f(x)$ to the e_{k-1}-th power and equating coefficients we obtain $b_k = \mp e_{k-1} a_k + T_{k-1}$, where

$$T_{k-1} = \sum_{\substack{s_0 + \cdots + s_{k-1} = e_{k-1} \\ s_1 + 2s_2 + \cdots + (k-1)s_{k-1} = k}} \frac{\mp e_{k-1}!}{s_0!\, s_1! \cdots s_{k-1}!}\, a_1^{s_1} a_2^{s_2} \cdots a_{k-1}^{s_{k-1}},$$

$$\frac{e_{k-1}!}{s_0! \cdots s_{k-1}!} \in Z.$$

The primes dividing $D(a_1^{s_1} \cdots a_{k-1}^{s_{k-1}})$ belong to the set $S_{k-1}(f)$. Hence if a prime $p/D(T_{k-1})$ then $p \in S_{k-1}(f)$.

If $p \in S'_k(f) - S'_{k-1}(f) = S'_k(f) - S_{k-1}(f)$ then $p/D(b_k)$ and $p \times D(T_{k-1})$. It follows that $p/D(e_{k-1}a_k)$ and hence that $p/D(a_k)$; i.e., that $p \in S_k(f) - S_{k-1}(f)$.

Conversely, if $p \in S_k(f) - S_{k-1}(f)$, it follows that $p/D(a_k)$. By the induction hypothesis, $p \times e_{k-1}$ and hence $p/D(e_{k-1}a_k)$, and since $p \times D(T_{k-1})$ it follows that $p/D(b_k)$; i.e., $p \in S'_k(f) - S_{k-1}(f) = S'_k(f) - S'_{k-1}(f)$. We have thus proved that $S'_k(f) - S'_{k-1}(f) = S_k(f) - S_{k-1}(f)$. It follows from this and the induction hypothesis that $S'_k(f) = S_k(f)$. Q.E.D.

2. Stability of Rational Power Series

Definition 2.1. We say that $f(x) \in Q^*[\![x]\!]$ is stable $\bmod x^{k+1}$ iff $d_i(f) = e_i(f)$, $1 \leqslant i \leqslant k$.

Notation. For a rational number q and a prime p, let $v_p(q)$ denote the exponent of p in the prime factorization of q.

Lemma 2.2. *Let* $f(x) = \mp 1 + \sum_{j=1}^{\infty} a_j x^j \in Q^*[\![x]\!]$. *If* $a_j \in Z$, $1 \leqslant j \leqslant r-1$, *then* $f(x)$ *is stable* $\bmod x^{2r}$.

Proof. Let $r \leqslant k \leqslant 2r-1$, $p \in S_k(f)$, and $\alpha = v_p(d_k(f)) = -\min_{r \leqslant j \leqslant k} v_p(a_j)$. Let a_i be the first coefficient $r \leqslant i \leqslant k$ for which $v_p(a_i) = -\alpha$. Let

$$f(x)^{p^u} = \mp 1 + \sum_{j=1}^{\infty} b_{u,j} x^j, \qquad b_{u,j} = \sum_{\substack{s_0 + \cdots + s_j = p^u \\ s_1 + \cdots + js_j = j}} T_f(n, k, s)$$

for $r \leqslant j \leqslant 2r-1$, where

$$T_f(n, k, s) = \frac{(p^u)!}{s_0! \cdots s_j!} a_1^{s_1} \cdots a_j^{s_j}.$$

Given each sequence $s = (s_0, s_1, ..., s_j)$ there exists at most one $r \leqslant t \leqslant j$ for which $s_t > 0$ and s_t is at most one. We claim the following.

Statement. For $r \leqslant j \leqslant i$ either $T_f(n, k, s) \in Z_p$ or $v_p T_f(n, k, s) \geqslant u - \alpha + 1$ for all sequences other than the sequence s given by $s_i = 1$, $s_0 = p^u - 1$, and $s_h = 0$, $h \neq 0, i$, for which

$$v_p T_f(n, k, s) = u - \alpha.$$

For $i \leqslant j \leqslant k$, either $T_f(n, k, s) \in Z_p$ or $v_p T_f(n, k, s) \geqslant u - \alpha$.

Proof. Let $r \leqslant j \leqslant i$. If s is the sequence given by $s_i = 1$, $s_0 = p^u - 1$, $s_h = 0$, $j \neq 0, i$, then $T_f(n, k, s) = p^u a_1^{s_1} \cdots a_{r-1}^{s_{r-1}} a_i$ and hence $v_p T_f(n, k, s) = u - \alpha$. For all other sequences, either $s_t = 0$ for $t \geqslant r$ in which case $T_f(n, k, s) \in Z_p$, or there exists precisely one $t \geqslant r$ for which $s_t = 1$. Let $\sigma = s_0 + s_1 + \cdots + s_{r-1}$. Then

$$T_f(n, k, s) = p^u \frac{\sigma!}{s_0! \cdots s_{r-1}!} a_1^{s_1} \cdots a_{r-1}^{s_{r-1}} a_t$$

and hence $v_p T_f(n, k, s) \geqslant u - \alpha + 1$.

Let $i \leqslant j \leqslant k$. If $s_t = 0$ for all $t \geqslant r$ then $T_f(n, k, s) \in Z_p$, and if there exists only one $t \geqslant r$ for which $s_t = 1$ then

$$T_f(n, k, s) = p^u \frac{\sigma!}{s_0! \cdots s_{r-1}!} a_1^{s_1} \cdots a_{r-1}^{s_{r-1}} a_t$$

and hence $v_p T_f(n, k, s) \geqslant u - \alpha$.

From the statement follows the

COROLLARY. *$b_{\alpha-1,j} \in Z_p$ for $j < i$ and $v_p(b_{\alpha-1,i}) = -1$ and $b_{\alpha,j} \in Z_p$ for $r \leqslant j \leqslant k$.*

From this we deduce that $v_p(e_j(f)) = \alpha$ for $i \leqslant j \leqslant k$, and in particular that $v_p(e_k(f)) = \alpha = v_p(d_k(f))$. Q.E.D.

COROLLARY 2.2. *If $e_{r-1}(f)/n$ then f^n is stable* mod x^{2r}.

Notation. Let $d_{n,k}(f) = d_k(f^n)$ and $e_{n,k}(f) = e_k(f^n)$. Then f^n is stable mod x^{k+1} iff $d_{n,i}(f) = e_{n,i}(f)$, $1 \leqslant i \leqslant k$.

3. RATIONAL D-SERIES

Let $f(x) = \mp 1 + \sum_{j=1}^{\infty} a_j x^j \in Q^*[\![x]\!]$, p be a prime, and a_{α_p} be the first coefficient whose denominator is divisible by p. Let $v_p(a_{\alpha_p}) = -\beta_p$. If the p-components of the denominators of all the remaining coefficients are bounded by the inequality $v_p(a_k) \geqslant -\beta_p[k/\alpha_p]$, we say that $f(x)$ is a rational series dominated by its first non-trivial coefficient with respect to the prime p. We formalize this in the following

DEFINITION 3.1. For each prime p, let $\alpha_p, \beta_p \in Z^+$ and $\alpha = (\alpha_p)$, $\beta = (\beta_p)$. We call $f(x) = \mp 1 + \sum_{j=1}^{\infty} a_j x^j \in Q^*[\![x]\!]$ a rational D-series of type (α, β) iff $v_p(a_{\alpha_p}) = -\beta_p$ and $v_p(a_k) \geqslant -\beta_p[k/\alpha_p]$ for $k \neq \alpha_p$.

We call $f(x)$ a strict D-series of type (α, β) iff $v_p(a_{\alpha_p}) = -\beta_p$ and $v_p(a_k) > -\beta_p[k/\alpha_p]$ for $k \neq \alpha_p$.

We call $f(x)$ a weak D-series of type (α, β) iff $v_p(a_k) \geqslant -\beta_p[k/\alpha_p]$ $\forall k \in Z^+$.

DEFINITION 3.2. For each prime p, let α_p, $\beta_p \in Z^+$ and $\alpha = (\alpha_p)$, $\beta = (\beta_p)$. We define numbers $M_k(\alpha, \beta)$ and $N_k(\alpha)$ by

$$v_p(M_k(\alpha, \beta)) = \sup_{1 \leqslant r \leqslant [k/\alpha_p]} (\beta_p r + v_p(r))$$

and

$$v_p(N_k(\alpha)) = \sup_{1 \leqslant r \leqslant [k/\alpha_p]} (1 + v_p(r)).$$

The following properties of $M_k(\alpha, \beta)$ and $N_k(\alpha)$ immediately follow from Definition 3.2:

(i) $M_{k-1}(\alpha, \beta)/M_k(\alpha, \beta)$ and if $k \not\equiv 0 \pmod{\alpha_p}$, $v_p(M_k(\alpha, \beta)) = v_p(M_{k-1}(\alpha, \beta))$.

(ii) $N_{k-1}(\alpha)/N_k(\alpha)$ and if $k \neq p^t\alpha_p$ for some $t \in Z^+$, $v_p(N_k(\alpha)) = v_p(N_{k-1}(\alpha))$.

(iii) $N_k(\alpha)/M_k(\alpha, \beta)$.

(iv) For a rational D-series of type (α, β) both numbers $M_k(\alpha, \beta)$ and $N_k(\alpha)$ are defined $\forall k \in Z^+$.

Let us note that the number M_k defined in [1] is $M_{k-1}(\alpha, \beta)$ in our notation, where $\alpha_p = p - 1$ and $\beta_p = 1$. Then [1, Theorem 1.7] can be interpreted to prove that $e_k((\log(1+x))/x) = e_k((-\log(1-x))/x) = M_{k+1}$.

We now improve [1, Lemma 6.1].

LEMMA 3.3. *Let p be a prime, s a positive integer, and $k = s\alpha_p$. Suppose that $v_p(n) \geqslant v_p(N_{k-1}(\alpha))$. Then $v_p(\binom{n}{r}) = v_p(n) - v_p(r)$ $(1 \leqslant r \leqslant s)$.*

Proof. For $1 \leqslant m < s$, $v_p(m) < v_p(N_{k-1}(\alpha)) = \sup_{1 \leqslant m' < s} (1 + v_p(m'))$. Hence if $v_p(n) \geqslant v_p(N_{k-1}(\alpha))$ then $v_p(m) < v_p(n)$ and thus $v_p(m) = v_p(n-m)$. Since $\binom{n}{r} = (n/r) \prod_{m=1}^{r-1} (n-m)/m$, it follows that for $1 \leqslant r \leqslant s$, $v_p(\binom{n}{r}) = v_p(n/r) = v_p(n) - v_p(r)$. Q.E.D.

Let $f(x)$ be a weak D-series of type (α, β) and let

$$f(x)^n = \mp 1 + \sum_{k=1}^{\infty} c_k x^k, \qquad c_k = \sum_{\substack{s_0 + \cdots + s_k = n \\ s_1 + 2s_2 + \cdots + ks_k = k}} T_f(n, k, s),$$

where

$$T_f(n, k, s) = \frac{\mp n!}{s_0!\, s_1! \cdots s_k!} a_1^{s_1} a_2^{s_2} \cdots a_k^{s_k}.$$

Let p be a prime. The sequences $s=(s_0, s_1, ..., s_k)$ fall into two cases:

Case I. $s_i > 0$ for some $i > 0$ and $i \neq 0 \pmod{\alpha_p}$.

Case II. If $s_i > 0$ for $i > 0$ then $i = 0 \pmod{\alpha_p}$.

If there exists a sequence s in Case II then $k = \sum_{i>0} is_i = 0 \pmod{\alpha_p}$
If $k = s\alpha_p$, let s_d be the distinguished sequence given by $s_0 = n - s$, $s_{\alpha_p} = s$, and $s_i = 0$ for $i \neq 0, \alpha_p$.

Let $M_k = M_k(\alpha, \beta)$ and $N_k = N_k(\alpha)$.

PROPOSITION 3.4. (i) *Let s be as in Case* I *and $f(x)$ be a weak D-series. If $v_p(n) \geqslant v_p(N_{k-2})$, then $v_p T_f(n, k, s) \geqslant v_p(n) - v_p(M_{k-1})$.*

(ii) *Let s be as in Case* II *and $k = s\alpha_p$.*
If $v_p(n) \geqslant v_p(N_{k-1})$, then

$$\begin{aligned} &v_p T_f(n, k, s) \\ &\quad \geqslant v_p(n) - (\beta_p s + v_p(s)) \qquad \text{if } f(x) \text{ is a weak D-series} \\ &\quad > v_p(n) - (\beta_p s + v_p(s)) \qquad \text{if } f(x) \text{ is a strict D-series and } s \neq s_d \\ &\quad = v_p(n) - (\beta_p s + v_p(s)) \qquad \text{if } f(x) \text{ is a D-series and } s = s_d. \end{aligned}$$

Proof. By induction on k.
It is true for $k=0$. Let $k \geqslant 1$ and assume it to be true for $k' < k$.

(i) Let s be as in Case I. Then there exists $i > 0$, $i \neq 0 \pmod{\alpha_p}$, and $s_i > 0$. Let $i_0 = \alpha_p[i/\alpha_p] < i$ and $i_0/\alpha_p = [i/\alpha_p]$. Let $a'_{i_0} = p^{\min(v_p(a_{i_0}), v_p(a_i))}$, $a'_j = a_j$ for $j \neq i_0$. $f'(x) = \mp 1 + \sum_{j=1}^{\infty} a'_j x^j$. Define the sequence s' by $s'_{i_0} = s_{i_0} + s_i$, $s'_i = 0$, $s'_j = s_j$ for $j \neq i_0, i$. Then $k' = s_1 + \cdots + i_0(s_{i_0} + s_i) + \cdots + ks_k < s_1 + \cdots + i_0 s_{i_0} + \cdots + is_i + \cdots + ks_k = k$.

$$T_f(n, k, s) = \binom{s_{i_0} + s_i}{s_i} \left(\frac{a_{i_0}}{a'_{i_0}}\right)^{s_{i_0}} \left(\frac{a_i}{a'_{i_0}}\right)^{s_i} T_{f'}(n, k', s')$$

$f'(x)$ is a weak D-series by construction and since $v_p(n) \geqslant v_p(N_{k-2}) \geqslant v_p(N_{k'-1})$, it follows from the induction hypothesis for k' (by analyzing whether s' lies in Case I or II) that $v_p T_{f'}(n, k', s') \geqslant v_p(n) - v_p(M_{k'}) \geqslant v_p(n) - v_p(M_{k-1})$. $(a_{i_0}/a'_{i_0})^{s_{i_0}}$, $(a_i/a'_{i_0})^{s_i} \in Z_p$ by definition and hence $v_p T_f(n, k, s) \geqslant v_p(n) - v_p(M_{k-1})$.

(ii) Let s be as in Case II. For all $i > 0$, $s_i > 0$, we have $i = 0 \pmod{\alpha_p}$. Let $k = s\alpha_p$. Put $\sigma = s_{\alpha_p} + \cdots + s_k$. Then $\alpha_p \sigma = \alpha_p s_{\alpha_p} + \cdots + \alpha_p s_j + \cdots + \alpha_p s_k \leqslant \alpha_p s_{\alpha_p} + \cdots + js_j + \cdots + ks_k = k$, $\sigma \leqslant k/\alpha_p = s$. Since $v_p(n) \geqslant v_p(N_{k-1})$, by Lemma 3.3, $v_p(\binom{n}{\sigma}) = v_p(n) - v_p(\sigma)$. By [1, Lemma 6.2],

$$v_p\left(\frac{\sigma!}{s_{\alpha_p}! \cdots s_k!}\right) \geqslant v_p(\sigma) - v,$$

where

$$v = \inf_{j \geqslant \alpha_p} s_j v_p\left(\frac{n!}{s_0!\, s_1! \cdots s_k!}\right) = v_p\binom{n}{\sigma} + v_p\left(\frac{!}{s_{\alpha_p}! \cdots s_k!}\right)$$
$$\geqslant v_p(n) - v_p(\sigma) + v_p(\sigma) - v = v_p(n) - v.$$

Since $s = s_{\alpha_p} + \cdots + (j/\alpha_p)\, s_j + \cdots + s s_k$, it follows that $v \leqslant v_p(s)$ and hence $v_p(n!/s_0!\, s_1! \cdots s_k!) \geqslant v_p(n) - v_p(s)$.

If $f(x)$ is a weak D-series,

$$v_p(a_{\alpha_p}^{s_{\alpha_p}} \cdots a_k^{s_k}) \geqslant -\left(s_{\alpha_p}\beta_p \frac{\alpha_p}{\alpha_p} + \cdots + s_j \beta_p \frac{j}{\alpha_p} + \cdots + s_k \beta_p \frac{k}{\alpha_p}\right)$$
$$= -\frac{\beta_p}{\alpha_p}(\alpha_p s_{\alpha_p} + \cdots + j s_j + \cdots + k s_k) = -\beta_p \frac{k}{\alpha_p} = -\beta_p s.$$

Hence $v_p T_f(n, k, s) \geqslant v_p(n) - v_p(s) - \beta_p s$.

If $f(x)$ is a strict D-series and $s \neq s_d$, there exists $i > \alpha_p$ such that $s_i > 0$ and $v_p(a_i) > -\beta_p(i/\alpha_p)$. Hence

$$v_p(a_{\alpha_p}^{s_{\alpha_p}} \cdots a_k^{s_k}) > -\left(s_{\alpha_p}\beta_p \frac{\alpha_p}{\alpha_p} + \cdots + s_i \beta_p \frac{i}{\alpha_p} + \cdots + s_k \beta_p \frac{k}{\alpha_p}\right) = -\beta_p s$$

and thus $v_p T_f(n, k, s) > v_p(n) - v_p(s) - \beta_p s$.

Finally, $T_f(n, k, s_d) = \binom{n}{s} a_{\alpha_p}^s$. Since $v_p(n) \geqslant v_p(N_{k-1})$ it follows from Lemma 3.3 that $v_p(\binom{n}{s}) = v_p(n) - v_p(s)$ and if $f(x)$ is a D-series, $v_p(a_{\alpha_p}^s) = s v_p(a_{\alpha_p}) = -\beta_p s$. Thus, $v_p T_f(n, k, s_d) = v_p(n) - v_p(s) - \beta_p s$. Q.E.D.

The main result of this paper is the following

THEOREM 3.5. 1. *Let $f(x)$ be a weak D-series of type (α, β). Then $e_k(f)/M_k(\alpha, \beta)$ $\forall k \in Z^+$.*

2. *Let $f(x)$ be a D-series of type (α, β), p a prime, $r \in Z^+$, and*

$$h = \left[\frac{\log r - \log \beta_p}{\log p}\right].$$

Then $v_p(e_k(f)) = v_p(M_k(\alpha, \beta)) = \beta_p p^r + r$ for $p^r \alpha_p \leqslant k < (p^r + p^h)\, \alpha_p$.

3. *Let $f(x)$ be a strict D-series of type (α, β). Then*

(i) $e_k(f) = M_k(\alpha, \beta)$ $\quad \forall k \in Z^+$.

(ii) *If n is divisible by $N_{k-1}(\alpha)$ then f^n is stable* mod x^{k+1}.

Proof. We prove all the statements except Statement 2 by induction on k. They are true for $k = 0$. Let $k \geqslant 1$ and assume them to be true for $(k-1)$. Let $M_k = M_k(\alpha, \beta)$ and $N_k = N_k(\alpha)$.

1. Suppose $f(x)$ is a weak D-series of type (α, β) and that n is divisible by M_k. By the induction hypothesis, $e_{k-1}(f)/M_{k-1}/M_k$ and thus $f(x)^n \in Z[\![x]\!] \pmod{x^k}$. It follows from Proposition 3.4 that $v_p T_f(n, k, s) \geqslant 0$ for all sequences s and all primes p. Hence $T_f(n, k, s) \in Z$ for all sequences s and thus $c_k \in Z$. Hence $f(x)^n \in Z[\![x]\!] \pmod{x^{k+1}}$ and thus $e_k(f)/M_k$.

3. Suppose $f(x)$ is a strict D-series of type (α, β) and that n is divisible by N_{k-1}. By the induction hypothesis, $e_{k-1}(f) = M_{k-1}$, $d_{n,k-1}(f) = e_{n,k-1}(f)$, and hence $v_p(d_{n,k-1}) = v_p(e_{n,k-1}(f)) = \max(v_p(M_{k-1}) - v_p(n); 0)$. Thus $df(x)^n \in Z[\![x]\!] \pmod{x^{k+1}}$ iff $v_p(d) \geqslant \max(v_p(M_{k-1}) - v_p(n); 0)$ and that $dc_k \in Z_p$ for all primes p. Let us assume therefore that $v_p(d) \geqslant \max(v_p(M_{k-1}) - v_p(n); 0)$ for all primes p. By (i) of Proposition 3.4, $dT_f(n,k,s) \in Z_p$ for all sequences s belonging to Case I. If $k \not\equiv 0 \pmod{\alpha_p}$, $dc_k \in Z_p$, and if $k = s\alpha_p$, it follows from (ii) of Proposition 3.4 that $dc_k \in Z_p$ iff

$$v_p(d) \geqslant \max(\beta_p s + v_p(s) - v_p(n); 0).$$

Hence $df(x)^n \in Z[\![x]\!] \pmod{x^{k+1}}$ iff

$$v_p(d) \geqslant \begin{cases} \max(v_p(M_{k-1}) - v_p(n); 0) \\ \qquad \text{if} \quad k \not\equiv 0 \pmod{\alpha_p} \\ \max(v_p(M_{k-1}) - v_p(n); \beta_p s + v_p(s) - v_p(n); 0) \\ \qquad \text{if} \quad k = s\alpha_p; \end{cases}$$

i.e., iff $v_p(d) \geqslant \max(v_p(M_k) - v_p(n); 0)$ in either case.

Hence $v_p(d_{n,k}(f)) = \max(v_p(M_k) - v_p(n); 0)$. Thus $d_{n,k}(f) = 1$ iff $v_p(d_{n,k}(f)) \geqslant 0$ for all primes p; i.e., iff $v_p(n) \geqslant v_p(M_k)$ for all primes p and iff n is divisible by M_k. Hence $e_k(f) = M_k$ and thus $v_p(e_{n,k}(f)) = \max(v_p(M_k) - v_p(n); 0)$ $\forall n, k \in Z^+$. It thus follows that if n is divisible by N_{k-1}, $d_{n,k}(f) = e_{n,k}(f)$.

2. Let $f(x)$ be a D-series of type (α, β), $r \in Z^+$, and p a prime, and put $k = p^r \alpha_p$, $s = p^r$. Let s be a sequence belonging to Case II and let $\sigma = s_{\alpha_p} + \cdots + s_k$, $v = \inf_{i \geqslant \alpha_p} v_p(s_i)$.

Since $s_{\alpha_p} + \cdots + (i/\alpha_p) s_i + \cdots + s s_k = p^r$, it follows that $v = v_p(s) = p^r$ iff $s_{\alpha_p} = p^r$, $s_0 = n - p^r$, $s_i = 0$ for $i \neq 0, \alpha_p$; i.e., iff $s = s_d$. Thus for $s \neq s_d$, $v < v_p(s) = r$ and

$$\begin{aligned} v_p\left(\frac{n!}{s_0!\, s_1! \cdots s_k!}\right) &= v_p\binom{n}{\sigma} + v_p\left(\frac{\sigma!}{s_{\alpha_p}! \cdots s_k!}\right) \\ &\geqslant v_p(n) - v_p(\sigma) + v_p(\sigma) - v \\ &= v_p(n) - v > v_p(n) - v_p(s), \end{aligned}$$

and hence $v_p T_f(n, k, s) > v_p(n) - (\beta_p s + v_p(s))$ for $s \neq s_d$, and by the same argument as in the strict D-case, we prove that

$$v_p(e_{p^r\alpha_p}(f)) = v_p(M_{p^r\alpha_p}) = \beta_p p^r + r.$$

Now let

$$h = \left[\frac{\log r - \log \beta_p}{\log p}\right];$$

i.e., h is the greatest integer such that $\beta_p p^h \leqslant r$ and let $p^r\alpha_p \leqslant k < (p^r + p^h)\alpha_p$. Let $\rho^r \leqslant s \leqslant [k/\alpha_p]$, $p^r \leqslant s < p^r + p^h$. Then $s = p^r + p^t u$ for $0 \leqslant t < h$, $(p, u) = 1$, $u < p^{h-t}$.

$$\begin{aligned}\beta_p s + v_p(s) &= \beta_p(p^r + p^t u) + t \leqslant \beta_p(p^r + p^t u) + \beta_p p^t(p^{h-t} - u)\\ &= \beta_p p^r + \beta_p p^h \leqslant \beta_p p^r + r,\end{aligned}$$

and hence $v_p(M_k) = v_p(M_{p^r\alpha_p}) = \beta_p p^r + r$. We deduce from Statement 1 that $v_p(e_k(f)) \leqslant v_p(M_k) = \beta_p p^r + r$. Also $v_p(e_k(f)) \geqslant v_p(e_{p^r\alpha_p}(f)) = \beta_p p^r + r$. Hence $v_p(e_k(f)) = \beta_p p^r + r$. Q.E.D.

Remark 1. Statement 3(ii) of Theorem 3.5 yields much more information on the stability of strict D-series than the one given by Lemma 2.2, which of course is valid for arbitrary rational series.

Remark 2. One can define a rational D-series to be strict at a prime p and prove a local version of Statement 3 of Theorem 3.5 to the effect that if $f(x)$ is a rational D-series of type (α, β) strict at a prime p then

(i) $v_p(e_k(f)) = v_p(M_k(\alpha, \beta))\ \forall k \in Z^+$.

(ii) If n is divisible by the p-primary component of $N_{k-1}(\alpha)$ then $v_p(d_{n,i}(f)) = v_p(e_{n,i}(f))$ $(1 \leqslant i \leqslant k)$.

4. Applications

LEMMA 4.1. *$(e^x - 1)/x$ is a D-series of type (α, β) where $\alpha_p = p - 1$ and $\beta_p = 1$ for all primes p.*

Proof. $(e^x - 1)/x = 1 + \sum_{n=1}^{\infty} x^n/((n+1)!)$.

$$\begin{aligned}&v_p(n+1)! = 0 \quad \text{for } n+1 < p; \text{ i.e., } \quad \text{for } n < p-1\\ &v_p(n+1)! = 1 \quad \text{for } n+1 = p; \text{ i.e., } \quad \text{for } n = p-1.\end{aligned}$$

Let $n+1=\sum_{i=0}^{k} a_i p^i$, $0 \leqslant a_i \leqslant p-1$ and $a_k > 0$. Then by [2, Lemma 3.1],

$$v_p(n+1)! = \frac{n+1-\sum_{i=0}^{k} a_i}{p-1} \leqslant \frac{n}{p-1}$$

and thus $v_p(n+1)! \leqslant [n/(p-1)]$, which shows that $(e^x-1)/x$ is a rational D-series of type (α, β) where $\alpha_p = p-1$ and $\beta_p = 1$. Q.E.D.

PROPOSITION 4.2. *Let $M_k = M_k(\alpha, \beta)$ for $\alpha_p = p-1$ and $\beta_p = 1$. Then*

1. $e_k(x/(e^x-1)) = e_k((e^x-1)/x)/M_k$, $\forall k \in Z^+$.
2. *Let p be a prime, $r \in Z^+$, and $h = [\log r/\log p]$. Then*

$$v_p\left(e_k\left(\frac{x}{e^x-1}\right)\right) = v_p\left(e_k\left(\frac{e^x-1}{x}\right)\right) = v_p(M_k) = p^r + r$$

for $p^r(p-1) \leqslant k < (p^r+p^h)(p-1)$.

Proof. The results for $(e^x-1)/x$ are immediate consequences of Lemma 4.1 and Theorem 3.5, and those for $x/(e^x-1)$ follow from those for $(e^x-1)/x$ since $e_k(x/(e^x-1)) = e_k((e^x-1)/x)$. Q.E.D.

PROPOSITION 4.3. *Let $M_k = M_k(\alpha, \beta)$ for $\alpha_p = p-1$ and $\beta_p = 1$ and let $\bar{M}_k = \prod_{p \text{ odd}} p^{v_p(M_k)}$ be the odd part of M_k. Then*

1. $e_k(x/\sin x) = e_k(\sin x/x) = 2\bar{e}_k$, *where $\bar{e}_k/\bar{M}_k$.*
2. *Let p be an odd prime, $r \in Z^+$, and $h = [\log r/\log p]$. Then $v_p(e_k(x/\sin x)) = v_p(e_k(\sin x/x)) = v_p(M_k) = p^r + r$ for $p^r(p-1) \leqslant k < (p^r+p^h)(p-1)$.*

Proof. $\sin x/x = 1 + \sum_{n=1}^{\infty} ((-1)^n x^{2n}/(2n+1)!)$. As in the proof of Lemma 4.1, $v_p(2n+1)! \leqslant [2n/(p-1)]$ and it thus follows that $\sin x/x$ is a weak D-series of type (α, β) where $\alpha_p = p-1$ and $\beta_p = 1$. We deduce from Theorem 3.5 that $e_k(\sin x/x)/M_k.\ e^{2ix} = 1 + \sum_{n=1}^{\infty} (2^n i^n x^n/n!)$. If we expand $n = \sum_{j=0}^{k} 2^j a_j$ for $a_j = 0, 1$ then $v_2(n!) = n - \sum_{j=0}^{k} a_j$ by [2, Lemma 3.1] and thus $v_2(2^n/n!) = n - v_2(n!) = \sum_{j=0}^{k} a_j > 0$. Hence $2/1 - e^{2ix}$ or, equivalently, $(1-e^{2ix})/2 \in Z_2[\![x]\!]$. It thus follows that $\sin^2 x = (1-\cos 2x)/2 \in Z_2[\![x]\!]$ or $(\sin x/x)^2 \in Z_2[\![x]\!]$. Hence $v_2(e_k(\sin x/x)) = 1$; i.e., $e_k(\sin x/x) = 2\bar{e}_k$ where $\bar{e}_k/\bar{M}_k$. Statement 2 for $\sin x/x$ follows from Theorem 3.5 and the Proposition for $x/\sin x$ follows from that for $\sin x/x$ since $e_k(x/\sin x) = e_k(\sin x/x)$.

5. The Group $D(\alpha, \beta; k)$

For each prime p let α_p, $\beta_p \in Z^+$ and $\alpha = (\alpha_p)$ and $\beta = (\beta_p)$ and let $\tilde{D}(\alpha, \beta)$ be the set of weak D-series of type (α, β).

Observation 5.1. $\tilde{D}(\alpha, \beta)$ is a group.

Proof. Let $f(x)$, $g(x) \in \tilde{D}(\alpha, \beta)$ and $h(x) = f(x)/g(x)$. Let $f(x) = \mp 1 + \sum_{i=1}^{\infty} a_i x^i$, $g(x) = \mp 1 + \sum_{j=1}^{\infty} b_j x^j$ $(a_i, b_j \in Q)$, $h(x) = \mp 1 + \sum_{k=1}^{\infty} c_k x^k$. We prove by induction on k that $v_p(c_k) \geqslant -\beta_p[k/\alpha_p]$. It is true for $k = 0$. Let $k \geqslant 1$ and assume it to be true for $(k-1)$. $f(x) = g(x)\,h(x)$ and, equating coefficients, $a_k = \mp b_k + b_{k-1}c_1 + \cdots + b_1 c_{k-1} \mp c_k$, $v_p(a_k) \geqslant -\beta_p[k/\alpha_p]$, $v_p(b_i) \geqslant -\beta_p[i/\alpha_p]$, and by the induction hypothesis $v_p(c_{k-i}) \geqslant -\beta_p[(k-i)/\alpha_p]$ $(1 \leqslant i \leqslant k)$, and thus $v_p(b_i c_{k-i}) \geqslant -\beta_p[i/\alpha_p] - \beta_p[(k-i)/\alpha_p]$.

Thus $v_p(c_k) \geqslant -\beta_p[k/\alpha_p]$ and hence $h(x) \in \tilde{D}(\alpha, \beta)$. Q.E.D.

$\tilde{D}(\alpha, \beta)$ contains $Z^*[\![x]\!]$. Let $D(\alpha, \beta)$ be the quotient group, $D(\alpha, \beta) = \tilde{D}(\alpha, \beta)/Z^*[\![x]\!]$. Let $\tilde{D}(\alpha, \beta{:}\, k)$ be the image of $\tilde{D}(\alpha, \beta)$ under the quotient map $Q[\![x]\!] \to Q[\![x]\!]/(x^{k+1})$. Then $\tilde{D}(\alpha, \beta; k)$ contains $Z_{k,X}$ which is the group of units of $Z[\![x]\!]/(x^{k+1})$. Let $D(\alpha, \beta; k)$ be the quotient group $D(\alpha, \beta; k) = \tilde{D}(\alpha, \beta; k)/Z_{k,X}$. Then $D(\alpha, \beta; k)$ is a finite subgroup of $M_k = Q_{k,X}/Z_{k,X}$. For each prime p let α^p, β^p be the vectors defined by $\alpha^p_p = \alpha_p$, $\beta^p_p = \beta_p$, $\alpha^p_{p'} = \beta^p_{p'} = 0$ for $p' \neq p$. Then the p-summand of the finite abelian group $D(\alpha, \beta; k)$ is $D(\alpha^p, \beta^p; k)$ and the latter can be shown to have order $\prod_{k' \leqslant k} p^{\beta_p}[k'/\alpha_p]$.

Proposition 5.2. *The highest invariant of $D(\alpha^p, \beta^p; k)$ is $p^{v_p(M_k(\alpha, \beta))}$.*

Proof. By Statement 1 of Theorem 3.5, the order of every element in $D(\alpha^p, \beta^p; k)$ divides $p^{v_p(M_k(\alpha, \beta))}$, and $D(\alpha^p, \beta^p; k)$ contains strict D-series of type (α^p, β^p) whose orders, by Statement 3(i) of Theorem 3.5, are equal to $p^{v_p(M_k(\alpha, \beta))}$. Thus the cyclic subgroups in the decomposition of $D(\alpha^p, \beta^p; k)$ of highest order have order $p^{v_p(M_k(\alpha, \beta))}$.

It is of interest to determine the groups $D(\alpha, \beta; k)$ completely.

6. Rational D-Series with Respect to the Star-Product

Let $Q^{(1)}[\![x]\!] = xQ[\![x]\!] = (f(x) \in Q[\![x]\!]/f(0) = 0)$. We can define on $Q^{(1)}[\![x]\!]$ a commutative product called the star product by the equation $x^i * x^j = x^{ij}$; i.e., if $f(x) = \mp x + \sum_{i=2}^{\infty} a_i x^i$, $g(x) = \mp x + \sum_{j=2}^{\infty} b_j x^j$ $(a_i, b_j \in Q)$, then $f(x) * g(x) = \mp x + \sum_{k=2}^{\infty} c_k x^k$ where $c_k = \sum_{ij=k} a_i b_j$. Let $Q^{(1)}_*[\![x]\!] = (f(x) \in Q^{(1)}[\![x]\!]/f'(0) = \mp 1)$. Let $Z^{(1)}[\![x]\!]$ and $Z^{(1)}_*[\![x]\!]$ be similarly defined. Let $Q^1_{k,X}$ be the image of $Q^{(1)}_*[\![x]\!]$ under the quotient map $Q^{(1)}[\![x]\!] \to Q^{(1)}[\![x]\!]/(x^{k+1})$ and $Z^1_{k,X}$ the image of $Z^{(1)}_*[\![x]\!]$ under the quotient map

$Z^{(1)}[\![x]\!] \to Z^{(1)}[\![x]\!]/(x^{k+1})$. Let e_k^* be the algebraic order in the quotient group $M_k^* = Q_{k,x}^{(1)}/Z_{k,x}^{(1)}$.

We can define rational D-series with respect to the star product and all our results have their counterparts in this new setting.

DEFINITION 6.1. For each prime p let $\alpha_p, \beta_p \in Z^+$, $\alpha = (\alpha_p)$, $\beta = (\beta_p)$. $f(x) = \mp x + \sum_{k=2}^{\infty} a_k x^k \in Q_*^{(1)}[\![x]\!]$ is called a rational D^*-series of type (α, β) iff $v_p(a_{\alpha_p}) = -\beta_p$ and $v_p(a_k) \geqslant -\beta_p[\log k/\log \alpha_p]$ for $k \neq \alpha_p$. Similarly, $f(x)$ is called a strict D^*-series of type (α, β) iff

$$v_p(a_{\alpha_p}) = -\beta_p \qquad \text{and} \qquad v_p(a_k) > -\beta_p \left[\frac{\log k}{\log \alpha_p}\right] \qquad \text{for} \quad k \neq \alpha_p.$$

$f(x)$ is called a weak D^*-series of type (α, β) iff $v_p(a_k) \geqslant -\beta_p[\log k/\log \alpha_p]$ $\forall k \in Z^+$.

DEFINITION 6.2. For each prime p, let $\alpha_p, \beta_p \in Z^+$ and $\alpha = (\alpha_p)$, $\beta = (\beta_p)$. The numbers $M_k^*(\alpha, \beta)$ and $N_k^*(\alpha)$ are defined by

$$v_p(M_k^*(\alpha, \beta)) = \sup_{1 \leqslant r \leqslant \left[\frac{\log k}{\log \alpha_p}\right]} (\beta_p r + v_p(r)), \qquad v_p(N_k^*(\alpha)) = \sup_{1 \leqslant r \leqslant \left[\frac{\log k}{\log \alpha_p}\right]} (1 + v_p(r)),$$

and with this terminology the analogue of Theorem 3.5 is the following

THEOREM 6.3. 1. *Let $f(x)$ be a weak D^*-series of type (α, β). Then $e_k^*(f)/M_k^*(\alpha, \beta)$.*

2. *Let $f(x)$ be a D^*-series of type (α, β), p be a prime, $r \in Z^+$, and $h = [(\log r - \log \beta_p)/\log p]$. Then $v_p(e_k^*(f)) = v_p(M_k^*(\alpha, \beta)) = \beta_p p^r + r$ for $\alpha_p^{p^r} \leqslant k < \alpha_p^{p^r + p^h}$.*

3. *Let $f(x)$ be a strict D^*-series of type (α, β). Then*

(i) $e_k^*(f) = M_k^*(\alpha, \beta)$, $\quad \forall k \in Z^+$.

(ii) *If n is divisible by $N_{k-1}^*(\alpha)$ then $f^n = \underbrace{f * f * \cdots * f}_{n}$ is stable* mod x^{k+1}.

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