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Duality for Ideals in the Grassmann Algebra

I. Dibag

Department of Mathematics, Bilkent University, 06533 Bilkent, Ankara, Turkey

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A duality is established between left and right ideals of a finite dimensional Grassmann algebra such that if under the duality a left ideal $\mathfrak{I}$ and a right ideal J correspond then $\mathfrak{I}$ is the left annihilator of J and J the right annihilator of $\mathfrak{I}$. Another duality is established for two-sided ideals of the Grassmann algebra where two ideals that correspond are annihilators of each other. The dual of the principal ideal generated by an exterior 2-form is completely determined.

INTRODUCTION

In a finite dimensional Grassmann algebra we define the left (right) annihilator of a right (left) ideal as the set of elements whose products on the left (right) with elements of the ideal are zero and it is a left (right) ideal. Theorem 1.1.5 establishes a duality between left and right ideals of the Grassmann algebra such that if under the duality a left ideal $\mathfrak{I}$ and a right ideal J correspond then $\mathfrak{I}$ is the left annihilator of J and J is the right annihilator of $\mathfrak{I}$. We define the annihilator of a two-sided ideal to be the intersection of its left and right annihilators and it is a two-sided ideal. Theorem 1.2.3 establishes a duality between two-sided ideals of the Grassmann algebra such that if under the duality two ideals correspond then they are annihilators of one another. We define an ideal to be proper if its left and right annihilators coincide and are equal to its annihilator and prove that a homogeneous ideal is proper.

A knowledge of the annihilator $K(\mathfrak{I})$ of an ideal $\mathfrak{I}$ enables us to characterize $\mathfrak{I}$ by means of exterior equations. If $k_1 \dots k_r$ are generators for $K(\mathfrak{I})$ then $\mathfrak{I} = \{\iota \in \bigwedge V \mid \iota \wedge k_1 = \cdots = \iota \wedge k_r = 0\}$. For this reason we compute the annihilator of the ideal generated by a linearly indepen-

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dent set of vectors in the underlying vector space and relate it to the factorization problem of an exterior form into a wedge product of k vectors and an exterior form. The second section of the paper is devoted to the determination of the annihilator of the principal ideal generated by an exterior 2-form. As a consequence given a 2-form μ and a form ω, we obtain a system of exterior equations whose satisfaction by ω is a necessary and sufficient condition for ω to factor into $\omega = \tau \wedge \mu$ for some exterior form τ. The global version of this factorization problem is briefly discussed.

1. ANNIHILATORS OF IDEALS IN THE GRASSMANN ALGEBRA

1.1. *Left and Right Annihilators*

Let V be an n-dimensional vector space over a ground field k with dual space V^* and $\bigwedge V = \bigoplus_{p=0}^{n} \bigwedge^p V$ and $\bigwedge^* V = \bigwedge V^* = \bigoplus_{p=0}^{n} \bigwedge^p V^*$ the corresponding Grassmann algebras.

1.1.1. DEFINITION. Let $\Im$ be a right ideal. Then $K_{\mathrm{L}}(\Im) = \{k \in \bigwedge V \mid k \wedge \iota = 0\ \forall \iota \in \Im\}$ is defined to be the left annihilator of $\Im$. It is readily verified that $K_{\mathrm{L}}(\Im)$ is a left ideal. The right annihilator $K_{\mathrm{L}}(J)$ of a left ideal J is analogously defined and is a right ideal.

1.1.2. *Left and right duality operators.* Let $x \in \bigwedge V$ and $L_x\colon \bigwedge V \to \bigwedge V$ be left multiplication by x and $\delta_x^{\mathrm{L}}\colon \bigwedge^* V \to \bigwedge^* V$ be its dual and let $\Omega \in \bigwedge^n V^*$ be a fixed dual volume element. We then define the left duality operator $*^{\mathrm{L}}\colon \bigwedge V \to \bigwedge^* V$ by $*^{\mathrm{L}}(x) = \delta_x^{\mathrm{L}}(\Omega)$. If $(\ ,\ )$ is the dual pairing between $\bigwedge V$ and $\bigwedge^* V$ then $(u, *^{\mathrm{L}}(x)) = (x \wedge u, \Omega)\ \forall x, u \in \bigwedge V$. If $\{e_1, \ldots, e_n\}$ is the basis for V, $\{e_1^*, \ldots, e_n^*\}$ the dual basis for $\bigwedge^* V$, $\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 0 \le i_1 < i_2 < \cdots < i_p \le n\}$ the induced basis for $\bigwedge V$ and $\{e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \mid 0 \le i_1 < i_2 < \cdots < i_p \le n\}$ the corresponding dual basis for $\bigwedge^* V$ and if $\Omega = e_1^* \wedge \cdots \wedge e_n^*$, it can be verified that $*^{\mathrm{L}}(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}) = (-1)^{\operatorname{sgn} \sigma} e_{j_1}^* \wedge \cdots \wedge e_{j_{n-p}}^*$, where $(j_1 j_2 \ldots j_{n-p})$ is the set of complementary indices and $\sigma = (i_1 \ldots i_p j_1 \ldots j_{n-p})$ is a permutation of $(1\,2\,\ldots\,n)$. $*^{\mathrm{L}}$ is thus an isomorphism. We can also define the right duality operator $*^{\mathrm{R}}$ using the right multiplication R_x instead of L_x and obtain analogous equations: $(u, *^{\mathrm{R}}(x)) = (u \wedge x, \Omega)$ and $*^{\mathrm{R}}(e_{i_1} \wedge \cdots \wedge e_{i_p}) = (-1)^{\operatorname{sgn} \tau} e_{j_1}^* \wedge \cdots \wedge e_{j_{n-p}}^*$, where $\tau = (j_1 \ldots j_{n-p} i_1 \ldots i_p)$ is a permutation of $(1\,2\,\ldots\,n)$. We note that $\operatorname{sgn} \tau = (-1)^{p(n-p)} \operatorname{sgn} \sigma$ and hence if $\epsilon\colon \bigwedge V \to \bigwedge V$ is the automorphism of $\bigwedge V$ defined by $\epsilon|_{\bigwedge^p V} = (-1)^{p(n-p)}$ then $*^{\mathrm{R}} = *^{\mathrm{L}} \circ \epsilon$ and $*^{\mathrm{L}} = *^{\mathrm{R}} \circ \epsilon$.

1.1.3. PROPOSITION. *Let $\Im$ be a right ideal. Then*

(i) $K_{\mathrm{R}}(K_{\mathrm{L}}(\Im)) = \Im$.

(ii) $\dim K_{\mathrm{L}}(\Im) + \dim \Im = \dim \wedge V = 2^n$.

(iii) *There exist commutative diagrams*

$$\begin{tikzcd} \wedge V \arrow[r, "*^{\mathrm{L}}"] \arrow[d] & \wedge^* V \arrow[r, "\text{rest.}^{\text{n}}"] & \Im^* \\ \wedge V/K_{\mathrm{L}}(\Im) \arrow[urr, "\approx"'] & & \end{tikzcd} \qquad \begin{tikzcd} \wedge V \arrow[r, "*^{\mathrm{R}}"] \arrow[d] & \wedge^* V \arrow[r, "\text{rest.}^{\text{n}}"] & K_{\mathrm{L}}^*(\Im) \\ \wedge V/\Im \arrow[urr, "\approx"'] & & \end{tikzcd}$$

Proof. Let $i\colon \Im \subset \wedge V$ and $j\colon K_{\mathrm{L}}(\Im) \subset \wedge V$ be the inclusions and $i^*\colon \wedge^* V \to K_{\mathrm{L}}^*(\Im)$ and $j^*\colon \wedge^* V \to K_{\mathrm{L}}^*(\Im)$ be the restriction maps. Since $*^{\mathrm{L}}$ is an isomorphism and i^* is onto it follows that the composite $i^* \circ *^{\mathrm{L}}$ is onto. $u \in \operatorname{Ker}(i^* \circ *^{\mathrm{L}})$ iff $(x, *^{\mathrm{L}}(u)) = (u \wedge x, \Omega)) = 0\ \forall x \in \Im$, which clearly shows that $K_{\mathrm{L}}(\Im) \subset \operatorname{Ker}(i^* \circ *^{\mathrm{L}})$ and if $u \notin K_{\mathrm{L}}(\Im)$ then there exists $x \in \Im$ such that $u \wedge x \neq 0$. Let us choose a basis $\{e_1, \dots, e_n\}$ for V and express $u \wedge x = \sum \lambda_{i_1 \dots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}$. Since $u \wedge x \neq 0$ it follows that $\lambda_{i_1 \dots i_p} \neq 0$ for some $(i_1, \dots, i_p)$. Let $(j_1, \dots, j_{n-p})$ be the set of complementary indices and $\tau = e_{j_1} \wedge \cdots \wedge e_{j_{n-p}}$. Then $y = x \wedge \tau \in \Im$ since $\Im$ is a right ideal and $(y, *^{\mathrm{L}}(u)) = (u \wedge x \wedge \tau, \Omega) = \pm \lambda_{i_1 \dots i_p} \neq 0$ and this gives a contradiction. Hence $\operatorname{Ker}(*^{\mathrm{L}} \circ i^*) = K_{\mathrm{L}}(\Im)$ and this proves the first part of (iii). If we take dimensions of both sides under the isomorphism, $\wedge V/K_{\mathrm{L}}(\Im) \xrightarrow{\approx} \Im^*$ we obtain (ii). By reasoning analogous to that used to prove the first diagram we deduce that $\Im \subseteq \operatorname{Ker}(j^* \circ *^{\mathrm{R}}) = K_{\mathrm{R}}(K_{\mathrm{L}}(\Im))$ and we obtain an isomorphism $\wedge V/K_{\mathrm{R}}(K_{\mathrm{L}}(\Im)) \xrightarrow{\approx} K_{\mathrm{L}}^*(\Im)$. Hence $\dim K_{\mathrm{R}}(K_{\mathrm{L}}(\Im) + \dim K_{\mathrm{L}}(\Im) = \dim \wedge V = \dim K_{\mathrm{L}}(\Im) + \dim \Im$ by (ii). Thus $\dim K_{\mathrm{R}}(K_{\mathrm{L}}(\Im)) = \dim \Im$ and this gives equality, i.e., $K_{\mathrm{R}}(K_{\mathrm{L}}(\Im)) = \Im$, which proves both (i) and the second diagram in (iii).

1.1.4. PROPOSITION. *Let $\Im$ be a left ideal. Then*

(i) $K_{\mathrm{L}}(K_{\mathrm{R}}(\Im)) = \Im$.

(ii) $\dim K_{\mathrm{R}}(\Im) + \dim \Im = \dim \wedge V = 2^n$.

(iii) *There exist commutative diagrams*

$$\begin{tikzcd} \wedge \mathrm{V} \arrow[r, "*^{\mathrm{R}}"] \arrow[d] & \wedge^* \mathrm{V} \arrow[r, "\text{rest.}^{\text{n}}"] & \Im^* \\ \wedge \mathrm{V}/\mathrm{K}_{\mathrm{R}}(\Im) \arrow[urr, "\approx"'] & & \end{tikzcd} \qquad \begin{tikzcd} \wedge \mathrm{V} \arrow[r, "*^{\mathrm{L}}"] \arrow[d] & \wedge^* \mathrm{V} \arrow[r, "\text{rest.}^{\text{n}}"] & \mathrm{K}_{\mathrm{R}}^*(\Im) \\ \wedge \mathrm{V}/\Im \arrow[urr, "\approx"'] & & \end{tikzcd}$$

Proof. Identical with that of Proposition 1.1.3.

1.1.5. THEOREM. *There exists a duality between left and right ideals of the Grassmann algebra such that if under the duality a left ideal $\mathfrak{I}$ and a right ideal J correspond then $\mathfrak{I} = K_L(J)$ and $J = K_R(\mathfrak{I})$.*

Proof. Let A and B be the set of left and right ideals of the Grassmann algebra, respectively. We define mappings $T: A \to B$ and $U: B \to A$ by $T(\mathfrak{I}) = K_R(\mathfrak{I})\ \forall \mathfrak{I} \in A$ and $U(J) = K_R(J)\ \forall J \in B$. Then $TU = 1$ by Proposition 1.1.3 and $UT = 1$ by Proposition 1.1.4.

1.1.6. COROLLARY. *Let $\mathfrak{I}$ be a two-sided ideal. Then the automorphism $\epsilon: \wedge V \to \wedge V$ interchanges $K_L(\mathfrak{I})$ and $K_R(\mathfrak{I})$.*

Proof. Let $i: \mathfrak{I} \subset \wedge V$ be the inclusion and $i^*: \wedge^* V \to \mathfrak{I}^*$ be the restriction maps. Then by Propositions 1.1.3 and 1.1.4, $K_L(\mathfrak{I}) = \operatorname{Ker}(i^* \circ *^L \circ \epsilon) = \epsilon[\operatorname{Ker}(i^* \circ *^R] = \epsilon(K_L(\mathfrak{I}))$.

1.2. *The Annihilator of a Two-Sided Ideal*

From now onward when we say ideal we shall mean a two-sided ideal.

1.2.1. DEFINITION. The annihilator $K(\mathfrak{I})$ of an ideal $\mathfrak{I}$ is defined by $K(\mathfrak{I}) = K_L(\mathfrak{I}) \cap K_R(\mathfrak{I})$.

1.2.2. PROPOSITION.

(i) *$K(\mathfrak{I})$ is an ideal left invariant under the automorphism ϵ.*

(ii) $K(K(\mathfrak{I})) = \mathfrak{I}$.

Proof.

(i) Let $k \in K(\mathfrak{I})$, $\omega \in \wedge V$, and $x \in \mathfrak{I}$. Then $(\omega \wedge k) \wedge x = \omega \wedge (k \wedge x) = 0$ since $k \in K(\mathfrak{I})$ and hence $\omega \wedge k \in K_L(\mathfrak{I})$. Also, $x \wedge (\omega \wedge k) = (x \wedge \omega) \wedge k = 0$ since $x \wedge \omega \in \mathfrak{I}$ and $k \in K_R(\mathfrak{I})$ and thus $\omega \wedge k \in K_R(\mathfrak{I})$. Hence $\omega \wedge k \in K_L(\mathfrak{I}) \cap K_R(\mathfrak{I}) = K(\mathfrak{I})$. This shows that $K(\mathfrak{I})$ is a left ideal. Similar argument shows that $K(\mathfrak{I})$ is also a right ideal. Hence $K(\mathfrak{I})$ is an ideal. The fact that ϵ leaves $K(\mathfrak{I})$ invariant follows from Corollary 1.1.6.

(ii) $K(K(\mathfrak{I})) = K_L(K(\mathfrak{I})) \cap K_R(K(\mathfrak{I})) \subseteq K_L(K_R(\mathfrak{I})) \cap K_R(K_L(\mathfrak{I})) = \mathfrak{I} \cap \mathfrak{I} = \mathfrak{I}$, by Propositions 1.1.3 and 1.1.4 the other inclusion trivially holds, and hence we have equality, i.e., $K(K(\mathfrak{I})) = \mathfrak{I}$.

1.2.3. THEOREM. *There exists a duality among ideals of the Grassmann algebra. Two ideals that correspond under this duality are annihilators of each other.*

Proof. Let S be the set of ideals of the Grassmann algebra and define a mapping ϕ: $S \to S$ by $\phi(\Im) = K(\Im)\ \forall \Im \in S$. Then, $\phi^2 = 1$ by Proposition 1.2.2.

The annihilator $K(\Im)$ of an ideal $\Im$ will be called the dual ideal of $\Im$. This is not to be confused with the vector space dual $\Im^*$ of $\Im$.

1.2.4. Definition. An ideal $\Im$ is called proper iff $K(\Im) = K_{\mathrm{L}}(\Im) = K_{\mathrm{R}}(\Im)$. Note that if an ideal is proper then $\dim K(\Im) + \dim \Im = \dim \bigwedge V = 2^n$.

1.2.5. Observation. A necessary and sufficient condition for an ideal $\Im$ to be proper is that the automorphism ϵ leave either $K_{\mathrm{L}}(\Im)$ or $K_{\mathrm{R}}(\Im)$ invariant.

Proof. It follows from Corollary 1.1.6.

1.2.6. Observation. If an ideal $\Im$ is proper so is $K(\Im)$.

Proof. $K_{\mathrm{L}}(K(\Im)) = K_{\mathrm{L}}(K_{\mathrm{R}}(\Im)) = \Im$ and $K_{\mathrm{R}}(K(\Im)) = K_{\mathrm{R}}(K_{\mathrm{L}}(\Im)) = \Im$ by Propositions 1.1.3 and 1.1.4 and thus $K_{\mathrm{L}}(K(\Im)) = K_{\mathrm{R}}(K(\Im)) = \Im$.

1.2.7. Lemma. *Let $\Im_1$ and $\Im_2$ be proper ideals. Then $\Im_1 \cap \Im_2$ and $\Im_1 + \Im_2$ are also proper and*

(i) $K(\Im_1 \cap \Im_2) = K(\Im_1) + K(\Im_2)$

(ii) $K(\Im_1 + \Im_2) = K(\Im_1) \cap K(\Im_2)$.

Proof. Let $k_i \in K(\Im)$ $(i = 1, 2)$, $k = k_1 + k_2$, $\iota \in \Im_1 \cap \Im_2$. Then $\iota \in \Im_1$ and $k_1 \wedge \iota = 0$ and similarly $k_2 \wedge \iota = 0$. Hence $k \wedge \iota = 0$ and thus $k \in K_{\mathrm{L}}(\Im_1 \cap \Im_2)$. Hence 1. $K(\Im_1) + K(\Im_2) \subseteq K_{\mathrm{L}}(\Im_1 \cap \Im_2)$. Replacing $\Im_i$ by $K(\Im_i)$ $(i = 1, 2)$ and using Proposition 1.2.2 we obtain $\Im_1 + \Im_2 \subseteq K_{\mathrm{L}}(K(\Im_1) \cap K(\Im_2))$. Taking dimensions of both sides gives $\dim \Im_1 + \dim \Im_2 - \dim(\Im_1 \cap \Im_2) \leq 2^n - \dim[K(\Im_1) \cap K(\Im_2)]$, i.e., $2^n - \dim(\Im_1 \cap \Im_2) \leq (2^n - \dim \Im_1) + (2^n - \dim \Im_2) - \dim[K(\Im_1) \cap K(\Im_2)]$, $\dim K_L(\Im_1 \cap \Im_2) \leq \dim K(\Im_1) + \dim K(\Im_2) - \dim[K(\Im_1) \cap K(\Im_2)]$, and $\dim K_L(\Im_1 \cap \Im_2) \leq \dim[K(\Im_1) + K(\Im_2)]$. We deduce from this and inclusion 1 that $K(\Im_1) + K(\Im_2) = K_{\mathrm{L}}(\Im_1 \cap \Im_2)$. Similarly $K(\Im_1) + K(\Im_2) = K_{\mathrm{R}}(\Im_1 \cap \Im_2)$ and hence $\Im_1 \cap \Im_2$ is proper and $K(\Im_1) + K(\Im_2) = K(\Im_1 \cap \Im_2)$, which proves (i). Replacing $\Im_i$ by $K(\Im_i)$ $(i = 1, 2)$ and taking K of both sides yield (ii).

1.2.8. Observation. If an ideal $\Im$ is multiplicatively generated by generators which lie in $\bigwedge^{\mathrm{ev}}(V) = \bigoplus_{i=0}^{[n/2]} \bigwedge^{2i} V$ then $\Im$ is proper.

Proof. Let $g_i \in \bigwedge^{\mathrm{ev}}(V)$ be the multiplicative generators $(1 \leq i \leq r)$. If $k \in K_{\mathrm{L}}(\Im)$, $g_i \wedge k = k \wedge g_i = 0$ $(1 \leq i \leq n)$ since g_i lies in the center of $\bigwedge V$ and thus $k \in K_{\mathrm{R}}(\Im)$. Hence $K_{\mathrm{L}}(\Im) \subseteq K_{\mathrm{R}}(\Im)$ and the reverse inclusion symmetrically follows.

1.2.9. LEMMA. *A homogeneous ideal is proper and its dual ideal is also homogeneous and proper.*

Proof. Let $\mathfrak{I}$ be a homogeneous ideal. Then $\mathfrak{I} = \oplus_{p=0}^{n} \mathfrak{I}_p$, where $\mathfrak{I}_p = \mathfrak{I} \cap \bigwedge^p V$. Let $k = \sum_{p=0}^{n} k_p \in K_{\mathrm{L}}(\mathfrak{I})$ for $k_p \in \bigwedge^p V$ and $x_q \in \mathfrak{I}_q$. Then $0 = k \wedge x_q = \sum_{p=0}^{n} k_p \wedge x_q$ and since $\bigwedge V = \oplus_{p=0}^{n} \bigwedge^p V$ is a direct sum decomposition, it follows that $k_p \wedge x_q = 0$. Varying x_q over $\mathfrak{I}_q$ for all $0 \leq q \leq n$ we see that $k_p \in K_{\mathrm{L}}(\mathfrak{I})$ and $K_{\mathrm{L}}(\mathfrak{I})$ is thus a homogeneous ideal. The automorphism ϵ hence leaves $K_{\mathrm{L}}(\mathfrak{I})$ invariant and hence $\mathfrak{I}$ is proper by Observation 1.2.5. $K(\mathfrak{I}) = K_{\mathrm{L}}(\mathfrak{I})$ is homogeneous and hence also proper.

1.2.10. *Remark.* Proposition 1.2.2 enables us to characterize proper ideals of the Grassmann algebra by means of exterior equations. Suppose $\mathfrak{I}$ is a proper ideal and $K(\mathfrak{I})$ is multiplicatively generated by generators $k_1, \ldots, k_r$. Then by (ii) of Proposition 1.2.2, $\mathfrak{I} = \{\omega \in \bigwedge V \mid \omega \wedge k_1 = \cdots = \omega \wedge k_r = 0\}$. This if course presupposes that $K(\mathfrak{I})$ is known. The determination of $K(\mathfrak{I})$ may be a formidable problem, as we shall see in Section 2.

1.2.11. LEMMA. *Let* $0 \neq x \in V$. *Then* $K[(x)] = (x)$.

Proof. Let U be a complementary subspace in V to the 1-dimensional subspace generated by x. There is an isomorphism, $\bigwedge U \xrightarrow{\approx} (x)$ given by $\Delta \to x \wedge \Delta$. Thus $\dim(x) = \dim \bigwedge U = 2^{n-1}$. Clearly $(x) \subseteq K[(x)]$ and $\dim K[(x)] = 2^n - \dim(x) = 2^n - 2^{n-1} = 2^{n-1}$. Hence $\dim K[(x)] = \dim(x)$ and thus $K[(x)] = (x)$.

1.2.12. COROLLARY. *Let* $0 \neq x \in V$ *and* $\omega \in \bigwedge V$. *Then* $\omega = \tau \wedge x$ *for some* $\tau \in \bigwedge V$ *iff* $\omega \wedge x = 0$.

1.2.13. PROPOSITION. *Let* $\{x_1, \ldots, x_k\}$ *be a linearly independent set of vectors in* V. *Then* $K[(x_1, \ldots, x_k)] = (x_1 \wedge \cdots \wedge x_k)$.

Proof. Let U be a complementary subspace in V to the k-dimensional subspace generated by $x_1, \ldots, x_k$. Then there is an isomorphism, $\bigwedge U \xrightarrow{\approx} (x_1 \wedge \cdots \wedge x_k)$ given by $\Delta \to x_1 \wedge \cdots \wedge x_k \wedge \Delta$ and thus $\dim(x_1 \wedge \cdots \wedge x_k) = \dim \bigwedge U = 2^{n-k}$. Also we have a direct-sum decomposition, $\bigwedge V = \bigwedge U \oplus (x_1 \wedge \cdots \wedge x_k)$ and hence $\dim(x_1 \wedge \cdots \wedge x_k) = 2^n - 2^{n-k}$. Clearly $(x_1 \wedge \cdots \wedge x_k) \subseteq K[(x_1 \wedge \cdots \wedge x_k)]$ and $\dim K[(x_1 \wedge \cdots \wedge x_k)] = 2^n - \dim(x_1 \wedge \cdots \wedge x_k) = 2^n - 2^k = \dim(x_1 \wedge \cdots \wedge x_k)$. Thus $K[(x_1 \wedge \cdots \wedge x_k)] = (x_1 \wedge \cdots \wedge x_k)$.

1.2.14. COROLLARY. *Let* $\{x_1, \ldots, x_k\}$ *be a linearly independent set of vectors in* V *and* $\omega \in \bigwedge V$. *Then* $\omega = \tau \wedge x_1 \wedge \cdots \wedge x_k$ *for some* $\tau \in \bigwedge V$ *iff* $x_1 \wedge \omega = \ldots = x_k \wedge \omega = 0$.

As a further corollary we recover [1, Theorem 1], e.g., 1.2.15. THEOREM. *Let* $\omega \in \bigwedge V$ *and* $R_\omega\colon \bigwedge V \to \bigwedge V$ *be right multiplication by* ω. *Then* ω *factors into the wedge product of* k 1-*vectors and an exterior form iff* $\dim \operatorname{Ker} R_\omega \geq k$.

1.3. *Abelian Ideals*

1.3.1. DEFINITION. A proper ideal $\mathfrak{I}$ is called an abelian ideal iff $\mathfrak{I} \subseteq K(\mathfrak{I})$.

1.3.2. DEFINITION. An abelian ideal $\mathfrak{I}$ is called a maximal abelian ideal iff $\mathfrak{I} = K(\mathfrak{I})$. Note that the dimension of a maximal abelian ideal is $\frac{1}{2}\dim \bigwedge V = 2^{n-1}$.

1.3.3. *Types of maximal ideals*

Let $\{e_1, \dots, e_n\}$ be a basis for V and $\mathfrak{I} = \bigoplus_{p=1}^{n} \bigwedge^p V$ the unique maximal ideal of $\bigwedge V$. Let $\mathfrak{I}_k = (e_1, \dots, e_{2k-1})$ be the ideal in $\bigwedge V$ multiplicatively generated by $e_1, \dots, e_{2k-1}$ $(1 \leq k \leq [(n+1)/2])$. Then its kth-power, $\mathfrak{I}_k^k$ is a maximal abelian ideal. $\mathfrak{I}_k^k$ and its images under automorphisms of V are called maximal abelian ideals of type k. Two maximal ideals of types k and l are isomorphisms iff $k = l$. Thus classes of different types of maximal abelian ideals form the non-isomorphic classes of maximal abelian ideals in $\bigwedge V$. Also, $\mathfrak{I}^k = \bigcup_{\sigma \in \mathrm{Aut}(V)} \sigma(\mathfrak{I}_k^k)$.

2. THE DUAL OF THE PRINCIPAL IDEAL GENERATED BY AN EXTERIOR 2-FORM

2.1. *The Principal Ideal* (μ)

Let μ be an exterior 2-form in V. Since $\mu \in \bigwedge^{\mathrm{ev}}(V)$, it follows from Observation 1.2.8 that the principal ideal (μ) generated by μ is a proper ideal.

2.2. *The Ideal* $\theta(\mu)$

Suppose $\operatorname{rank}(\mu) = 2s$. Then there exists a linearly independent set $\{x_1, \dots, x_s, y_1, \dots, y_s\}$ of vectors in V such that $\mu = x_1 \wedge y_1 + \cdots + x_s \wedge y_s$. Define $\mu_j = x_j \wedge y_j$ $(1 \leq j \leq n)$ so that $\mu = \mu_1 + \cdots + \mu_s$. Then $(\mu_i - \mu_j) \wedge (\mu_i + \mu_j) = (\mu_i + \mu_j) \wedge (\mu_i - \mu_j) = 0$ and $(\mu_i - \mu_j)^2 = -2\mu_i \wedge \mu_j = 2x_i \wedge y_i \wedge x_j \wedge y_j$. Take all possible partitions $(i_1 j_1) \cdots (i_r j_r)(k_1 k_2 \cdots k_{s-2r})$, $i_k \leq j_k$ $(1 \leq k \leq r)$, $i_1 < \cdots < i_r$, $k_1 < \cdots < k_{s-2r}$, for all $0 \leq r \leq [s/2]$ and let $\theta(\mu)$ be the homogeneous ideal multiplicatively generated by generators $g_\alpha = (\mu_{i_1} - \mu_{j_1}) \wedge (\mu_{i_2} - \mu_{j_2})$

$\wedge \cdots \wedge (\mu_{i_r} - \mu_{j_r}) \wedge v_{k_1} \wedge \cdots \wedge v_{k_{s-2r}}$ in $\bigwedge^s V$, where v_{k_j} is either x_{k_j} or y_{k_j} $(1 \le j \le s-2r)$. $\theta(\mu)$ is a homogeneous ideal and is thus proper by Lemma 1.2.9. Also, $\theta(\mu) \subseteq K[(\mu)]$.

2.2.1. LEMMA. *Every element $\omega \in \theta(\mu)$ has an expression of the form*

$$\omega = \sum_{\alpha} \lambda_\alpha \wedge g_\alpha$$

for $\lambda_\alpha \in \bigwedge V$.

Proof. Every element $\omega \in \theta(\mu)$ by definition has an expression $\omega = \sum_\alpha a_\alpha \wedge g_\alpha \wedge b_\alpha$, $a_\alpha, b_\alpha \in \bigwedge V$; $g_\alpha = (\mu_{i_1} - \mu_{j_1}) \wedge (\mu_{i_2} - \mu_{j_2}) \wedge \cdots \wedge (\mu_{i_r} - \mu_{j_r}) \wedge v_{k_1} \wedge \cdots \wedge v_{k_{s-2r}}$. Let $b_\alpha = \sum_{p=0}^n \tau_{\alpha,p}$, where $\tau_{\alpha,p} \in \bigwedge^p V$. Then $g_\alpha \wedge \tau_{\alpha,p} = (-1)^{p(s-2r)} \tau_{\alpha,p} \wedge g_\alpha$. Define $b'_\alpha = \sum_{p=0}^n (-1)^{p(s-2r)} \tau_{\alpha,p}$. Then $g_\alpha \wedge b_\alpha = \sum_{p=0}^n g_\alpha \wedge \tau_{\alpha,p} = \sum (-1)^{p(s-2r)} \tau_{\alpha,p} \wedge g_\alpha = b'_\alpha \wedge g_\alpha$ and hence $a_\alpha \wedge g_\alpha \wedge b_\alpha = a_\alpha \wedge b'_\alpha \wedge g_\alpha$. Putting $\lambda_\alpha = a_\alpha \wedge b'_\alpha$ yields the result.

2.2.2. *Equivalence classes of generators*

We define an equivalence relation on the set of generators. Let $g_\alpha = (\mu_{i_1} - \mu_{j_1}) \wedge (\mu_{i_2} - \mu_{j_2}) \wedge \cdots \wedge (\mu_{i_r} - \mu_{j_r}) \wedge v_{k_1} \wedge \cdots \wedge v_{k_{s-2r}}$ and $g_\beta = (\mu_{i'_1} - \mu_{j'_1}) \wedge (\mu_{i'_2} - \mu_{j'_2}) \wedge \cdots \wedge (\mu_{i'_t} - \mu_{j'_t}) \wedge u_{k'_1} \wedge \cdots \wedge u_{k'_{s-2t}}$ be two generators. Then $g_\alpha \sim g_\beta$ iff $r = t$, $(i'_1 j'_1 \cdots i'_t j'_t)$ is a permutation of $(i_1 j_1 \cdots i_r j_r)$, $k'_j = k_j$, and $u_{k'_j} = v_{k_j}$ $(1 \le j \le s-2r)$. The equivalence classes are denoted by $\Delta(v_{k_1}, v_{k_2}, \ldots, v_{k_{s-2r}})$ for $0 \le r \le [s/2]$ and v_{k_j} is either x_{k_j} or y_{k_j}. If s is even, $\Delta(\phi)$ denotes the equivalence class of generators, $g_\alpha = (\mu_{i_1} - \mu_{j_1}) \wedge (\mu_{i_2} - \mu_{j_2}) \wedge \cdots \wedge (\mu_{i_{s/2}} - \mu_{j_{s/2}})$. Let $\Delta(v_{k_1}, v_{k_2}, \ldots, v_{k_{s-2r}})$ be an equivalence class and let u_{k_j} be the complement of v_{k_j} in the set $\{x_{k_j}, y_{k_j}\}$, i.e., $\{v_{k_j}, u_{k_j}\} = \{x_{k_j}, y_{k_j}\}$ $(1 \le j \le s-2r)$. Then $\Delta(u_{k_1}, u_{k_2}, \ldots, u_{k_{s-2r}})$ is called the "dual" class of $\Delta(v_{k_1}, v_{k_2}, \ldots, v_{k_{s-2r}})$.

2.2.3. LEMMA. *Let g_α and g_β be two generators for $\theta(\mu)$. If their equivalence classes are not dual then $g_\alpha \wedge g_\beta = 0$.*

Proof. Since g_α and g_β do not belong to dual equivalence classes there exists k_1 such that WLG $x_{k_1} \mid g_\alpha$ but $y_{k_1} \nmid g_\beta$. If $x_{k_1} \mid g_\beta$ then obviously $g_\alpha \wedge g_\beta = 0$. Suppose there exists $k_2 \ne k_1$ such that $(\mu_{k_1} - \mu_{k_2}) \mid g_\beta$. Thus $g_\alpha \wedge g_\beta$ contains $x_{k_1} \wedge (\mu_{k_1} - \mu_{k_2}) = -x_{k_1} \wedge \mu_{k_2}$ as a factor. If g_α has either x_{k_2} or y_{k_2} at the k_2th-place then $g_\alpha \wedge g_\beta$ contains the wedge product of this with $x_{k_1} \wedge \mu_{k_2}$, which is zero and hence $g_\alpha \wedge g_\beta = 0$. So let us assume WLG that there exists $k_3 \ne k_2$ such that g_α contains $(\mu_{k_2} - \mu_{k_3})$ as a factor. Then $g_\alpha \wedge g_\beta$ contains $-x_{k_1} \wedge \mu_{k_2} \wedge (\mu_{k_2} - \mu_{k_3}) = x_{k_1} \wedge \mu_{k_2} \wedge \mu_{k_3}$ as a factor, which would be zero if $k_3 = k_1$. We thus assume that (k_1, k_2, k_3) are all distinct. Continuing in this manner, we find either that $g_\alpha \wedge g_\beta = 0$ or that there exist distinct integers $k_1, k_2, k_3, \ldots$ such that $g_\alpha = x_{k_1} \wedge (\mu_{k_2} - \mu_{k_3}) \wedge (\mu_{k_4} - \mu_{k_5}) \wedge \cdots$ $g_\beta = (\mu_{k_1} - \mu_{k_2}) \wedge \mu_{k_3} - \mu_{k_4}) \wedge \ldots$ and this is a contradiction since $\deg(g_\alpha) \neq \deg(g_\beta)$.

2.2.4. LEMMA. *Let V be a vector space with basis $\{e_1, \ldots, e_2\}$. Define the isomorphism $\phi\colon V \to V^*$ by $\phi(e_i) = e_i^*$, where e_i^* is the dual of e_i. If $U \subset V$ is any subspace of V then the composite map $\psi\colon U \subset V \to^{\phi} V^* \to^{\text{rest.}^n} U^*$ is an isomorphism.*

Proof. Let $\{u_1, \ldots, u_m\}$ be a basis for U $(m \leq n)$ and express $u_i = \sum_{j=1}^n a_{ij} e_j$ $(1 \leq i \leq m)$, where $A = (a_{ij})$ is an $(m \times n)$-matrix with linearly independent rows and has rank m. Let A^{t} be the transpose of A. Then $B = AA^{\mathrm{t}}$ is a non-singular $(m \times m)$-matrix. Let $B = (b_{ij})$. Then $\psi(u_i)(u_j) = (\sum_{k=1}^n a_{ik} e_k^*)(\sum_{l=1}^n a_{jl} e_l) = \sum_{k=1}^n a_{ik} a_{jk} = b_{ij}$, i.e., $\psi(u_i) = \sum_{j=1}^m b_{ij} u_j^*$. Since B is nonsingular, it follows that $\{\psi(u_i)\}$ is a linearly independent set and hence is a basis for U^*. Thus ψ is an isomorphism.

2.2.5. DEFINITION. Let $V = \{x_1, y_1, \ldots, x_{2n}, y_{2n}\}$ be a $4n$-dimensional vector space. $\mu_j = x_j \wedge y_j$ $(1 \leq j \leq 2n)$. Let T be the subspace of $\bigwedge^{2n} V$ spanned by $\{(\mu_{i_1} - \mu_{j_1}) \wedge (\mu_{i_2} - \mu_{j_2}) \wedge \cdots \wedge (\mu_{i_n} - \mu_{j_n})\}$, $i_k \leq j_k$ $(1 \leq k \leq n)$, $i_1 < \cdots < i_n$ as $(i_1, j_1)(i_2, j_2) \cdots (i_n, j_n)$ runs through the set of $(2, 2, \ldots, 2)$ partitions of $(1, 2, \ldots, 2n)$. Define $K_T(T) = \{t \in T \mid t \wedge t' = 0\ \forall t' \in T\}$.

2.2.6. LEMMA. $K_T(T) = 0$.

Proof. Let $\phi\colon V \to V^*$ be defined by $\phi(x_i) = x_i^*$ and $\phi(y_i) = y_i^*$ and $\bigwedge \phi\colon \bigwedge V \to \bigwedge^* V$ be the induced map which maps the induced basis for $\bigwedge V$ into the dual basis for $\bigwedge^* V$. Let $*\colon \bigwedge V \to \bigwedge^* V$ be the (left) duality operator. Then an easy computation shows that

$$\begin{aligned} * & \left[\left\{(\mu_{i_1} - \mu_{j_1}) \wedge \cdots \wedge (\mu_{i_n} - \mu_{j_n})\right\}\right] \\ & = \mp\left(\bigwedge \phi\right)\left[\left\{(\mu_{i_1} - \mu_{j_1}) \wedge \cdots \wedge (\mu_{i_n} - \mu_{j_n})\right\}\right]. \end{aligned} \tag{1}$$

Let $\iota\colon T \subset \bigwedge V$ be the inclusion and $\iota^*\colon \bigwedge^* V \to T^*$ be the restriction map. Define $\alpha, \psi\colon T \to T^*$ by $\alpha = \iota^* \circ * \circ \iota$ and $\psi = \iota^* \circ \phi \circ \iota$. Applying ι^* to Eq. (1) yields

$$\begin{aligned} \alpha & \left[\left\{(\mu_{i_1} - \mu_{j_1}) \wedge \cdots \wedge (\mu_{i_n} - \mu_{j_n})\right\}\right] \\ & = \mp\psi\left[\left\{(\mu_{i_1} - \mu_{j_1}) \wedge \cdots \wedge (\mu_{i_n} - \mu_{j_n})\right\}\right]. \end{aligned} \tag{2}$$

α maps $K_T(T)$ to zero. However, ψ is an isomorphism by Lemma 2.2.4 and thus α is an isomorphism by Eq. (2). Hence $K_T(T) = 0$.

2.2.7. *Remark.* Let $U_j = \{x_j, y_j\}$ $(1 \le j \le s)$ and let U be a complementary subspace to $U_1 \oplus U_2 \oplus \cdots \oplus U_s$ in V. In the following proposition we shall regard $\mu_1 + \mu_2 + \cdots + \mu_{s-1} \in \bigwedge^2(U_1 \oplus U_2 \oplus \cdots \oplus U_{s-1} \oplus U)$ and $\theta(\mu_1 + \mu_2 + \cdots + \mu_{s-1}) \subset \bigwedge(U_1 \oplus U_2 \oplus \cdots \oplus U_{s-1} \oplus U)$ and similarly for the other terms.

2.2.8. PROPOSITION. $\theta(\mu) \cap K(\theta(\mu)) = \theta(\mu_1 + \cdots + \mu_{s-1}) \otimes \bigwedge^2 U_s + \theta(\mu_1 + \cdots + \mu_{s-2} + \mu_s) \otimes \bigwedge^2 U_{s-1} + \cdots + \theta(\mu_2 + \cdots + \mu_s) \otimes \bigwedge^2 U_1$.

Proof. Let $g \in \theta(\mu_1 + \cdots + \mu_{s-1}) \otimes \bigwedge^2 U_s$. Then $g = g_\alpha \wedge \mu_s$ for $g_\alpha \in \theta(\mu_1 + \cdots + \mu_{s-1})$. $g_\alpha = (\mu_{i_1} - \mu_{j_1}) \wedge \cdots \wedge (\mu_{i_r} - \mu_{j_r}) \wedge v_{k_1} \wedge \cdots \wedge v_{s-1-2r}$, where $0 \le r \le [(s-1)/2]$ and $(i_1 j_1)(i_2 j_2) \cdots (i_r j_r)$ $(k_1 \ldots k_{s-1-2r})$ is a partition of $(1\,2\,\ldots\, s-1)$. $g = g_\alpha \wedge \mu_s = g_\alpha \wedge x_s \wedge y_s = \mp x_s \wedge (\mu_{i_1} - \mu_{j_1}) \wedge \cdots \wedge (\mu_{i_r} - \mu_{j_r}) \wedge v_{k_1} \wedge \cdots \wedge v_{s-2r} \wedge y_s \in \theta(\mu)$, hence RHS $\subseteq$ LHS.

Conversely, let $g \in \theta(\mu) \cap K(\theta(\mu))$. By Lemma 2.2.1 we can write

$$g = \sum_\alpha a_\alpha \wedge g_\alpha = \sum_{\Delta(v_{k_1}, v_{k_2}, \ldots, v_{k_{s-2r}})} \sum_{g_\alpha \in \Delta(v_{k_1}, v_{k_2}, \ldots, v_{k_{s-2r}})} a_\alpha \wedge g_\alpha .$$

Fix an equivalence class $\Delta(v_{k_1}, v_{k_2}, \ldots, v_{k_{s-2r}})$ of generators and let $\Delta(u_{k_1}, u_{k_2}, \ldots, u_{k_{s-2r}})$ be the "dual" equivalence class and let $g_\beta \in \Delta(u_{k_1}, u_{k_2}, \ldots, u_{k_{s-2r}})$. Then $g_\alpha \wedge g_\beta = 0\ \forall g_\alpha \notin \Delta(v_{k_1}, v_{k_2}, \ldots, v_{k_{s-2r}})$ by Lemma 2.2.3. Then

$$0 = g \wedge g_\beta = \sum_{g_\alpha \in \Delta(v_{k_1}, v_{k_2}, \ldots, v_{k_{s-2r}})} a_\alpha \wedge g_\alpha \wedge g_\beta \tag{1}$$

Let $(i_1, i_2, \ldots, i_{2r})$, $1 \le i_1 < i_2 < \cdots < i_{2r} \le s$, be the complementary indices to $(k_1, k_2, \ldots, k_{s-2r})$ in $(1\,2\,\ldots\, s)$. Let W be the $4r$-dimensional subspace of V spanned by $\{x_{i_1}, y_{i_1}, \ldots, x_{i_{2r}}, y_{i_{2r}}\}$, $\mu_j = x_{i_j} \wedge y_{i_j} \in \bigwedge^2 W$ $(1 \le j \le 2r)$. Let T be the subspace of $\bigwedge^{2r} W$ generated by $\{(\mu_{l_1} - \mu_{m_1}) \wedge \cdots \wedge (\mu_{l_r} - \mu_{m_r})\}$ as $(l_1 m_1) \ldots (l_r m_r)$ runs through the set $(2, 2, \ldots, 2)$ partitions of $(i_1, i_2, \ldots, i_{2r})$. Then $g_\alpha = t_\alpha \wedge v_{k_1} \wedge \cdots \wedge v_{k_{s-2r}}$ and $g_\beta = t_\beta \wedge u_{k_1} \wedge \cdots \wedge u_{k_{s-2r}}$ for some $t_\alpha, t_\beta \in T$. Define $\Omega' = v_{k_1} \wedge \cdots \wedge v_{k_{s-2r}} \wedge u_{k_1} \wedge \cdots \wedge u_{k_{s-2r}} = \mp x_{k_1} \wedge y_{k_1} \wedge \cdots \wedge x_{k_{s-2r}} \wedge y_{k_{s-2r}} = \mp \mu_{k_1} \wedge \mu_{k_2} \wedge \cdots \wedge \mu_{k_{s-2r}}$. Then $g_\alpha \wedge g_\beta = t_\alpha \wedge t_\beta \wedge \Omega'$ and define $\Omega'' = \mp x_{i_1} \wedge y_{i_1} \wedge \cdots \wedge x_{i_{2r}} \wedge y_{i_{2r}} = \mp \mu_{i_1} \wedge \mu_{i_2} \wedge \cdots \wedge \mu_{i_{2r}}$. Then $t_\alpha \wedge t_\beta = n_{\alpha\beta} \Omega''$ for unique integers $n_{\alpha\beta}$. The matrix $(n_{\alpha\beta})$ is non-singular by Lemma 2.2.6. Define $\Omega = \Omega' \wedge \Omega'' = x_1 \wedge y_1 \wedge \cdots \wedge x_{2r} \wedge y_{2r} = \mu_1 \wedge$

$\mu_2 \wedge \cdots \wedge \mu_{2r}$. Then $g_\alpha \wedge g_\beta = n_{\alpha\beta}\Omega$. Substituting into Eq. (1) yields

$$0 = \sum_{g_\alpha \in \Delta(v_{k_1}, v_{k_2}, \ldots, v_{k_{s-2r}})} n_{\alpha\beta} a_\alpha \wedge \Omega.$$

Since the matrix $(n_{\alpha\beta})$ is non-singular, we deduce that $a_\alpha \wedge \Omega = 0$ $\forall g_\alpha \in \Delta(v_{k_1}, v_{k_2}, \ldots, v_{k_{s-2r}})$, i.e., $a_\alpha \wedge x_1 \wedge y_1 \wedge \cdots \wedge x_{2r} \wedge y_{2r} = 0$. It follows from Proposition 1.2.13 that $a_\alpha = a_{\alpha,1} \wedge x_1 + a'_{\alpha,1} \wedge y_1 + \cdots + a_{\alpha,2r} \wedge x_{2r} + a'_{\alpha,2r} \wedge y_{2r}$. Then $a_\alpha \wedge g_\alpha = (a_{\alpha,1} \wedge x_1 + a'_{\alpha,1} \wedge y_1 + \cdots + a_{\alpha,2r} \wedge x_{2r} + a'_{\alpha,2r} \wedge y_{2r}) \wedge (\mu_{l_1} - \mu_{m_1}) \wedge \cdots \wedge (\mu_{l_r} - \mu_{m_r}) \wedge v_{k_1} \wedge \cdots \wedge v_{k_{s-2r}}$.

Take the term $x_1 \wedge (\mu_{l_1} - \mu_{m_1}) \wedge \cdots \wedge (\mu_{l_r} - \mu_{m_r}) \wedge v_{k_1} \wedge \cdots \wedge v_{k_{s-2r}}$. Suppose $1 \in \{k_1, k_2, \ldots, k_{s-2r}\}$ and assume WLG that $k_1 = 1$. Either $v_{k_1} = x_1$, in which case the term is zero or $v_{k_1} = y_1$ and the term equals $(\mu_{l_1} - \mu_{m_1}) \wedge \cdots \wedge (\mu_{l_r} - \mu_{m_r}) \wedge v_{k_1} \wedge \cdots \wedge v_{k_{s-2r}} \wedge \mu_1 \in \theta(\mu_2 + \cdots + \mu_s) \otimes \bigwedge^2 U_1$. Now suppose that $1 \notin \{k_1, k_2, \ldots, k_{s-2r}\}$ and the term equals $x_1 \wedge (\mu_{l_1} - \mu_{m_1}) \wedge \cdots \wedge (\mu_{l_r} - \mu_{m_r}) \wedge v_{k_1} \wedge \cdots \wedge v_{k_{s-2r}} = (\mu_{l_2} - \mu_{m_2}) \wedge \cdots \wedge (\mu_{l_r} - \mu_{m_r}) \wedge x_1 \wedge v_{k_1} \wedge \cdots \wedge v_{k_{s-2r}} \wedge \mu_{m_1} \in \theta(\mu_1 + \cdots + \hat{\mu}_{m_1} + \cdots + \mu_s) \otimes \bigwedge^2 U_{m_1}$ and similar for the other terms. This shows that $a_\alpha \wedge g_\alpha \in$ RHS and hence that $g \in$ RHS and this shows that LHS $\subseteq$ RHS and hence equality, i.e., LHS $=$ RHS.

2.3. *Duality between* (μ) *and* $\theta(\mu)$

2.3.1. LEMMA. $(\mu) \cap (\mu_s) = [K[(\mu_1 + \cdots + \mu_{s-1})] + (\mu_1 + \cdots + \mu_{s-1})] \otimes \bigwedge^2 U_s$.

Proof. Let U be a complementary subspace to $U_1 \oplus \cdots \oplus U_s$ in V and let $\omega \in [K[(\mu_1 + \cdots + \mu_{s-1})] + (\mu_1 + \cdots + \mu_{s-1})] \otimes \bigwedge^2 U_s$. Then $\omega = \omega_1 \wedge \mu_s + \omega_2 \wedge \mu_s$ for $\omega_1 \in K[(\mu_1 + \cdots + \mu_{s-1})]$ and $\omega_2 = \tau \wedge (\mu_1 + \cdots + \mu_{s-1})$ for $\tau \in \bigwedge(U_1 \oplus \cdots \oplus U_{s-1} \oplus U)$. Then $\omega_1 \wedge \mu = \omega_1 \wedge (\mu_1 + \cdots + \mu_{s-1}) + \omega_1 \wedge \mu_s = \omega_1 \wedge \mu_s$ and $\omega_2 \wedge \mu_s = \tau \wedge (\mu_1 + \cdots + \mu_{s-1}) \wedge \mu_s = \tau \wedge \mu \wedge \mu_s$ and thus $\omega = \omega_1 \wedge \mu + \tau \wedge \mu \wedge \mu_s = (\omega_1 + \tau \wedge \mu_s) \wedge \mu \in (\mu)$. Also, $\omega \in (\mu_s)$. Hence $\omega \in (\mu) \cap (\mu_s)$. Conversely, let $\omega \in (\mu) \cap (\mu_s)$. Then, $\omega = \tau \wedge \mu$ for some $\tau \in \bigwedge V$ and since $\bigwedge V = \bigwedge(U_1 \oplus \cdots \oplus U_{s-1} \oplus U) \otimes \bigwedge U_s$. We can write $\tau = \tau_0 + \tau_1 \wedge x_s + \tau_2 \wedge y_s + \tau_3 \wedge \mu_s$ for $\tau_i \in \bigwedge(U_1 \oplus \cdots \oplus U_{s-1} \oplus U)$ $(1 \le i \le 3)$. Thus $\omega = \tau_0 \wedge \mu + \tau_1 \wedge x_s \wedge \mu + \tau_2 \wedge y_s \wedge \mu + \tau_3 \wedge \mu_s \wedge \mu$. $\omega \in (\mu_s) = K[(x_s, y_s)]$ by Proposition 1.2.13 and thus $0 = \omega \wedge \mu_s = \tau_0 \wedge \mu \wedge \mu_s = [\tau_0 \wedge (\mu_1 + \cdots + \mu_s)] \wedge \mu_s \in \bigwedge(U_1 \oplus \cdots \oplus U_{s-1} \oplus U) \otimes \bigwedge U_s$. Hence $\tau_0 \wedge (\mu_1 + \cdots + \mu_{s-1}) = 0$, i.e., $\tau_0 \in K[(\mu_1 + \cdots + \mu_{s-1})]$. Also,

$\tau_0 \wedge \mu = \tau_0 \wedge (\mu_1 + \cdots + \mu_{s-1} + \mu_s) = \tau_0 \wedge \mu_s$. $0 = \omega \wedge x_s + \tau_2 \wedge y_s \wedge \mu \wedge x_s = -\tau_2 \wedge \mu \wedge \mu_s = -[\tau_2 \wedge (\mu_1 + \cdots + \mu_{s-1})] \wedge \mu_s$. Hence $\tau_2 \wedge (\mu_1 + \cdots + \mu_{s-1}) = 0$ and $\tau_2 \wedge y_s \wedge \mu = \tau_2 \wedge (\mu_1 + \cdots + \mu_{s-1}) \wedge y_s = 0$. Similarly, $\tau_1 \wedge x_s \wedge \mu = 0$. $\tau_3 \wedge \mu_s \wedge \mu = \tau_3 \wedge (\mu_1 + \cdots + \mu_{s-1}) \wedge \mu_s = \alpha \wedge \mu_s$, where $\alpha \in (\mu_1 + \cdots + \mu_{s-1})$. Thus $\omega = (\tau_0 + \alpha) \wedge \mu_s \in [K[(\mu_1 + \cdots + \mu_{s-1})] + (\mu_1 + \cdots + \mu_{s-1})] \otimes \bigwedge^2 U_s$.

2.3.2. LEMMA. $K[\theta(\mu)] \cap (\mu_s) = K[\theta(\mu_1 + \cdots + \mu_{s-2}) \otimes \bigwedge^2 U_{s-1} + \cdots + \theta(\mu_2 + \cdots + \mu_{s-1}) \otimes \bigwedge^2 U_1] \otimes \bigwedge^2 U_s$.

Proof. Let $\omega \in K[\theta(\mu)] \cap (\mu_s)$. Then $\omega = \tau \wedge \mu_s$ for $\tau \in \bigwedge(U_1 \oplus \cdots \oplus U_{s-1} \oplus U)$, where U is a complementary subspace to $U_1 \oplus U_2 \oplus \cdots \oplus U_s$ in V. Let $g_\alpha = (\mu_{i_1} - \mu_{j_1}) \wedge \cdots \wedge (\mu_{i_r} - \mu_{j_r}) \wedge v_{k_1} \wedge \cdots \wedge v_{k_{s-2r}}$ be a generator for $\theta(\mu)$, where $0 \leq r \leq [s/2]$, $(i_1 j_1) \ldots (i_r j_r)(k_1 k_2 \ldots k_{s-2r})$ is a partition of $(1\,2\,\ldots\,s)$ and v_{k_j} is either x_{k_j} or y_{k_j}. If $s \in \{k_1, \ldots, k_{s-2r}\}$ then $g_\alpha \wedge \omega = 0$ anyway; so the generators which contribute non-trivially are those for which $s \in \{i_1 j_1 \ldots i_r j_r\}$. Suppose g_α is such a generator and assume WLG that $j_r = s$. Then

$$\begin{aligned} 0 = g_\alpha \wedge \omega &= (\mu_{i_1} - \mu_{j_1}) \wedge \cdots \wedge (\mu_{i_r} - \mu_{j_r}) \wedge \mu_s \wedge v_{k_1} \wedge \cdots \wedge v_{k_{s-2r}} \wedge \tau \\ &= (\mu_{i_1} - \mu_{j_1}) \wedge \cdots \wedge (\mu_{i_{r-1}} - \mu_{j_{r-1}}) \wedge \mu_{i_r} \wedge \mu_s \wedge v_{k_1} \wedge \cdots \wedge v_{k_{s-2r}} \wedge \tau \\ &= g_\beta \wedge \mu_{i_r} \wedge \tau, \text{ where } g_\beta \in \theta\big(\mu_1 + \cdots + \hat{\mu}_{i_r} + \cdots + \mu_{s-1}\big) \\ &\qquad\qquad (1 \leq i_r \leq s-1). \end{aligned}$$

Thus $\tau \in K[\theta(\mu_1 + \cdots + \hat{\mu}_{i_r} + \cdots + \mu_{s-1}) \otimes \bigwedge^2 U_{i_r}] \subseteq K[\theta(\mu_1 + \cdots + \mu_{s-2}) \otimes \bigwedge^2 U_{s-1} + \cdots + \theta(\mu_2 + \cdots + \mu_{s-1})] \otimes \bigwedge^2 U_1$ and hence $\omega \in K[\theta(\mu_1 + \cdots + \mu_{s-2}) \otimes \bigwedge^2 U_{s-1} + \cdots + \theta(\mu_2 + \cdots + \mu_{s-1}) \otimes \bigwedge^2 U_1] \otimes \bigwedge^2 U_s$ and the argument can be reversed so as to prove the converse.

2.3.3. THEOREM. $K[\theta(\mu)] = (\mu)$.

By using the duality between and ideal and its annihilator, we can also state Theorem 2.3.3 in an equivalent form.

2.3.3. THEOREM*. $K[(\mu)] = \theta(\mu)$.

Proof. Clearly $(\mu) \subseteq K[\theta(\mu)]$.

The converse will be proved by induction on $s = \operatorname{rank}(\mu)$. For $s = 1$, $\mu = x \wedge y$ and $\theta(\mu) = (x, y) = K[(x \wedge y)] = K[(\mu)]$. Let $s > 1$ and assume the induction hypothesis for $(s-1)$. Let $\omega = x_1 \wedge y_1 + \cdots + x_s \wedge y_s$, $U_j = \{x_j, y_j\}$ $(1 \leq j \leq s)$ and let U be a complementary subspace to $U_1 \oplus \cdots \oplus U_s$ in V and regard $\theta(\mu_1 + \cdots + \mu_{s-1}) \subset \bigwedge(U_1 \oplus \cdots \oplus U_{s-1} \oplus U)$. Let $\mathscr{I}_s$ denote the restriction of (x_s, y_s) to $\bigwedge U_s$. Then

$\theta(\mu_1 + \cdots + \mu_{s-1}) \cap (x_s, y_s) = \theta(\mu_1 + \cdots + \mu_{s-1}) \otimes \mathscr{I}_s \subseteq \theta(\mu)$. Thus

$$\begin{aligned}
K[\theta(\mu)] &\subseteq K[\theta(\mu_1 + \cdots + \mu_{s-1}) \cap (x_s, y_s)] \\
&= K[(\theta(\mu_1 + \cdots + \mu_{s-1})] + K[(x_s, y_s)] \qquad \text{by Lemma 1.2.7.} \\
&= (\mu_1 + \cdots + \mu_{s-1}) + (x_s \wedge y_s) \\
&\qquad \text{by the induction hypothesis and by Proposition 1.2.13,}
\end{aligned}$$

where $(\mu_1 + \cdots + \mu_{s-1}) \subset \bigwedge(U_1 \oplus \cdots \oplus U_{s-1} \oplus U)$. Let $\omega \in K[(\theta(\mu)]$. We can write $\omega = \tau \wedge (\mu_1 + \cdots + \mu_{s-1}) + \tau' \wedge \mu_s$ or $\omega = \tau \wedge \mu + \alpha \wedge \mu_s$, where $\alpha = \tau' - \tau$.

$$\begin{aligned}
\alpha \wedge \mu_s &\in K[\theta(\mu)] \cap (\mu_s) \\
&= K\big[\theta(\mu_1 + \cdots + \mu_{s-2}) \otimes \textstyle\bigwedge^2 U_{s-1} + \cdots \\
&\qquad + \theta(\mu_2 + \cdots + \mu_{s-1}) \otimes \textstyle\bigwedge^2 U_1\big] \otimes \textstyle\bigwedge^2 U_s \qquad \text{by Lemma 2.3.2.} \\
&= K\big[\theta(\mu_1 + \cdots + \mu_{s-1}) \cap K[\theta(\mu_1 + \cdots + \mu_{s-1})]\big] \otimes \textstyle\bigwedge^2 U_s \\
&\qquad \text{by Proposition 2.2.8.} \\
&= \big[K[\theta(\mu_1 + \cdots + \mu_{s-1})] + \theta(\mu_1 + \cdots + \mu_{s-1})\big] \otimes \textstyle\bigwedge^2 U_s \\
&\qquad \text{by Lemma 1.2.7 and Proposition 1.2.2.} \\
&= \big[(\mu_1 + \cdots + \mu_{s-1}) + K[(\mu_1 + \cdots + \mu_{s-1})]\big] \otimes \textstyle\bigwedge^2 U_s \\
&\qquad \text{by the induction hypothesis} \\
&= (\mu) \cap (\mu_s) \qquad \text{by Lemma 2.3.1.}
\end{aligned}$$

Thus $\omega = \tau \wedge \mu + \alpha \wedge \mu_s$, where $\alpha \wedge \mu_s \in (\mu)$. Hence $\omega \in (\mu)$.

2.3.4. *Remark.* The vectors $\{x_1, \ldots, x_s, y_1, \ldots, y_s\}$ were used in the definition of the ideal $\theta(\mu)$. However, Theorem 2.3.3 showed that $\theta(\mu)$ is independent of the choice of these vectors.

2.3.5. DEMONSTRATION. Let $\mu = x_1 \wedge y_1 + x_2 \wedge y_2 + x_3 \wedge y_3$ be of rank 6 and $\omega \in \bigwedge V$. Then according to Theorem 2.3.3, ω factors into a wedge product $\omega = \tau \wedge \mu$ for some $\tau \in \bigwedge V$ iff ω satisfies the following system of exterior equations.

1.	$\omega \wedge x_1 \wedge x_2 \wedge x_3 = 0$	8.	$\omega \wedge y_1 \wedge y_2 \wedge y_3 = 0$
2.	$\omega \wedge x_1 \wedge x_2 \wedge y_3 = 0$	9.	$\omega \wedge (x_1 \wedge y_1 - x_2 \wedge y_2) \wedge x_3 = 0$
3.	$\omega \wedge x_1 \wedge y_2 \wedge x_3 = 0$	10.	$\omega \wedge (x_1 \wedge y_1 - x_2 \wedge y_2) \wedge y_3 = 0$
4.	$\omega \wedge x_1 \wedge y_2 \wedge y_3 = 0$	11.	$\omega \wedge (x_1 \wedge y_1 - x_3 \wedge y_3) \wedge x_2 = 0$

5. $\omega \wedge y_1 \wedge x_2 \wedge x_3 = 0$
6. $\omega \wedge y_1 \wedge x_2 \wedge y_3 = 0$
7. $\omega \wedge y_1 \wedge y_2 \wedge x_3 = 0$

12. $\omega \wedge (x_1 \wedge y_1 - x_3 \wedge y_3) \wedge y_2 = 0$
13. $\omega \wedge x_1 \wedge (x_2 \wedge y_2 - x_3 \wedge y_3) = 0$
14. $\omega \wedge y_1 \wedge (x_2 \wedge y_2 - x_3 \wedge y_3) = 0$

2.4. *The Global Problem of Factorization*

Let ζ be a vector bundle over a topological space X and μ a 2-form and ω a form on ζ. At each point $x \in X$, define $\theta_x(\mu)$ as an ideal of $(\bigwedge \zeta)_x$ and put $\theta(\mu) = \bigcup_{x \in X} \theta_x(\mu)$. If μ is of constant rank then $\theta(\mu)$ is a subbundle of $\bigwedge \zeta$. Suppose $\omega_x \in K[\theta_x(\mu)]\ \forall x \in X$. Then by Theorem 2.3.3, ω factors into $\omega_x = \tau_x \wedge \mu_x$ for some $\tau_x \in (\bigwedge \zeta)_x$ and this can be done locally in some neighborhood of every point. The question is, what primary and higher obstructions will be hit for a global factorization of this form, i.e., for the existence of a continuous form τ on ζ such that $\omega = \tau \wedge \mu$?

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