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Defects of Irreducible Characters of p-Soluble Groups

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We prove a refinement of the p-soluble case of Robinson's conjectural local characterization of the defect of an irreducible character. © 1998 Academic Press

1. INTRODUCTION

Some recent refinements of Alperin's weight conjecture by Dade [3] and Robinson [10] indicate undiscovered riches in the rapport between Clifford theory and the local theory of group representations. The p-soluble case may be a good place to start looking at subproblems arising from these conjectures, because it is here that Clifford theory works best.

We fix a prime p and a finite group G. Let ν_p denote the p-adic valuation on the rational numbers. Thus $\nu_p(z) = \log_p z_p$ for any rational number z. When we speak of an irreducible character of G, we shall always be referring to an absolutely irreducible Frobenius character. Recall that the *defect* of an irreducible character χ of G is the natural number $d(\chi) := \nu_p(|G|/\chi(1))$.

The principle of local theory—only a vague statement can even hope to encompass the plethora of sometimes conflicting approaches—is that the positive defect irreducible characters of G (and many other aspects of positive-defect p-blocks) can be related to "local" objects, objects which are in some way associated with nontrivial p-subgroups of G. In the context of Alperin's conjecture, the normalizers of the nontrivial p-subgroups would have to be considered "local." But in some sense (this point of view is taken in much of Puig's work, say [8, 9]), the less our objects

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depend on G itself, and on the inclusions of the p-subgroups in G, the "more local" our objects are. It is in this sense that the following conjecture is "very local." The conjecture, "locally" characterizing the defect of an irreducible character, is part of the consequence [10, 5.1] of the refinement [10, 4.1] of Alperin's conjecture.

Conjecture 1.A (*Robinson*). Let λ be an irreducible character of a central p-subgroup L of G, and let χ be an irreducible character of G lying over λ and lying in a p-block with defect group D. Then there exists some P with $C_D(P) \leq P \leq D$, and an irreducible character η of P lying over λ such that $d(\chi) = d(\eta)$.

When $L = 1$, the conjecture simplifies attractively and still has interesting consequences as noted in [10]. However, both here and in [10], the full generality is needed for inductive arguments.

The p-soluble case probably follows already from some (as yet, incompletely published) work of Dade. Indeed, Dade [3, p. 98] writes:

> ... we can use Clifford theory for a normal subgroup N of G to reduce theorems and conjectures to the case where G is simple. Eventually we shall perform such a reduction for a strengthened form of our conjecture.

Dade informs me that his reductions subsume a proof of the p-soluble case of the strengthened form of his conjecture. Meanwhile, Robinson [10, p. 324] writes:

> It would appear from the results of this paper and [3] that Conjecture 4.1 is equivalent to Dade's weight conjecture 17.10 in [3], although at first sight, Conjecture 4.1 seems stronger.

In this note, a clearer and more direct proof of a refinement of the p-soluble case of the conjecture shows that by no coincidence (at least in the p-soluble case) do the conjecture and the theorem of Knörr, Picaronny, and Puig in [1, 1.1] both provide us with a subgroup P of a defect group D such that $C_D(P) = Z(P)$.

Let κ be the field of fractions of a characteristic-zero, integrally closed, complete local commutative noetherian ring $\mathscr{O}$ whose residue field k has characteristic p. (For instance, $\mathscr{O}$ may be the completion of the integral closure of the p-adic completion of the rational integers). We shall not repeat, here, G-algebra theory introduced in Puig [7, 9]; the books by Külshammer [5] and Thévenaz [12] each contain all the terminology we shall need. Given a subgroup $H \leq G$, and an interior G-algebra A, we write A^H for the H-fixed subalgebra of A, write Tr_H^G for the relative trace map $A^H \to A^G$, and write A_H^G for the image of Tr_H^G. When A is $\mathscr{O}$-free, we define $\kappa A := \kappa \otimes_{\mathscr{O}} A$ as an interior G-algebra over κ. The induction map from characters of H to characters of G, and the induction functor from interior H-algebras to interior G-algebras are both denoted by Ind_H^G.

Given an irreducible character χ of G, we write e_χ for the centrally primitive idempotent of κG fixing a simple κG-module X affording χ. Then the action κGe_χ on X engenders an interior G-algebra isomorphism $\kappa Ge_\chi \cong \mathrm{End}_\kappa(X)$. Puig has suggested (personal communication) that the "local invariants" of χ be identified with the "local invariants" (such as the defect groups or source algebras) of the $\mathscr{O}$-free primitive interior G-algebra $\mathscr{O}Ge_\chi$ (a subring of κGe_χ). We shall adapt this idea.

We must review a fairly well-known G-algebra-theoretic version of a Clifford-theoretic construction; all the notation in the following discussion will be needed in the proof of the theorem below. Further details, including a demonstration that the construction is both possible and essentially unique, may be found in (for instance) [2, Section 4]. Let R be a normal subgroup of G, let $\underline{G} := G/R$, let ψ be an irreducible character of R inertial in G, and let $e := e_\psi$, which is a central idempotent of both κR and κG. Given an irreducible κG-character χ, then χ lies over ψ if and only if $e_\chi = e_\chi e$.

We realize $(\kappa Ge)^R$ as a twisted group algebra $\widetilde{k\underline{G}}$ of $\underline{G}$ over κ with $\underline{G}$-grading $\widetilde{\kappa\underline{G}} = \bigoplus_{\underline{g} \in \underline{G}} \kappa\tilde{\underline{g}}$, where each $\tilde{\underline{g}} \in (g\kappa Re)^R$. Let κ^* denote the multiplicative group of κ. Let $\hat{R}$ be the cyclic subgroup of κ^* such that $|\hat{R}|$ is the order of the element of $H^2(\underline{G}, \kappa^*)$ associated with $\widetilde{\kappa\underline{G}}$. By Schmid [11, 7.3], $|\hat{R}|$ divides the exponent of R. It is also well known that $|\hat{R}|$ divides $|G : R|$. We insist the elements $\tilde{\underline{g}}$ be chosen such that $\hat{R}$ contains all the values of the factor set $\underline{\alpha}$ of $\underline{G}$ given by $\tilde{\underline{g}}\tilde{\underline{h}} = \alpha(\underline{g}, \underline{h})\widetilde{\underline{gh}}$ for $\underline{g}, \underline{h} \in \underline{G}$. Let

$$1 \to \hat{R} \to \hat{G} \to \underline{G} \to 1$$

be the central extension determined up to equivalence by the condition that $\underline{G}$ has a section $\{\hat{\underline{g}} \colon \underline{g} \in \underline{G}\}$ in $\hat{G}$ satisfying $\hat{\underline{g}}\hat{\underline{h}} = \underline{\alpha}(\underline{g}, \underline{h})\widehat{\underline{gh}}$. The inclusion $\hat{R} \hookrightarrow \kappa$ is an irreducible character $\hat{\psi}$ of $\hat{R}$. Let $\hat{e} := e_{\hat{\psi}}$. Identifying $\kappa\hat{G}\hat{e}$ with $\widetilde{\kappa\underline{G}}$ via the correspondence $\hat{\underline{g}}\hat{e} \leftrightarrow \tilde{\underline{g}}$, the algebra isomorphism

$$\sigma_G \colon \widetilde{\kappa\underline{G}} \otimes_\kappa \kappa Re \xrightarrow{\sim} \kappa Ge \qquad \text{such that } a \otimes b \mapsto ab,$$

may be rewritten as

$$\sigma_G \colon \kappa\hat{G}\hat{e} \otimes_\kappa \kappa Re \xrightarrow{\sim} \kappa Ge \qquad \text{such that } \hat{\underline{g}}\hat{e} \otimes b \mapsto \tilde{\underline{g}}b.$$

Let α be the factor set of G such that $\alpha(g,h) := \underline{\alpha}(g,h)^{-1}$ for $g, h \in G$, where $\underline{g}, \underline{h}$ denote the images of g, h, respectively, in $\underline{G}$. Let $\widetilde{\kappa G}$ be a twisted group algebra with G-grading $\widetilde{\kappa G} = \bigoplus_{g \in G} \kappa\tilde{g}$ and such that each $\tilde{g}\tilde{h} = \alpha(g,h)\widetilde{gh}$. It is easy to show that $\tilde{g}^{-1}\tilde{g}\tilde{h}^{-1}h = \alpha(g,h)\underline{\widetilde{gh}}^{-1}gh$. The algebra map θ: $\widetilde{\kappa G} \to \kappa Re$ given by $g \mapsto \tilde{g}^{-1}g$ satisfies $\tilde{g}\theta(\tilde{g}) = ge = \theta(\tilde{g})\tilde{g}$. Regarding $\kappa\hat{G}\hat{e} \otimes_\kappa \kappa Re$ as an interior G-algebra with structural map $\bar{g} \mapsto \hat{\bar{g}}\hat{e} \otimes \theta(\tilde{g})$, then σ_G is an interior G-algebra isomorphism.

The Fong correspondence $\chi \leftrightarrow \hat{\chi}$ between the irreducible characters χ of G lying over ψ, and the irreducible characters $\hat{\chi}$ of $\hat{G}$ lying over $\hat{\psi}$ is characterized by the condition $\sigma_G(e_{\hat{\chi}} \otimes e) = e_\chi$. We also note that if $\hat{R} \leq \hat{H} \leq \hat{g}$ and $R \leq H \leq G$ such that $\hat{H}/\hat{R} = H/R$, then σ_G restricts to an interior H-algebra isomorphism σ_H: $\kappa\hat{H}\hat{e} \otimes_\kappa \kappa Re \xrightarrow{\sim} \kappa He$.

For a normal subgroup K of G, an irreducible character μ of K, and a natural number d, let $k(G,d,\mu)$ denote the number of irreducible characters of G with defect d lying over μ.

LEMMA. *Let K be a normal subgroup of p-power index in G. Given a Sylow p-subgroup P of G, a Sylow p-subgroup L of K central in G, an irreducible character μ of K inertial in G, and an irreducible character λ of L lying under μ, then $k(G,d,\mu) = k(P,d,\lambda)$.*

Proof. Letting $R := O_{p'}(G)$, then $K = LR$, and $\mu = \lambda \otimes \psi$ for some irreducible character ψ of R inertial in G. In the notation above, $\hat{R}$ is trivial, and we have evident isomorphisms $\kappa P \cong \kappa\hat{G}\hat{e}$ and $\kappa L \cong \kappa Le = \kappa\hat{K}\hat{e}$. Thus σ_G and σ_K may be regarded as isomorphisms $\kappa P \otimes \kappa Re \cong \kappa Ge$ and $\kappa L \otimes \kappa Re \cong \kappa Ke$, respectively. Since $\sigma_G(e_\lambda \otimes e) = \sigma_K(e_\lambda \otimes e) = e_\mu$, the Fong correspondence $\chi \leftrightarrow \hat{\chi}$ restricts to a defect-preserving bijective correspondence between the irreducible characters χ of G lying over μ, and the irreducible characters $\hat{\chi}$ of P lying over λ. ∎

The proof of the lemma can be recast using the methods in Isaacs [4, Chapter 13]: ψ extends uniquely, as in [4, 13.3], to an irreducible character $\tilde{\psi}$ of G such that the matrices representing the elements of P have determinant unity. It can be shown that the condition $\tilde{\chi} = \tilde{\chi} \otimes \tilde{\psi}$ specifies a defect-preserving bijective correspondence between the irreducible characters $\tilde{\chi}$ of G lying over μ, and the irreducible characters $\hat{\chi}$ of P lying over λ.

THEOREM. *Suppose that G is p-soluble. Let λ be an irreducible character of a central p-subgroup L of G, and let χ be an irreducible character of G lying over λ and lying in a p-block with defect group D. Then there exists some P with $C_D(P) \leq P \leq D$, an irreducible character η of P lying over λ such that $d(\chi) = d(\eta)$, and an $\mathcal{O}$-free primitive interior G-algebra A with defect group P such that $\kappa A \cong \kappa Ge_\chi$.*

Proof. First note that, given any $\mathcal{O}$-free interior G-algebra A such that $\kappa A \cong \kappa Ge_\chi$, then $A^G \cong \mathcal{O}$, and in particular, A must be primitive. Furthermore, Picaronny–Puig [6, Proposition 1], the theorem of Knörr, Picaronny, and Puig in [1, 1.1], and the transitivity property of defect groups together imply that any such A has a defect group P satisfying $C_D(P) \leq P \leq D$. In particular, since $O_p(G) \leq D$, we have $O_p(Z(G)) \leq P$. It suffices to prove that there exists an $\mathcal{O}$-free interior G-algebra A such that $\kappa A \cong \kappa G e_\chi$, and an irreducible character η of a defect group P of A such that η lies over λ, and $d(\chi) = d(\eta)$. No generality is lost in assuming that $L = O_p(Z(G))$.

We argue by a double induction, first on $|G\colon Z(G)|$, and second on $|G|$. We may assume that G is nonabelian, because otherwise we put $A = \mathcal{O} G e_\chi$, whereupon we can let P be the Sylow p-subgroup of G, and put $\eta = \mathrm{Res}^G_P(\chi)$. Since $O_p(G) O_{p'}(G) \nleq Z(G)$, we can choose R such that $R \nleq Z(G)$ and either $R = O_p(G)$ or $R = L O_{p'}(G)$. Let ψ be an irreducible character of R lying under χ, and let T be the inertia group of ψ in G. Clearly, $L \leq R$ and ψ lies over λ.

Suppose that $T \neq G$. As is well known, there exists a unique irreducible character χ' of T lying over ψ such that $\chi = \mathrm{Ind}^G_T(\chi')$. By induction, we may assume that there exists an $\mathcal{O}$-free interior T-algebra A' such that $\kappa A' \cong \kappa T e_{\chi'}$, and an irreducible character η of a defect group P of A' such that η lies over λ, and $d(\chi') = d(\eta)$. Putting $A := \mathrm{Ind}^G_T(A')$, then $\kappa A \cong \mathrm{Ind}^G_T(\kappa A') \cong \kappa G e_\chi$. A Mackey decomposition argument shows that A has defect group P. We have $d(\chi) = d(\eta)$ because $\chi(1) = |G\colon T|\, \chi'(1)$. The assertion is now proved in the case $T \neq G$.

Henceforth, we assume that $T = G$, and again consider the central extension $\hat{G}$ of $\underline{G}$ by $\hat{R}$. The cyclic group $\hat{R}$ is the direct product $\hat{R} = \hat{L}\hat{L}'$ of a p-subgroup $\hat{L}$ and a p'-subgroup $\hat{L}'$. The linear character $\hat{\psi}$ of $\hat{R}$ decomposes as a tensor product $\hat{\psi} = \hat{\lambda} \otimes \hat{\lambda}'$ of linear characters $\hat{\lambda}, \hat{\lambda}'$ of $\hat{L}, \hat{L}'$, respectively. Let $\hat{\chi}$ be the irreducible character of $\hat{G}$ lying over $\hat{\psi}$ such that $\sigma_G(e_{\hat{\chi}} \otimes e) = e_\chi$. Then $\hat{\chi}$ lies over $\hat{\lambda}$. Now, if $R = L O_{p'}(G)$, then $Z(G) < R$, hence $|\hat{G}\colon Z(\hat{G})| \leq |G : R| < |G : Z(G)|$. On the other hand, if $R = O_p(G)$, then $\hat{R} = O_p(\hat{G})$, hence $Z(\hat{G}) = \hat{R}.O_{p'}(\hat{G}))$, and $|\hat{G} : Z(\hat{G})| \leq |\underline{G} : Z(\underline{G}) < |G\ : Z(G)|$. Induction allows us to assume that there exists an $\mathcal{O}$-free interior $\hat{G}$-algebra $\hat{A}$ such that $\kappa \hat{A} \cong \kappa G e_{\hat{\chi}}$, and an irreducible character $\hat{\eta}$ of a defect group $\hat{P}$ of $\hat{A}$ such that $\hat{\eta}$ lies over $\hat{\lambda}$, and $d(\hat{\chi}) = d(\hat{\eta})$.

The $\mathcal{O}$-linear span Θ of $\{\theta(\tilde{g})\colon g \in G\}$ in $\kappa R e$ is an algebra over $\mathcal{O}$ because the values of α are elements of $\mathcal{O}$. For all $g, h \in G$, we have

$$\begin{aligned} h\theta(\tilde{g})h^{-1} &= \underline{\tilde{h}}\theta(\tilde{h})\theta(\tilde{g})\theta(\tilde{h})^{-1}\underline{\tilde{h}}^{-1} \\ &= \alpha(h,h^{-1})^{-1}\alpha(h,g)\alpha(hg,h^{-1})\theta\left({}^{\tilde{h}}g\right), \end{aligned}$$

which is an $\mathcal{O}$-multiple of $\theta({}^{\tilde{h}}g)$. So Θ acquires the structure of a G-algebra by restriction from $\kappa R e$. For all $x \in R$, we have $\theta(\tilde{x}) = xe$, so

$\mathscr{O}Re \subseteq \Theta \subseteq \kappa Re$. We define $A := \hat{A} \otimes_{\mathscr{O}} \Theta$, and write $g1_A = \tilde{g}1_{\hat{A}} \otimes \theta(\tilde{g})$ to specify the way in which we regard A as an interior G-algebra. Then $\kappa A \cong \kappa\hat{A} \otimes_{\kappa} \kappa Re \cong \kappa Ge_{\chi}$ as interior G-algebras.

We have $A^R = \hat{A} \otimes e$. More generally, given $\hat{R} \leq \hat{H} \leq \hat{G}$ and $R \leq H \leq G$ such that $\hat{H}/\hat{R} = H/R$, then $A^H = \hat{A}^{\hat{H}} \otimes e$. For all $a \in \hat{A}^{\hat{H}}$, we have $\mathrm{Tr}_{\hat{H}}^{\hat{G}}(a) \otimes e = \mathrm{Tr}_H^G(a \otimes e)$, so $A_H^G = \hat{A}_{\hat{H}}^{\hat{G}} \otimes e$. In particular, $1_A \in A_H^G$ if and only if $1_{\hat{A}} \in \hat{A}_{\hat{H}}^{\hat{G}}$. We deduce that A has a defect group P satisfying $\hat{P}\hat{R}/\hat{R} = PR/R$.

Suppose that $R = LL'$, where $L' := O_{p'}(G)$. By comments in the first paragraph of the argument, $\hat{L} \leq O_p(Z(\hat{G})) \leq \hat{P}$ and $L \leq P$. So $\hat{P}\hat{L}'/\hat{R} = PL'/R$. Since $\hat{\eta}$ lies over $\hat{\lambda}$, the irreducible character $\hat{\gamma} := \hat{\eta} \otimes \hat{\lambda}'$ of $\hat{P}\hat{L}'$ lies over $\hat{\psi}$. Let γ be the irreducible character of PL' lying over ψ such that $\sigma_{PL'}(e_{\hat{\gamma}} \otimes e) = e_{\gamma}$. Since $\gamma(1)/\hat{\gamma}(1) = \psi(1) = \chi(1)/\hat{\chi}(1)$ and $|G : PL'| = |\hat{G} : \hat{P}\hat{L}'|$, we have $d(\chi) - d(\hat{\chi}) = d(\gamma) - d(\hat{\gamma})$. Also, $\nu_p(|\hat{P}\hat{L}'|) = \nu_p(|\hat{P}|)$ and $\nu_p(\hat{\gamma}(1)) = \nu_p(\hat{\eta}(1))$, so $d(\hat{\gamma}) = d(\hat{\eta}) = d(\hat{\chi})$, hence $d(\chi) = d(\gamma)$. By the lemma, there exists an irreducible character η lying over λ such that $d(\eta) = d(\gamma)$. This completes the argument in the case $R = LL'$.

We may now assume that $R = O_p(G)$. Then $\hat{R}$ is a p-group, hence $\hat{R} = \hat{L} \leq \hat{P}$ and $\hat{\psi} = \hat{\lambda}$. We claim that $R \leq P$. Choosing an element $c \in A^P$ such that $\mathrm{Tr}_P^G(c) = 1_A$, let us write $c = \sum_i a_i \otimes b_i$ with each $a_i \in \hat{A}$, each $b_i \in \Theta$, and the elements a_i linearly independent. For all $g \in G$, we have ${}^g(a_i \otimes b_i) = {}^g a_i \otimes {}^g b_i$. Each a_i is fixed by $R \cap P$, so by the linear independence of the a_i, each b_i is fixed by $R \cap P$. Mackey decomposition gives

$$\sum_{RgP \subseteq G} \sum_i {}^g a_i \otimes \mathrm{Tr}_{R \cap {}^g P}^R({}^g b_i) = \sum_{RgP \subseteq G} \mathrm{Tr}_{R \cap {}^g P}^R({}^g c) = 1_A.$$

Since $\Theta^R = \mathscr{O}e$, at least one of the terms $\mathrm{Tr}_{R \cap {}^g P}^R({}^g b_i)$ must be an $\mathscr{O}^*$-multiple of e. Then $e \in \Theta_{R \cap {}^g P}^R$. For each $h \in G$, the elements h and $\theta(\tilde{h})$ have the same conjugation action on Θ. In particular, $\theta(\tilde{h})J(\mathscr{O}R)e = J(\mathscr{O}R)e\theta(\tilde{h})$, so $J := J(\mathscr{O}R)\Theta$ is an ideal of Θ. The quotient Θ/J inherits, from Θ, the structure of an interior R-algebra over k, and $1_{\Theta/J} \in (\Theta/J)_{R \cap {}^g P}^R$. But each element of R maps to 1_{Θ} modulo J. So the conjugation action of R on Θ/J is trivial. Hence $R \cap {}^g P = R$, and the claim holds. Therefore $\hat{P}/\hat{R} = P/R$.

Let η be the irreducible character of P lying over ψ such that $\sigma_P(e_{\hat{\eta}} \otimes e) = e_{\eta}$. Then η lies over λ. Since $\eta(1) = \hat{\eta}(1)\psi(1)$ and $\chi(1) = \hat{\chi}(1)\psi(1)$, we have

$$d(\eta) - d(\hat{\eta}) = \nu_P(|P|) - \nu_p(|\hat{P}|) + \nu_p(\psi(1)),$$

$$d(\chi) - d(\hat{\chi}) = \nu_p(|G|) - \nu_p(|\hat{G}|) + \nu_p(\psi(1)).$$

But $|G : P| = |\hat{G} : \hat{P}|$ and $d(\hat{\chi}) = d(\hat{\eta})$, hence $d(\chi) = d(\eta)$. ∎

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