

Some Identities and Asymptotics for Characters of the Symmetric Group

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1. INTRODUCTION

This paper is motivated by the problem (arising from quantum gravity [9, 2]) of counting combinatorial types of triangulations Σ of a Riemann surface X with given degrees of vertices. Let us color 2-simplexes of the barycentric subdivision Σ' in black and white, so that adjacent simplexes are of different color, and consider a mapping $\pi: X \rightarrow \mathbb{P}^1$ onto the Riemann sphere \mathbb{P}^1 which send white simplexes onto the northern hemisphere, black simplexes onto the southern hemisphere, and centers k -simplexes of Σ into an equatorial point y_k , $k = 0, 1, 2$. Then $\deg \pi = 3n$, n is the number of 2-simplexes in Σ , and π is unramified outside y_0, y_1, y_2 . It is easy to see that points of X over y_1 and y_2 have ramification index 2 (respectively, 3), while ramification indices of points over y_0 are just degrees of vertices of Σ . So the problem of counting triangulations reduces to the problem of counting ramified coverings $\pi: X \rightarrow Y$ of Riemann surfaces with given ramification indices.

In Section 2 we recall a connection between ramified coverings $\pi: X \rightarrow Y$ of a Riemann surface Y of genus g_Y and irreducible characters χ of the symmetric group S_n , $n = \deg \pi$. The starting point is the following formula, essentially owing to Hurwitz [6] (see also [10, 4, 2]):

$$\sum_{\pi: X \rightarrow Y} \frac{1}{|\text{Aut } \pi|} = \frac{|C_1| |C_2| \cdots |C_k|}{(n!)^{2-2g_Y}} \sum_{\chi} \frac{\chi(g_1) \chi(g_2) \cdots \chi(g_k)}{(\chi(1))^{k-(2-2g_Y)}}, \quad (1.1)$$

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where the first sum extends over all coverings of degree n with fixed conjugacy classes $C_i \subset S_n$ of monodromy $g_i \in C_i$ around ramification points $y_i \in Y$, while the second sum runs over all irreducible characters of the symmetric group S_n . Following Serre we will refer to the first sum in (1.1) as *Eisenstein number of coverings*.

We call the covering $\pi: X \rightarrow Y$ and formula (1.1) *elliptic*, *parabolic*, or *hyperbolic* if the Euler characteristic $2 - 2g_X$ is positive, zero, or negative. By the Riemann–Hurwitz formula the type depends on the sign of the quantity

$$\frac{1}{n}(2 - 2g_X) = \frac{1}{l_1} + \frac{1}{l_2} + \cdots + \frac{1}{l_k} - k + 2 - 2g_Y, \quad (1.2)$$

where l_i is the mean value of cycle length of the monodromy g_i .

In Section 3 we consider *elliptic* and *parabolic* coverings for which the transformations g_i consist of cycles of the same length ($= l_i$).

In the elliptic case such a covering is a disjoint union of factorizations $\mathbb{P}^1 \rightarrow \mathbb{P}^1/G$ by a finite group of Möbius transformations G . For each such group G the first sum in (1.1) may be easily evaluated. For example, the identity

$$\sum_{\chi} \frac{\chi(\sigma_2)\chi(\sigma_3)\chi(\sigma_5)}{\chi(1)} = \frac{(30n)!2^{30n}(20n)!3^{20n}(12n)!5^{12n}}{(60n)!n!(60)^n} \quad (1.3)$$

comes from the icosahedral group (Theorem 3.3). Here $\sigma_d \in S_{60n}$ is a permutation splitting in cycles of length d , and the sum extends over all irreducible characters of S_{60n} .

Under the same condition, *parabolic* formulas (1.1) relate to root systems of rank 2. They give identities like

$$\sum_{\chi \in S_{3n}} \frac{\chi(\sigma_3)^3}{\chi(1)} = \frac{3^{3n}n!^3}{(3n)!} \left(\text{coefficient at } q^n \text{ in } \prod_{k=1}^{\infty} (1 - q^k)^{-1/3} \right), \quad (1.4)$$

which corresponds to root system A_2 (Theorem 3.9). In general under our assumptions parabolic character sums (1.1) in essence coincide with Fourier coefficients of some negative power of Dedekind η function. There is an explicit Rademacher–Zukerman formula [11] for these coefficients, similar to that of for partition function $p(n)$.

Geometrically, these two types of identities correspond to coverings (1.1) of positive or zero Euler characteristic, but in the most interesting case of negative Euler characteristic, almost nothing is known about the corresponding character sums.

Of course in the *hyperbolic* case we have no chance to get an explicit formula similar to (1.3) or (1.4). The main problem we keep in mind is an asymptotic formula for the character sum in (1.1). We have not yet succeeded completely in solving this problem, mainly because we do not know leading terms of the sum. Let us observe that in view of Riemann–Hurwitz formula (1.2), the character sum in (1.1) may be written as

$$\sum_{\chi} \chi(1)^{(2-2g_X)/n} \frac{\chi(g_1)}{\chi(1)^{1/l_1}} \frac{\chi(g_2)}{\chi(1)^{1/l_2}} \cdots \frac{\chi(g_k)}{\chi(1)^{1/l_k}}. \tag{1.5}$$

If genus g_X of Riemann surface X is fixed, while $n \rightarrow \infty$, then the first factor $\chi(1)^{(2-2g_X)/n}$ decreases or increases as a power of n . So in essence the problem reduces to studying for large n the ratio $\chi(g)/(\dim \chi)^{1/l}$, where l is the mean value of cycles length $g \in S_n$. This ratio may be compared with (or opposed to) normalized character $\chi(g)/\dim \chi$ treated in [15, 16, 12].

As a contribution to this problem, we give in Section 4 an asymptotic formula for characters of S_n under some restrictions. More specifically, consider a sequence of diagrams $\beta: b_1 \geq b_2 \geq \cdots \geq b_m$ such that the ratio $b_i/n = \beta_i$ is a constant, and a sequence of permutations $\sigma_\alpha \in S_n$ with constant relative multiplicity $a_k/n = \alpha_k$ of k cycles. The main result of this section is the asymptotic formula for $n \rightarrow \infty$ (Theorem 4.2),

$$\chi_\beta(\sigma_\alpha) \sim \frac{\exp(n\omega)}{(2\pi n)^{(m-1)/2}} \frac{\prod_{i < j} (1 - \exp(\tau_j - \tau_i))}{\sqrt{(1/m) \sum_{i=1}^m H_{ii}(\tau)}}, \tag{1.6}$$

where we suppose $\beta_i \neq \beta_j$ and lengths of all cycles involved in $\sigma_\alpha \in S_n$ are coprime. Here $\omega = \omega(\alpha, \beta)$ is minimum of the function

$$L_{\alpha, \beta}(t) = \sum_k \alpha_k \log(\exp(kt_1) + \exp(kt_2) + \cdots + \exp(kt_m)) - \sum_{i=1}^m \beta_i t_i, \tag{1.7}$$

$t \in \mathbb{R}^m,$

and $H_{ii}(\tau)$ are principle minors of Hessian $\|(\partial^2/\partial t_i \partial t_j)L_{\alpha, \beta}\|$ at the point of minimum $\tau \in \mathbb{R}^m$ (which exists, satisfies $\tau_1 > \tau_2 > \cdots > \tau_m$, and is unique up to transformation $\tau_i \mapsto \tau_i + c$). Note that $\beta \mapsto \omega(\alpha, \beta)$ is just the Legendre transform of the first sum in (1.7).

The asymptotic formula may be extended to diagrams β with multiple rows by making use of an integral, which is a special case of Macdonald conjecture. The resulting formula differs from (1.6) only in pre-exponential factor, which becomes more complicated. To avoid technicalities we con-

sider only the most degenerate case of a rectangular diagram (Proposition 4.4).

The second restriction (lengths of cycles are coprime) is more subtle. Let q be the greatest common divisor of cycle lengths in σ_α . Then by Littlewood theorem $\chi_\beta(\sigma_\alpha) \neq 0$ only if the diagram β (or character χ_β) is divisible by q . In the last case

$$\chi_\beta(\sigma_\alpha) = \pm \text{Ind}^{S_{n/q}} [\chi_{\beta_1} \times \chi_{\beta_2} \times \cdots \times \chi_{\beta_q}](\sigma_{\alpha/q}) := \pm \chi_{\beta_1, \beta_2, \dots, \beta_q}(\sigma_{\alpha/q}),$$

where diagrams $\beta_1, \beta_2, \dots, \beta_q$, $|\beta_1| + |\beta_2| + \cdots + |\beta_q| = n/q$ form so-called q quotients of β , and cycle lengths of $\sigma_{\alpha/q}$ are that of σ_α divided by q (see [7] for details). Our approach may be applied to $\chi_{\beta_1, \beta_2, \dots, \beta_q}(\sigma_{\alpha/q})$ as well. As result we can see that for $q \mid \beta$ the critical exponent of $\chi_\beta(\sigma_\alpha)$ is given by the same formula $\omega = \min L_{\alpha\beta}$.

In conclusion we give a useful estimation of the critical value ω (Theorem 4.6),

$$\frac{H(\beta)}{l} \leq \omega \leq \frac{H(\beta)}{l_{\min}},$$

where l and l_{\min} are mean value and minimal value of cycle lengths in σ_α , and $H(\beta) = -\sum \beta_i \log \beta_i$ is an entropy function. It follows that for large n ,

$$\chi_\beta(1)^{1/l} \leq \chi_\beta(\sigma_\alpha) \leq \chi_\beta(1)^{1/l_{\min}}$$

provided diagram β is not rectangular and not all cycle lengths of σ_α are equal.

2. CONNECTION WITH CHARACTERS

As explained in the Introduction, the problem of counting triangulations is a particular case of counting ramified coverings with prescribed ramification indices [2]. In this section we recall some results on interrelation of the last problem with characters of symmetric groups [10, 4, 2]. This connection follows from two basic facts:

Fact 1. Topologically, coverings $\pi: X \rightarrow Y$ of degree n unramified outside k points $y_i \in Y$ are classified by conjugacy classes of homomorphisms $\pi_1 \rightarrow S_n$ of the fundamental group $\pi_1 = \pi_1(Y \setminus \{y_1, y_2, \dots, y_k\})$, which is known to be defined by the unique relation

$$g_1 g_2 \cdots g_k [f_1, h_1] [f_2, h_2] \cdots [f_g, h_g] = 1,$$

where $g = g_Y$ is the genus of Y and the brackets denote the commutator $[f, h] = fhf^{-1}h^{-1}$. Thus the coverings of Riemann sphere $\pi: X \rightarrow \mathbb{P}^1$ of given degree n and ramification indices are parametrized by solutions of the equation

$$g_1 g_2 \cdots g_k = 1, \quad g_i \in C_i, \tag{2.1}$$

up to conjugacy, where cycle lengths of the conjugacy class $C_i \subset S_n$ are equal to ramification indices of points in fibers $\pi^{-1}(y_i)$. For an arbitrary Riemann surface Y of genus g the number of coverings is equal to the number of solutions up to conjugacy of the equation

$$g_1 g_2 \cdots g_k [f_1, h_1] \cdots [f_g, h_g] = 1, \quad f_i, h_i \in S_n, g_i \in C_i. \tag{2.2}$$

Fact 2. The Burnside theorem gives the number of solutions of the equations (2.1) and (2.2) for an arbitrary group G in terms of irreducible characters [1],

$$\begin{aligned} & \#\{g_1 g_2 \cdots g_k = 1 \mid g_i \in C_i\} \\ &= \frac{|C_1| |C_2| \cdots |C_k|}{|G|} \sum_{\chi} \frac{\chi(g_1) \chi(g_2) \cdots \chi(g_k)}{\chi(1)^{k-2}}, \\ & \#\{g_1 g_2 \cdots g_k [f_1, h_1] \cdots [f_g, h_g] = 1\} \\ &= \frac{|C_1| |C_2| \cdots |C_k|}{|G|^{1-2g}} \sum_{\chi} \frac{\chi(g_1) \chi(g_2) \cdots \chi(g_k)}{\chi(1)^{k+2g-2}}, \end{aligned} \tag{2.3}$$

where the sums extend over all irreducible characters χ of G and $g_i \in C_i$ are elements from fixed conjugacy classes C_i .

Combination of these two results gives the following formula for Eisenstein number of ramified coverings [4, 2].

THEOREM 2.1. *In the previous notation the following formula for Eisenstein number of coverings $\pi: X \rightarrow \mathbb{P}^1$ with prescribed ramification indices holds:*

$$\sum_{\pi: X \rightarrow \mathbb{P}^1} \frac{1}{|\text{Aut } \pi|} = \frac{|C_1| |C_2| \cdots |C_k|}{(n!)^2} \sum_{\chi} \frac{\chi(g_1) \chi(g_2) \cdots \chi(g_k)}{(\chi(1))^{k-2}}. \tag{2.4}$$

Proof. In view of the Burnside formula (2.3) it is sufficient to show that

$$\#\{g_1 g_2 \cdots g_k = 1 \mid g_i \in C_i \subset S_n\} = \sum_{\pi: X \rightarrow \mathbb{P}^1} \frac{n!}{|\text{Aut } \pi|}. \tag{2.5}$$

According to Fact 1, a solution $\{g_1, g_2, \dots, g_k\}$ of the equation in the left-hand side of (2.5) corresponds to a ramified covering $\pi: X \rightarrow \mathbb{P}^1$ and

$$\text{Aut } \pi \cong C(g_1, g_2, \dots, g_k),$$

where $C(g_1, g_2, \dots, g_k)$ is the centralizer of the set $\{g_1, g_2, \dots, g_k\}$ in S_n . Hence the number of solutions conjugate to $\{g_1, g_2, \dots, g_k\}$ is equal to

$$[S_n: C(g_1, g_2, \dots, g_k)] = \frac{n!}{|\text{Aut } \pi|}$$

and (2.5) follows. ■

The same arguments give a similar result for Eisenstein number of coverings of an arbitrary Riemann surface Y .

THEOREM 2.2. *In the previous notation,*

$$\sum_{\pi: X \rightarrow Y} \frac{1}{|\text{Aut } \pi|} = \frac{|C_1| |C_2| \cdots |C_k|}{(n!)^{2-2g_Y}} \sum_{\chi} \frac{\chi(g_1) \chi(g_2) \cdots \chi(g_k)}{(\chi(1))^{k-(2-2g_Y)}}. \quad (2.6)$$

Theorems 2.1 and 2.2 may be used in both directions, i.e., information on coverings may be transformed into information on characters and vice versa. When the structure of the covering is known, it is easier to carry information on coverings to characters. In the next we give some examples.

3. EXPLICIT FORMULAE

There exist several remarkable cases in which the number of coverings can be evaluated explicitly. In essence these are the elliptic and parabolic coverings $\pi: X \rightarrow Y$ with equal ramification indices in each fiber $\pi^{-1}(y)$. Let us consider such a covering $\pi: X \rightarrow \mathbb{P}^1$ of Riemann sphere unramified outside y_1, y_2, \dots, y_k with ramification index m_i in points over y_i . To be elliptic or hyperbolic it should satisfy the inequality (cf. (1.2))

$$\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k} \geq k - 2.$$

It turns out that all such coverings may be explicitly described in terms of finite groups of Möbius transformations or affine Coxeter groups. Since the structure of these groups is known, explicit formulae for (2.4) can be obtained.

3.1. Elliptic Case

In this case we have strict inequality,

$$\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k} > k - 2, \tag{3.1}$$

which admits the following values of m_i :

Cyclic: $k = 2, m_1 = m_2 = m$;

Dihedral: $k = 3, m_1 = m_2 = 2, m_3 = m$;

Tetrahedral: $k = 3, m_1 = 2, m_2 = m_3 = 3$;

Octahedral: $k = 3, m_1 = 2, m_2 = 3, m_3 = 4$;

Icosahedral: $k = 3, m_1 = 2, m_2 = 3, m_3 = 5$.

Remark 3.1. The fundamental group of a sphere with two punctures is \mathbb{Z} . Hence in the case $k = 2$ there should be $m_1 = m_2$, and we may disregard other solutions of (3.1). To each of these cases corresponds a unique covering that may be described in terms of a finite group of Möbius transformations we use to label solutions of (3.1).

PROPOSITION 3.2. *Let $\pi: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a covering unramified outside y_1, \dots, y_k with ramification index m_i in each point over y_i . Then π is isomorphic to factorization $\pi_G: \mathbb{P}^1 \rightarrow \mathbb{P}^1/G \cong \mathbb{P}^1$ by a finite group G of Möbius transformations.*

Proof. Let m_1, m_2, \dots, m_k be one of the above solutions of (3.1). Then the group $G = G(m_1, m_2, \dots, m_k)$ defined by relations

$$g_1^{m_1} = g_2^{m_2} = \cdots = g_k^{m_k} = g_1 g_2 \cdots g_k = 1$$

is finite and isomorphic to the rotation group of the polyhedron we use for labeling the solution. Now observe that the only transitive permutation representation of G for which the element g_i splits into cycles of length m_i is regular. Hence the covering with ramification indices m_i is unique. It may be constructed from realization of G as a group of Möbius transformations. ■

In combination with Theorem 2.1 this implies

THEOREM 3.3. *The following identities hold:*

Cyclic:
$$\sum_{\chi} \chi(\sigma_m)^2 = \left(\frac{n}{m}\right)! m^{n/m};$$

$$\text{Dihedral: } \sum_{\chi} \frac{\chi(\sigma_2)^2 \chi(\sigma_m)}{\chi(1)} = \frac{[(n/2)!2^{n/2}]^2 (n/m)!m^{n/m}}{n!(2m)^{n/2m} (n/2m)!};$$

Tetrahedral:

$$\sum_{\chi} \frac{\chi(\sigma_3)^2 \chi(\sigma_2)}{\chi(1)} = \frac{[(n/3)!3^{n/3}]^2 (n/2)!2^{n/2}}{n!(12)^{n/12} (n/12)!};$$

Octahedral:

$$\sum_{\chi} \frac{\chi(\sigma_2)\chi(\sigma_3)\chi(\sigma_4)}{\chi(1)} = \frac{(n/2)!2^{n/2}(n/3)!3^{n/3}(n/4)!4^{n/4}}{n!(24)^{n/24}(n/24)!};$$

Icosahedral:

$$\sum_{\chi} \frac{\chi(\sigma_2)\chi(\sigma_3)\chi(\sigma_5)}{\chi(1)} = \frac{(n/2)!2^{n/2}(n/3)!3^{n/3}(n/5)!5^{n/5}}{n!(60)^{n/60}(n/60)!},$$

where the summations are taken over all irreducible characters χ of S_n and σ_m denotes a permutation consisting of n/m cycles of length m .

Proof. Let as before m_1, m_2, \dots, m_k be one of the above solutions of (3.1), let G be the corresponding group of Möbius transformations, and let $\pi: X \rightarrow \mathbb{P}^1$ be a covering with ramification indices m_i . Then each component X_i of X should be elliptic, i.e., isomorphic to \mathbb{P}^1 , and by the previous proposition the restriction $\pi_i: X_i \rightarrow \mathbb{P}^1$ is isomorphic to the factorization $\mathbb{P}^1 \rightarrow \mathbb{P}^1/G$. So for each n such that $|G|$ divides n there exists only one such covering π and $\text{Aut } \pi$ is a wreath product of G and symmetric group S_m , $m = n/|G|$, of permutation of the components of X . As result the left-hand side of the formula (2.4) contains only one term,

$$\frac{1}{m!|G|^m}, \quad m = \frac{n}{|G|},$$

and the theorem follows. ■

3.2. Parabolic Case

In this case we have

$$\sum_{i=1}^k \frac{1}{m_i} = k - 2 \tag{3.2}$$

with the solutions

$$A_1 \times A_1: \quad k = 4, \quad m_1 = m_2 = m_3 = m_4 = 2;$$

$$B_2: \quad k = 3, \quad m_1 = 2, \quad m_2 = m_3 = 4;$$

$$G_2: \quad k = 3, \quad m_1 = 2, \quad m_2 = 3, \quad m_3 = 6;$$

$$A_2: \quad k = 3, \quad m_1 = m_2 = m_3 = 3.$$

Let as before $G = G(m_1, m_2, \dots, m_k)$ be a group defined by the relations

$$g_1^{m_1} = g_2^{m_2} = \dots = g_k^{m_k} = g_1 g_2 \dots g_k = 1,$$

where m_1, m_2, \dots, m_k is one of the solutions of (3.2). It turns out that the group G has a remarkable geometrical interpretation [3] as the group of even elements in affine Coxeter group $\Gamma = \Gamma(m_1, m_2, \dots, m_k)$ generated by reflections in the sides of a plane k -gon with angles π/m_i . In other words, Γ is an affine Weyl group of the root system used for labeling solutions of (3.2). In this interpretation the generators g_i are just rotations by angle $2\pi/m_i$ around vertices of the k -gon. Let $T \subset G$ be the (normal) subgroup of translations. T is the unique maximal torsion-free subgroup of G and index $\mu = [G:T]$ is finite.

PROPOSITION 3.4. *Connected parabolic coverings $\pi: X \rightarrow \mathbb{P}^1$ with constant ramification indices m_i in singular fibers $\pi^{-1}(y_i)$, $i = 1, 2, \dots, k$, have degree $n = m\mu$ divisible by $\mu = [G:T]$. Such coverings are parametrized by conjugacy classes of translation subgroups $H \subset T$ of index $m = [T:H]$.*

Proof. There exists a one-to-one correspondence between connected coverings $\pi: X \rightarrow \mathbb{P}^1$ of degree n and conjugacy classes of subgroups in $\pi_1(\mathbb{P}^1 \setminus \{y_1, \dots, y_k\})$ of index n . If the monodromy around y_i has order m_i , then we can deal with subgroups $H \subset G$ of G instead of π_1 . We claim that the generators g_i split into cycles of the same length m_i in G/H if and only if $H \subset T$, i.e., H consists of translations. Really

$$g_i \text{ splits into cycles of length } m_i \Leftrightarrow g_i^k g H \neq g H, \quad k \neq 0(m_i), \forall g \in H$$

$$\Leftrightarrow g^{-1} g_i^k g \notin H,$$

i.e., H contains no elements conjugate to g_i^k . But from the above geometrical interpretation of G we know that all elements of finite order are conjugate to some g_i^k . Hence H is torsion-free and the result follows. ■

PROPOSITION 3.5. *In the above notation the Eisenstein number of parabolic connected coverings $\pi: X \rightarrow \mathbb{P}^1$ of degree $n\mu$ is equal to*

$$\sum_{\pi: X \rightarrow \mathbb{P}^1} \frac{1}{|\text{Aut } \pi|} = \frac{1}{n\mu} \sum_{d|n} d,$$

where $\mu = [G:T]$, T is the translation subgroup of $G = G(m_1, \dots, m_k)$.

Proof. We will need the following information:

(i) If G acts transitively on a set Y , then

$$\text{Aut}_G(Y) \cong N_G(G_y)/G_y,$$

where $N_G(G_y)$ is the normalizer of stabilizer G_y of a point $y \in Y$.

(ii) Let $H \subset T$ be a sublattice of index n in a lattice T of rank 2. Then

$$\#\{H \subset T \mid [T:H] = n\} = \sum_{d|n} d$$

(see [14]).

Using (i) and Proposition 3.4 we can write the Eisenstein number of connected parabolic coverings as

$$\sum_{\pi: X \rightarrow \mathbb{P}^1} \frac{1}{|\text{Aut } \pi|} = \sum_{H \subset G} \frac{1}{[N_G(H):H]} = \frac{1}{n\mu} \sum_{H \subset G} [G:N_G(H)], \quad (3.3)$$

where summation on the right-hand side extends over conjugacy classes of torsion-free subgroups $H \subset G$ of index $n\mu$. Since such a subgroup H is contained in the translation group T , which is a lattice of rank 2, we can use (ii) and end the proof as

$$\frac{1}{n\mu} \sum_{H \subset G} [G:N_G(H)] = \frac{1}{n\mu} \#\{H \subset T \mid [T:H] = n\} = \frac{1}{n\mu} \sum_{d|n} d.$$

■

So we have a simple formula for the Eisenstein number of *connected* parabolic coverings. The number of all coverings may be easily deduced from here (cf. [4]).

LEMMA 3.6. *For coverings $\pi: X \rightarrow Y$ with given constant ramification indices m_i in singular fibers $\pi^{-1}(y_i)$, the identity*

$$\sum_{\pi} \frac{q^{\deg \pi}}{|\text{Aut } \pi|} = \exp \left(\sum_{\text{connected } \pi} \frac{q^{\deg \pi}}{|\text{Aut } \pi|} \right), \quad (3.4)$$

holds, where the first sum runs over all coverings $\pi: X \rightarrow Y$ (connected or not), while the second sum extends only over coverings with connected surface x .

Proof. Let $X = \sqcup_i d_i X_i$ be a disjoint union of pairwise nonisomorphic connected components X_i of multiplicity d_i and let $\pi_i = \pi|_{X_i}$. Then

$|\text{Aut } \pi| = \prod_i |\text{Aut } \pi_i|^{d_i} (d_i)!$ and $\text{deg } \pi = \sum_i d_i \text{deg } \pi_i$. Hence the left-hand side of (3.4) may be written as

$$\sum_{\pi} \prod_i \frac{q^{d_i \text{deg } \pi_i}}{|\text{Aut } \pi_i|^{d_i} d_i!} = \prod_{\text{connected } \pi} \sum_{d \geq 0} \frac{q^d \text{deg } \pi}{|\text{Aut } \pi|^d d!} = \exp \left(\sum_{\text{connected } \pi} \frac{q^{\text{deg } \pi}}{|\text{Aut } \pi|} \right).$$

■

Combining this lemma with the previous proposition, we get the following formula.

COROLLARY 3.7. *The Eisenstein number of elliptic coverings $\pi: X \rightarrow \mathbb{P}^1$ of degree n with given constant ramification indices m_i in singular fibers $\pi^{-1}(y_i)$ is equal to*

$$\sum_{\pi: X \rightarrow \mathbb{P}^1} \frac{1}{|\text{Aut } \pi|} = \text{coefficient at } q^n \text{ in } \prod_{k=1}^{\infty} (1 - q^{k\mu})^{-1/\mu},$$

where $\mu = [G:T]$ and T is the translation subgroup of $G = G(m_1, m_2, \dots, m_k)$.

Proof. Let $N = n\mu, n \geq 1$. Then

$$\begin{aligned} \sum_{\text{connected } \pi} \frac{q^{\text{deg } \pi}}{|\text{Aut } \pi|} &= \sum_N q^N \sum_{\text{deg } \pi = N} \frac{1}{|\text{Aut } \pi|} = \sum_N q^N \frac{1}{N} \sum_{d|N} d \\ &= \frac{1}{\mu} \sum_{m \geq 1} \sum_{d \geq 1} \frac{q^{\mu md}}{d} = -\frac{1}{\mu} \sum_{m \geq 1} \log(1 - q^{\mu m}). \end{aligned} \tag{3.5}$$

In the second equality we use Proposition 3.5. The corollary follows by taking exponents and applying Lemma 3.6. ■

Here are values of the index $\mu = [G:T]$ for each solution of (3.2):

$A_1 \times A_1:$	$\mu = 2,$
$B_2:$	$\mu = 4,$
$G_2:$	$\mu = 6,$
$A_2:$	$\mu = 3,$

Remark 3.8. The right-hand side of (3.5) in essential is a negative power of Dedekind η function, so we may apply the Rademacher analytic formula [11] for its Fourier coefficients. As a simple corollary we get the

following asymptotics for nonzero coefficients of the series $\prod_{k=1}^{\infty} (1 - q^{k\mu})^{-1/\mu}$:

$$a_n \sim \sqrt{12\mu} \frac{\exp(\pi/6\mu\sqrt{24n})}{(24n)^{3/4+1/4\mu}}, \quad n \equiv 0 \pmod{\mu}. \quad (3.6)$$

Combining Corollary 3.7 with Theorem 2.2, we get a number of “strange” identities.

THEOREM 3.9. *The following identities hold:*

$A_1 \times A_1$:

$$\sum_{\chi \in S_{2n}} \frac{\chi(\sigma_2)^4}{\chi(1)^2} = \frac{2^{4n}(n!)^4}{((2n)!)^2} \left(\text{coeff. at } q^n \text{ in } \prod_{k=1}^{\infty} (1 - q^k)^{-1/2} \right),$$

B_2 :

$$\sum_{\chi \in S_{4n}} \frac{\chi(\sigma_2)\chi(\sigma_4)^2}{\chi(1)} = \frac{2^{6n}(2n)!(n!)^2}{(4n)!} \left(\text{coeff. at } q^n \text{ in } \prod_{k=1}^{\infty} (1 - q^k)^{-1/4} \right),$$

G_2 :

$$\sum_{\chi \in S_{6n}} \frac{\chi(\sigma_2)\chi(\sigma_3)\chi(\sigma_6)}{\chi(1)} = \frac{2^{4n}3^{3n}(3n)!(2n)!n!}{(6n)!} \times \left(\text{coeff. at } q^n \text{ in } \prod_{k=1}^{\infty} (1 - q^k)^{-1/6} \right),$$

$$A_2: \quad \sum_{\chi \in S_{3n}} \frac{\chi(\sigma_3)^3}{\chi(1)} = \frac{3^{3n}(n!)^3}{(3n)!} \left(\text{coeff. at } q^n \text{ in } \prod_{k=1}^{\infty} (1 - q^k)^{-1/3} \right),$$

where the permutation σ_m splits into cycles of length m .

Remark 3.10. It is known that $\chi_{\lambda}(\sigma_m) = 0$ except the Young diagram λ is divisible by m . In the last case the value of character is given by the hook formula [18]

$$\chi_{\lambda}(\sigma_m) = \pm \frac{(n/m)!}{\prod_{m\text{-hooks}} h_{ij}/m}, \quad (3.7)$$

where h_{ij} are hook lengths of λ . The first identity just means that

$$\sum_{\text{domino diagrams}} \left[\frac{\text{product of odd hooks}}{\text{product of even hooks}} \right]^2 = \text{coeff. at } q^n \text{ in } \prod_k (1 - q^k)^{-1/2},$$

where the sum is extended over all *even* diagrams of order $2n$ (i.e., diagrams which may be tiled by n dominos).

Let us observe that in view of estimation (3.6) the character sums of the theorem are of *subexponential* growth in n , while the dimension $\chi(1)$ generically is *superexponential*. Hence the first identity implies that $\chi(1)^{1/2}$ in general is a good estimation for $\chi(\sigma_2)$. We may suppose that a similar result is valid for $\chi(\sigma_m)$, as suggested by formula (3.7), and moreover

$$|\chi(g)| < \chi(1)^{1/d+\varepsilon}$$

for “general” character χ and $n \rightarrow \infty$. Here d is the mean value of cycle length of $g \in S_n$. The characters we consider in the next section have exponential growth of dimension and thus are not general.

4. ASYMPTOTIC FORMULAE

Let $\pi: X \rightarrow Y$ be a ramified covering of degree n , of surface Y of genus g_Y by surface X of genus g_X , ramified over k points y_1, \dots, y_k in Y . The Riemann–Hurwitz formula connecting the genus of X and Y may be written in the form

$$2g_X - 2 = n \left[2g_Y - 2 + k - \frac{1}{l_1} - \frac{1}{l_2} \dots - \frac{1}{l_k} \right], \tag{4.1}$$

where l_i is the mean value of ramification indices in the fiber $\pi^{-1}(y_i)$ (equal to the mean value of the cycle length of the monodromy g_i around y_i). Hence, Theorem 2.2 may be written as

$$\begin{aligned} & \sum_{\pi: X \rightarrow Y} \frac{1}{|\text{Aut } \pi|} \\ &= \frac{|C_1| \dots |C_k|}{(n!)^{2-2g_Y}} \sum_{\chi} \chi(1)^{(2-2g_X)/n} \frac{\chi(g_1)}{\chi(1)^{1/l_1}} \frac{\chi(g_2)}{\chi(1)^{1/l_2}} \dots \frac{\chi(g_k)}{\chi(1)^{1/l_k}}. \end{aligned}$$

If the genus g_X is fixed, then the first factor $\chi(1)^{(2-g_X)/n}$ decreases or increases at most as some power of n . So the main problem is in

estimation of the quotient $\chi(g)/\chi(1)^{1/l}$, where l is the mean value of cycle length of $g \in S_n$, as $n \rightarrow \infty$.

Let us denote by

$$\beta: \quad b_1 \geq b_2 \geq \dots \geq b_m \quad (4.2)$$

the Young diagram of $\chi = \chi_\beta$ and by $\sigma_\alpha \in S_n$ an element with cycle structure

$$\alpha: \quad 1^{a_1} 2^{a_2} \dots s^{a_s}. \quad (4.3)$$

In subsequent text, we restrict ourselves to sequences of characters χ_β and elements $\sigma_\alpha \in S_n$ subject to the following conditions:

Condition 1. Diagrams β have a fixed number of rows and permutations σ_α contain only cycles of bounded length.

Condition 2. Lengths of rows in β increase linearly with n , i.e., $b_i = \beta_i n$.

Condition 3. Multiplicities of cycles in $\sigma_\alpha \in S_n$ increase linearly with n , i.e., $a_k = \alpha_k n$.

Under these conditions we will find asymptotics for $\chi_\beta(\sigma_\alpha)$ and $\chi_\beta(\sigma_\alpha)/\chi_\beta(1)^{1/d}$. It follows that the last quotient exponentially increases for $n \rightarrow \infty$, provided not all rows of β are equal and not all cycles of σ_α are of the same length.

4.1. Frobenius Formula

Let z_1, z_2, \dots, z_m be independent variables and let

$$\begin{aligned} \delta &= (m-1, m-2, \dots, 0), \\ z^{\beta+\delta} &= z_1^{b_1+m-1} z_2^{b_2+m-2} \dots z_m^{b_m}, \\ s_j &= z_1^j + \dots + z_m^j, \\ \Delta &= \Delta(z_1, \dots, z_m) = \prod_{i < j} (z_i - z_j). \end{aligned}$$

In this notation we have the following Frobenius formula for irreducible characters:

$$\chi_\beta(\sigma_\alpha) = \text{coefficient at } z^{\beta+\delta} \text{ in } \Delta \prod_j s_j^{a_j}. \quad (4.4)$$

Making use of the residue theorem we can rewrite it in integral form,

$$\chi_\beta(g) = \frac{1}{(2\pi i)^m} \int_{|z_1|=1} \cdots \int_{|z_m|=1} g(z) e^{n \cdot \omega(z)} dz, \quad (4.5)$$

where

$$z = (z_1, \dots, z_m) \in \mathbb{C}^m$$

$$g(z) = \frac{\Delta(z_1, \dots, z_m)}{z_1^m z_2^{m-1} \cdots z_m} = \prod_{i < j} \left(1 - \frac{z_j}{z_i}\right), \quad (4.6)$$

$$\omega(z) = \sum_{k=1}^n \alpha_k \log s_k - \sum_{i=1}^m \beta_i \log z_i.$$

4.2. Asymptotics of Characters

According to general principles [5], in order to find asymptotics of the integral (4.5), we have to deform the surface of integration S in such a way that it passes through a critical point z_0 of $\omega(z)$ with maximal value $\text{Re } \omega$. The critical points of $\omega(z)$ are given by the equation $d\omega(z) = 0$, which in coordinates looks like the system of nonlinear algebraic equations

$$\sum_k k \alpha_k \frac{z_i^k}{z_1^k + \cdots + z_m^k} = \beta_i, \quad i = 1, \dots, m. \quad (4.7)$$

The system (4.7) is homogeneous in z and thus by the Bezout theorem has a lot of complex solutions. The first problem is to understand which of them is responsible for the asymptotics. For this the following result is crucial.

THEOREM 4.1. *For $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_m \geq 0$ the system (4.7) has, up to proportionality, unique positive real solution $x = (x_1, x_2, \dots, x_m)$ and for this solution, $x_1 \geq x_2 \geq \cdots \geq x_m \geq 0$.*

Proof. Let us put

$$z_i = e^{t_i}, \quad t_i \in \mathbb{R},$$

so that in variable t_i ,

$$\omega(e^{t_i}) = \sum_{k=1}^n \alpha_k \log(\exp(kt_1) + \cdots + \exp(kt_m)) - \sum_{i=1}^m \beta_i t_i$$

and the equation for critical points of w takes the form

$$\sum_k k \alpha_k \frac{\exp(kt_i)}{\exp(kt_1) + \cdots + \exp(kt_m)} = \beta_i, \quad i = 1, 2, \dots, m. \quad (4.8)$$

The proof may be divided into three steps.

Step 1. ω is a convex function of $t = (t_1, \dots, t_m)$, i.e.

$$\text{Hess}(\omega) = \sum_{i,j} \frac{\partial^2 \omega}{\partial t_i \partial t_j} X_j X_i \geq 0, \quad \forall X_i, X_j \in \mathbb{R}.$$

Proof. Really, since

$$\text{Hess}(\omega) = \sum_{k=1}^n \alpha_k \sum_{i,j} \frac{\partial^2}{\partial t_i \partial t_j} \log(\exp(kt_1) + \cdots + \exp(kt_m)) X_j X_i,$$

it suffices to show that

$$h(t) = \sum_{i,j} \frac{\partial^2}{\partial t_i \partial t_j} \log(\exp(t_1) + \cdots + \exp(t_m)) X_j X_i \geq 0. \quad (4.9)$$

A simple calculation gives

$$h(t) = \frac{\sum_i \exp(t_i) X_i^2}{\sum_i \exp(t_i)} - \left(\frac{\sum_i \exp(t_i) X_i}{\sum_i \exp(t_i)} \right)^2 \geq 0 \quad (4.10)$$

and equality is possible only for $X_1 = X_2 = \cdots = X_m$. ■

From Step 1 it follows that

Step 2. Restriction of $\text{Hess}(\omega)$ on the hyperplane $\sum_i X_i = 0$ is positive and hence the mapping

$$\Omega: (t_1, \dots, t_m) \rightarrow \left(\frac{\partial f}{\partial t_1}, \dots, \frac{\partial f}{\partial t_m} \right) = (\beta_1, \dots, \beta_m),$$

$$f(t) = \sum_k \alpha_k \log(\exp(kt_1) + \cdots + \exp(kt_m)),$$

is locally invertible on the hyperplane $\sum t_i = 0$.

Step 3. The mapping

$$\Omega: (\log x_1, \log x_2, \dots, \log x_m) \rightarrow (\beta_1, \beta_2, \dots, \beta_m),$$

$$x_1 + x_2 + \dots + x_m = 1, \quad x_i \geq 0,$$

gives a homeomorphism between simplexes

$$\Delta_x = \{x \mid x_1 + x_2 + \dots + x_m = 1, x_i \geq 0\},$$

$$\Delta_\beta = \{\beta \mid \beta_1 + \beta_2 + \dots + \beta_m = 1, \beta_i \geq 0\}.$$

Proof. We will proceed by induction on m . Without loss of generality we may suppose restriction of Ω on a face of the simplex Δ_x ,

$$\Omega_j: (\log x_1, \dots, \log x_{i-1}, 0, \log x_{i+1}, \dots, \log x_m)$$

$$\rightarrow (\beta_1, \dots, \beta_{i-1}, 0, \beta_{i+1}, \dots, \beta_m),$$

to be a homeomorphism. Then Ω induces a homeomorphism of the boundaries $\partial\Omega: \partial\Delta_x \rightarrow \partial\Delta_\beta$, and by Brouwer theorem the mapping Ω is surjective. Combining this with Step 2 we find out that Ω is an unramified covering of Δ_β . Since the simplex Δ_β is simply connected, Ω is in fact a homeomorphism. ■

The theorem follows from the Step 3, because $\Omega(t_1, \dots, t_m)$ is just the left-hand side of (4.8). ■

It turns out that the positive critical point x from Theorem 4.1 is responsible for asymptotics of characters $\chi_\beta(\sigma_\alpha)$.

THEOREM 4.2. *Suppose in addition to Conditions 1–3 that rows of Young diagrams β have distinct lengths $\beta_i \neq \beta_j$ and lengths of all cycles involved in $\sigma_\alpha \in S_n$ are coprime. Then, as $n \rightarrow \infty$ the characters $\chi_\beta(\sigma_\alpha)$ have the asymptotics*

$$\chi_\beta(\sigma_\alpha) \sim \frac{e^{nw(x)\sqrt{m}}}{(2\pi n)^{(m-1)/2}} \prod_{i < j} \left(1 - \frac{x_j}{x_i}\right) \frac{1}{\sqrt{\sum_{i=1}^m H_{ii}}},$$

where x is a positive critical point from Theorem 4.1,

$$\omega(x) = \sum_{k=1}^n \alpha_k \log(x_1^k + x_2^k + \dots + x_m^k) - \sum_{i=1}^m \beta_i \log x_i, \quad (4.11)$$

and H_{ii} is i th principle minor of the form

$$\text{Hess}(\omega) = \sum_{k=1}^n k^2 \alpha_k \left(\frac{\sum_{i=1}^m x_i^k dt_i^2}{\sum_i x_i^k} - \left(\frac{\sum_{i=1}^m x_i^k dt_i}{\sum_{i=1}^m x_i^k} \right)^2 \right). \quad (4.12)$$

Proof. Step 2 below is crucial. It ensures that only critical points proportional to the positive one are essential for asymptotics. The rest is an exercise in the multidimensional saddle point method [5], with a minor complication due to nonisolated critical points.

Step 1. Deformation of the surface of integration. By deformation of the contour in integral (4.5) we can write

$$\chi_\beta(g) = \frac{1}{(2\pi i)^m} \int_{|z_1|=x_1} \cdots \int_{|z_m|=x_m} g(z) e^{n\omega(z)} dz. \quad (4.13)$$

The surface of integration now passes through the positive critical point x .

Step 2. The asymptotics of integral (4.13) depends on an arbitrary small neighborhood of the set of critical points $z = \lambda x$, $|\lambda| = 1$ proportional to the positive one x . It suffices to show that the maximum of $\operatorname{Re}(\omega(z))$ on the contour $|z_i| = \exp(\tau_i)$ is attained only at λx . In order to see this, write

$$\operatorname{Re}(\omega(z)) = \sum_k \alpha_k \log |z_1^k + z_2^k + \cdots + z_m^k| - \sum_i \beta_i \log |z_i|. \quad (4.14)$$

Evidently,

$$|z_1^k + z_2^k + \cdots + z_m^k| \leq |z_1|^k + |z_2|^k + \cdots + |z_m|^k = x_1^k + x_2^k + \cdots + x_m^k$$

and equality is possible only for collinear z_i^k . In the last case,

$$z_i = \lambda \varepsilon_i x_i, \quad \varepsilon_i^k = 1.$$

This should be valid for all k that enter (4.14) with nonzero coefficient $\alpha_k \neq 0$, i.e., for all cycle lengths of $\sigma_\alpha \in S_n$ which are supposed to be coprime. As a result, we get $\varepsilon_i = 1$, and hence any critical point z with maximal value of $\operatorname{Re} \omega(z)$ should be proportional to x .

Step 3. The final formula. Let us write the positive critical point in the form

$$(x_1, x_2, \dots, x_m) = (\exp(\tau_1), \exp(\tau_2), \dots, \exp(\tau_m))$$

and put in integral (4.13),

$$z_j = \exp(\tau_j + it_j), \quad -\pi \leq t_j \leq \pi.$$

Then

$$\chi_\beta(\sigma_\alpha) = \frac{1}{(2\pi)^m} \int_{-\pi}^\pi \cdots \int_{-\pi}^\pi \prod_{i < j} \left(1 - \frac{\exp(\tau_j + it_j)}{\exp(\tau_i + it_i)} \right) \exp(n\omega(t)) dt,$$

where $\omega(t)$ denotes $\omega(e^{\tau+it})$. Let

$$F(t) = \prod_{i < j} \left(1 - \frac{\exp(\tau_j + it_j)}{\exp(\tau_i + it_i)} \right) \exp(n\omega(t)).$$

Observe that

$$F(t_1, \dots, t_m) = F(t_1 + a, \dots, t_m + a), \quad \forall a \in \mathbb{R},$$

so that $F(t)$ is a constant on lines parallel to the main diagonal $t_1 = t_2 = \cdots = t_m$ of the m -dimensional cube $-\pi < t_i < \pi$. According to Step 2 the asymptotics of the integral depends only on an arbitrary small neighborhood of the main diagonal. Hence asymptotically

$$\chi_\beta(\sigma_\alpha) \sim \frac{2\pi\sqrt{m}}{(2\pi)^m} \int_{H_\varepsilon} F(t) dt,$$

where integration is over a neighborhood of origin in the hyperplane $H: \sum_i t_i = 0$ orthogonal to the diagonal, and the extra multiplier $2\pi\sqrt{m}$ is equal to the length of the diagonal. The asymptotics of the last integral were determined by the isolated critical point $t = 0$, so by the saddle point method we get

$$\begin{aligned} \chi_\beta(\sigma_\alpha) &\sim \frac{e^{n\omega(x)\sqrt{m}}}{(2\pi n)^{(m-1)/2}} \prod_{i < j} \left(1 - \frac{x_j}{x_i} \right) \frac{1}{\sqrt{\det \text{Hess}(\omega)|_H(x)}} \\ &= \frac{e^{n\omega(x)\sqrt{m}}}{(2\pi n)^{(m-1)/2}} \prod_{i < j} \left(1 - \frac{x_j}{x_i} \right) \frac{1}{\sqrt{\sum_{i=1}^m H_{ii}}}, \end{aligned}$$

where H_{ii} are the principal minors of the Hessian $\text{Hess}(\omega)$. ■

EXAMPLE 4.3. As a simple illustration of Theorem 4.2, let us deduce the asymptotic formula for dimension,

$$\chi_\beta(1) \sim \frac{e^{nH(\beta)} \prod_{i < j} (1 - \beta_j/\beta_i)}{(2\pi n)^{(m-1)/2} \sqrt{\beta_1 \beta_2 \cdots \beta_m}},$$

where $H(\beta) = -\sum_i \beta_i \log \beta_i$ is the entropy function.

Proof. Really, for $\sigma_\alpha = 1$ the critical point has coordinates $x_i = \beta_i$, and hence the critical exponent is

$$\omega(x) = \log(\beta_1 + \beta_2 + \dots + \beta_m) - \sum_{i=1}^m \beta_i \log \beta_i = H(\beta),$$

since $\sum_{i=1}^m \beta_i = 1$. The Hessian in the case under consideration,

$$\text{Hess}(\omega) = \sum_i \beta_i dt_i^2 - \sum_{i,j} \beta_i \beta_j dt_i dt_j,$$

has principal minors $H_{ii} = \beta_1 \beta_2 \dots \beta_m$ and the result follows. ■

4.3. Multiple Rows

The saddle point method used in the previous section was based on comparing integral $\int g(x)e^{n\omega(x)} dx$ near a critical point with the Gaussian

$$\int_{\mathbb{R}^n} \exp(-\frac{1}{2}x^2) dx = (2\pi)^{n/2}.$$

This works only if the pre-exponential factor $g(x)$ does not vanish at the critical point. Otherwise the asymptotics heavily depends on the structure of zeros of $g(x)$ near the critical point, and we need another “reference” integral instead of the Gaussian one. This is the difficulty that appears for diagrams with multiple rows. In this section we show how the difficulty may be overridden by using as reference the integral

$$\int_{\mathbb{R}^n} |\Delta(x)|^2 \exp(-\frac{1}{2}x^2) dx = (2\pi)^{n/2} \prod_{k=1}^n k!, \tag{4.15}$$

where as usual $\Delta(x) = \prod_{i < j} (x_i - x_j)$. This integral is a special case of the Macdonald conjecture for the root system A_n , which in turn follows from the Selberg integral [13] (see [8] for details). To avoid technicalities we consider only the most degenerate case of rectangular diagrams.

PROPOSITION 4.4. *Assume all the notation and conditions of Theorem 4.2, but suppose the diagram $\beta = \square$ is rectangular of m rows. Then, as $n \rightarrow \infty$, character $\chi_{\square}(\sigma_\alpha)$ has the asymptotics*

$$\chi_{\square}(\sigma_\alpha) \sim \frac{m^{\#\{\text{cycles}\} + 1/2}}{(2\pi)^{(m-1)/2}} \left[\frac{m}{d_2 \cdot \#\{\text{cycles}\}} \right]^{(m^2-1)/2} \prod_{k=1}^{m-1} k!,$$

where d_2 is the mean square of cycle lengths in σ_α .

Proof. For the rectangular diagram the positive critical point is $x_1 = x_2 = \dots = x_n = 1$, which by (4.11) corresponds to the critical value

$$\omega(x) = \sum_k \alpha_k \log m = \frac{1}{d} \log m,$$

where d is the mean value of cycle lengths. We start with integral representation (4.15) from the proof of Theorem 4.2, which for the above critical value gives

$$\begin{aligned} \chi_\beta(\sigma_\alpha) &\sim \frac{\sqrt{m}}{(2\pi)^{m-1}} \int_{H_\varepsilon} \frac{\Delta(z_1, z_2, \dots, z_m)}{z_1^{m-1} z_2^{m-2} \dots z_{m-1}} \exp(n\omega(z)) dt \\ &\sim \frac{\sqrt{m} \cdot m^{n/d}}{(2\pi)^{m-1}} \int_H \frac{\Delta(z_1, z_2, \dots, z_m)}{z_1^{m-1} z_2^{m-2} \dots z_{m-1}} \exp\left(-\frac{n}{2} \mathcal{H}(t)\right) dt, \end{aligned} \quad (4.16)$$

where the first integral is taken over an arbitrary small neighborhood H_ε of the origin in the hyperplane $H: t_1 + t_2 + \dots + t_n = 0$. Here we use the notation $z = e^{it}$ and

$$\begin{aligned} \mathcal{H}(t) &= \sum_k k^2 \alpha_k \left[\frac{t_1^2 + t_2^2 + \dots + t_m^2}{m} - \left(\frac{t_1 + t_2 + \dots + t_m}{m} \right)^2 \right] \\ &= \frac{d_2}{md} (t_1^2 + t_2^2 + \dots + t_m^2) \end{aligned}$$

is the Hessian of the function $\omega(e^t)$ restricted to the hyperplane H . The crucial observation is that Hessian $\mathcal{H}(t)$ is a symmetric function of t , while

$$\Delta(z_1, z_2, \dots, z_n) = \prod_{i < j} (z_i - z_j)$$

is skew symmetric,

$$\Delta(\sigma \cdot z) = \text{sgn } \sigma \Delta(z), \quad \sigma \in S_n.$$

Hence by symmetrization we can change the factor

$$\frac{1}{z_1^{m-1} z_2^{m-2} \dots z_{m-1}} = \frac{1}{z_1^{m-1} z_2^{m-2} \dots z_{m-1}}$$

under the integrals in (4.16) by the sum

$$\frac{1}{m!} \sum_{\text{permutations}} \operatorname{sgn}(i_1, i_2, \dots, i_m) \overline{z_{i_1}^{m-1} z_{i_2}^{m-2} \dots z_{i_{m-1}}} = \frac{1}{m!} \overline{\Delta(z_1, z_2 \dots z_m)},$$

which is equal to the Vandermond determinant. So (4.16) reduces to the integral

$$\begin{aligned} \chi_{\square}(\sigma_{\alpha}) &\sim \frac{\sqrt{m} \cdot m^{n/d}}{(2\pi)^{m-1} m!} \int_H |\Delta(z_1, z_2, \dots, z_m)|^2 \exp\left(-\frac{n}{2} \mathcal{H}(t)\right) dt \\ &\sim \frac{\sqrt{m} \cdot m^{n/d}}{(2\pi)^{m-1} m!} \int_H |\Delta(t_1, t_2, \dots, t_m)|^2 \exp\left(-\frac{n}{2} \mathcal{H}(t)\right) dt, \end{aligned} \tag{4.17}$$

where in the second line we use that $|\Delta(z)|^2 \sim |\Delta(t)|^2$ near the critical point $t = 0$. The last integral is of the type (4.15), except the integration is over hyperplane $H: \sum t_i = 0$, rather than the whole space \mathbb{R}^n . Note that $\Delta(t)$ is a constant on any line orthogonal to H and hence

$$\begin{aligned} \int_{\mathbb{R}^n} \Delta^2(t) \exp\left(-\frac{1}{2} t^2\right) dt &= \int_H \Delta(\tau)^2 \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(\tau^2 + \sigma^2)\right) d\sigma d\tau \\ &= \sqrt{2\pi} \int_H \Delta(\tau)^2 \exp\left(-\frac{1}{2} \tau^2\right) d\tau, \end{aligned} \tag{4.18}$$

where τ and σ are tangent and normal to H components of t . Easy calculation using (4.17), (4.18), and (4.15) ends the proof. ■

COROLLARY 4.5. *For a rectangular diagram \square and $\sigma_{\alpha} \neq 1$,*

$$\lim_{n \rightarrow \infty} \frac{\chi_{\square}(\sigma_{\alpha})}{\chi_{\square}(1)^{1/d}} = 0,$$

where d is the mean value of cycle lengths in σ .

4.4. Estimation of ω

By Theorem 4.2 character $\chi_{\beta}(\sigma_{\alpha})$ increases exponentially as $e^{n\omega}$ while $n \rightarrow \infty$. Here we give two estimations for critical value ω . Let us recall that for dimension $\chi_{\beta}(1)$ the critical exponent is just entropy (see Example 4.3)

$$H(\beta) = - \sum_i \beta_i \log \beta_i.$$

THEOREM 4.6. *The critical exponent ω in asymptotic of character $\chi_\beta(\sigma_\alpha)$ satisfies the inequalities*

$$\frac{H(\beta)}{l} \leq \omega \leq \frac{H(\beta)}{l_{\min}},$$

where l and l_{\min} are mean value and minimum of cycle lengths in σ_α . The left equality holds only if all cycles of σ_α are of the same length or diagram β is rectangular, while the right one holds only for substitutions σ_α with equal cycle lengths and for trivial character $\chi_\beta = 1$.

Proof. By Theorem 4.1, ω is the unique critical value of the function

$$\begin{aligned} \omega(x) &= \sum_k \alpha_k \log(x_1^k + \dots + x_m^k) - \sum_i \beta_i \log x_i \\ &= \sum_k \alpha_k \left[\log(x_1^k + \dots + x_m^k) - \sum_i \beta_i \log x_i^k \right] \end{aligned} \tag{4.19}$$

for positive x_i . Since by Step 1 in the Proof of Theorem 4.2, the Hessian of this function is positive, ω is the global minimum:

$$\omega \leq \omega(x), \quad \forall x > 0. \tag{4.20}$$

To get the upper bound for ω let us put $x_i = \beta_i^\delta$ in (4.20):

$$\omega \leq \sum_k \alpha_k \log(\beta_1^{k\delta} + \beta_2^{k\delta} + \dots + \beta_m^{k\delta}) + \delta H(\beta). \tag{4.21}$$

For $\delta = 1/l_{\min}$ we have $\beta_1^{k\delta} + \beta_2^{k\delta} + \dots + \beta_m^{k\delta} \leq 1$ and hence the sum in (4.21) is not positive and the upper bound follows.

The lower bound for the function (4.19), and hence for ω , follows from inequality

$$\log(y_1 + y_2 + \dots + y_m) - \sum_i \beta_i \log y_i \geq H(\beta). \tag{4.22}$$

To prove (4.22), observe that the function in the left-hand side has a unique critical point given by the linear system

$$y_1 = \beta_i(y_1 + y_2 + \dots + y_m)$$

with solution $y_i = \lambda \beta_i$. The corresponding critical value is just entropy $H(\beta)$.

Thus we get from (4.19) and (4.22) that $\omega \geq H(\beta)/l$ with equality possible only if

$$x_i^k = \lambda_k \beta_i$$

for all k that enter in (4.19) with nonzero coefficient $\alpha_k \neq 0$. There are two possibilities.

1. There exist two cycles of different lengths, say k and r . Then

$$x_i^k = \lambda_k \beta_i, \quad x_i^r = \lambda_r \beta_i,$$

which implies $x_1 = x_2 = \cdots = x_m$ and $\beta_1 = \beta_2 = \cdots = \beta_m$, i.e., β is a rectangular diagram.

2. All cycles are of the same length k . Then $x_i^k = \lambda \beta_i$ and $\omega = H(\beta)$ for any diagram β . ■

By combining Theorem 4.6 with Example 4.3 we get upper and lower bounds of a character in terms of its dimension (cf. Corollary 4.5).

COROLLARY 4.7. *For large n , the inequalities*

$$\chi_\beta(\mathbf{1})^{1/l} < \chi_\beta(\sigma_\alpha) < \chi_\beta(\mathbf{1})^{1/l_{\min}}$$

hold, provided diagram β is not rectangular and not all cycle lengths of σ_α are equal.

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