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Multisource Bayesian sequential binary hypothesis testing problem

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Abstract We consider the problem of testing two simple hypotheses about unknown local characteristics of several independent Brownian motions and compound Poisson processes. All of the processes may be observed simultaneously as long as desired before a final choice between hypotheses is made. The objective is to find a decision rule that identifies the correct hypothesis and strikes the optimal balance between the expected costs of sampling and choosing the wrong hypothesis. Previous work on Bayesian sequential hypothesis testing in continuous time provides a solution when the characteristics of these processes are tested separately. However, the decision of an observer can improve greatly if multiple information sources are available both in the form of continuously changing signals (Brownian motions) and marked count data (compound Poisson processes). In this paper, we combine and extend those previous efforts by considering the problem in its multisource setting. We identify a Bayes optimal rule by solving an optimal stopping problem for the likelihood-ratio process. Here, the likelihood-ratio process is a jump-diffusion, and the solution of the optimal stopping problem admits a two-sided stopping region. Therefore, instead of using the variational arguments (and smooth-fit principles) directly, we solve the problem by patching the solutions of a sequence of optimal stopping problems for the pure diffusion part of the likelihood-ratio process. We also provide a numerical algorithm and illustrate it on several examples.

Keywords Bayesian sequential identification · Jump-diffusion processes · Optimal stopping

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1 Introduction

On some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(X_t^{(i)})_{t\geq 0}$, $1 \leq i \leq d$ be d independent Brownian motions with constant drifts $\mu^{(i)}$, $1 \leq i \leq d$, and $(T_n^{(j)}, Z_n^{(j)})_{n\geq 1}$, $1 \leq j \leq m$ be m independent compound Poisson processes independent of the Brownian motions. For every $1 \leq j \leq m$, $(T_n^{(j)})_{n\geq 1}$ are the arrival times, and $(Z_n^{(j)})_{n\geq 1}$ are the marks on some measurable space $(E, \mathcal{E})$, with arrival rates $\lambda^{(j)}$ and mark distributions $\nu^{(j)}(\cdot)$ on $(E, \mathcal{E})$.

Suppose that $\mu^{(i)}$, $1 \leq i \leq d$ and $(\lambda^{(j)}, \nu^{(j)})_{1\leq j\leq m}$ are unknown, but exactly one of the following two simple hypotheses,

$$\begin{aligned} H_0\colon &\quad \left\{ \begin{array}{c} \mu^{(i)} = \mu_0^{(i)}, \quad 1 \leq i \leq d \\ \left(\lambda^{(j)}, \nu^{(j)}\right) = \left(\lambda_0^{(j)}, \nu_0^{(j)}\right), \quad 1 \leq j \leq m \end{array} \right\}, \\ H_1\colon &\quad \left\{ \begin{array}{c} \mu^{(i)} = \mu_1^{(i)}, \quad 1 \leq i \leq d \\ \left(\lambda^{(j)}, \nu^{(j)}\right) = \left(\lambda_1^{(j)}, \nu_1^{(j)}\right), \quad 1 \leq j \leq m \end{array} \right\}, \end{aligned} \tag{1.1}$$

is correct for some known $\mu_0^{(i)}, \mu_1^{(i)}$ for every $1 \leq i \leq d$, and $(\lambda_0^{(j)}, \nu_0^{(j)})$, $(\lambda_1^{(j)}, \nu_1^{(j)})$ for every $1 \leq j \leq m$, where probability measures $\nu_0^{(j)}$ and $\nu_1^{(j)}$ on $(E, \mathcal{E})$ are equivalent. Let Θ be the index of correct hypothesis, which is a $\{0, 1\}$-valued random variable with prior distribution

$$\mathbb{P}\{\Theta = 1\} = 1 - \mathbb{P}\{\Theta = 0\} = \pi$$

for some known $\pi \in (0, 1)$.

The problem is to find a stopping time τ and a terminal decision rule d which depend only on the observations of Brownian motions $(X_n^{(i)})_{n\geq 0}$, $1 \leq i \leq d$ and compound Poisson processes $(T_n^{(j)}, Z_n^{(j)})_{n\geq 1}$, $1 \leq j \leq m$, and which minimizes the Bayes risk

$$R_{\tau,d}(\pi) := \mathbb{E}\big[\tau + 1_{\{\tau<\infty\}}(a 1_{\{d=0,\Theta=1\}} + b 1_{\{d=1,\Theta=0\}})\big], \tag{1.2}$$

where a and b are known positive constants and correspond to the costs of making wrong terminal decisions. If such a decision rule (τ, d) exists, then it strikes optimal balance between the expected total sampling cost and the expected cost of selecting the wrong hypothesis.

Sequential hypothesis testing problems have been studied extensively in the literature due to their practical applications in different fields. These include target detection in radar and sonar systems, threat identification in homeland security, fault identification and isolation in industrial processes, testing the riskiness of financial assets; see, e.g., Marcus and Swerling (1962), Fu (1968), Veeravalli and Baum (1996), Ernisse et al. (1997), Dragalin et al. (2000), Lai (2001), and the references therein.

The non-Bayes formulation of the sequential hypothesis testing problem has been studied by many authors, both in discrete- and continuous-time, and can be found in the recent reviews and contributions made by Lai (2000, 2001), Dragalin et al. (1999, 2000), Lorden (1977). In the Bayesian framework, sequential hypothesis testing problems were studied in discrete-time for the identification of the distribution of i.i.d. observations by Wald and Wolfowitz (1950), Blackwell and Girshick (1979), Zacks (1971), Shiryaev (1978). In continuous-time, on the other hand, Bayesian formulations are studied and solved for the identification of the drift of a Brownian motion in Shiryaev (1978), Peskir and Gapeev (2004), Shiryaev and Zhitlukhin (2011), for the identification of the drift term of more general diffusion processes in Shiryaev and Gapeev (2011), for the identification of the arrival rate of a simple Poisson process in Peskir and Shiryaev (2000, 2006), and for the identification of the arrival rate and mark distribution of a compound Poisson process in Gapeev (2002), Dayanik and Sezer (2006), Dayanik et al. (2008b), Ludkovski and Sezer [2012, Sect. 5.2]. The problem has not been addressed earlier for joint identification of local characteristics of concurrently observed independent Brownian motions and compound Poisson processes, and its solution is the main contribution of this paper.

Multisource detection problems with Brownian motions and compound Poisson processes appear naturally in many fields. In finance, for example, stock prices are modeled with exponential Brownian motions, and the firm defaults or credit derivatives are modeled with compound Poisson processes. Stock prices, firm defaults, and credit derivatives move together in a random economic environment, which may either be in recession or in a booming state. For a fund's asset manager interested in the best short-term financial plans, it is important to identify as quickly and confidently as possible if the economy is in recession or in a booming regime. For that purpose, the asset manager can trace the stock prices of certain companies operating in different industries or even in different countries. It is then reasonable to expect that those stock price processes change independently conditionally on the common market indicators of a recession or a booming economy. In addition to those stocks, the asset manager may also trace the number of defaults in some other key industries. To the extend that the conditionally independent industry default processes and the stock return rate processes of companies operating in different and diverse industries can be modeled with compound Poisson processes and Brownian motions with arrival rates, mark distributions, and drift rates modulated by the given common unobserved market indicators of economic state, our paper provides a simple formulation and solution for the asset manager's problem. Similar problem also arises when we test the reliability of mechanical systems; we can monitor both the occurrence/depth of cracks as marked count data and the vibrations in the system as continuously changing signals for a better diagnosis.

When the observations come from both several Brownian motions and compound Poisson processes simultaneously, finding the best multisource sequential identification rule becomes a very difficult dynamic programming problem because of an unfavorable second-order integro-differential operator. Instead of following the standard variational arguments, we develop an alternative solution method that takes specifically into account the special structure of the sample-paths of a suitable sufficient statistic for the problem. This method enables us to solve the problem completely for the most general case—without any need for specific simplifying assumptions about the relationships between drift rates, arrival rates, and mark distributions.

We show that an optimal decision rule (τ, d) always exists. The optimal stopping time τ is when the likelihood-ratio process

$$L_t := \exp\left\{\sum_{i=1}^{d}\big(\mu_1^{(i)} - \mu_0^{(i)}\big)\big(X_t^{(i)} - X_0^{(i)}\big) - \frac{t}{2}\sum_{i=1}^{d}\Big[\big(\mu_1^{(i)}\big)^2 - \big(\mu_0^{(i)}\big)^2\Big]\right\}$$
$$\times \exp\left\{\sum_{j=1}^{m}\sum_{n:0<T_n^{(j)}\le t}\log\left(\frac{\lambda_1^{(j)}}{\lambda_0^{(j)}}\frac{\mathrm{d}\nu_1^{(j)}}{\mathrm{d}\nu_0^{(j)}}\big(Z_n^{(j)}\big)\right) - t\sum_{j=1}^{m}\big(\lambda_1^{(j)} - \lambda_0^{(j)}\big)\right\}$$

exits for the first time a bounded interval $(\phi_1(1-\pi)/\pi, \phi_2(1-\pi)/\pi)$ for some suitable constants $0 < \phi_1 < b/a < \phi_2 < \infty$, and optimal terminal decision rule d is to choose the null hypothesis if $\pi L_\tau/(1-\pi) \le b/a$ and the alternative hypothesis otherwise. We describe a provably convergent numerical method to calculate both the minimum Bayes risk and the decision boundaries ϕ_1 and ϕ_2 of the optimal stopping rule τ. The minimum Bayes risk is shown to be the uniform limit of a decreasing sequence of successive approximations, which are obtained by applying a contraction mapping iteratively to a suitable initial function. The maximum absolute difference between successive approximations is bounded by an explicit bound, which decays at a known exponential rate with the number of iterations. Thus, one can always determine the necessary number of iterations ex-ante for any desired level of accuracy in the approximations of the minimum Bayes risk and optimal decision boundaries.

We address the problem by reducing it to the optimal stopping of the likelihood-ratio process, which we solve later. The likelihood-ratio process is a jump-diffusion with an infinitesimal generator which is a second-order integro-differential operator, and the conventional method of variational inequalities is very unlikely to succeed. Instead, we solve the problem by means of a jump operator, which is obtained by applying the dynamic programming principle at the jump times. The role of this operator is to patch, at successive jump times, the solutions of suitably modified optimal stopping problems for pure diffusion part. The latter modified problems are solved easily and directly by the potential-theoretic methods developed by Dayanik and Karatzas (2003) and Dayanik (2008). A similar sequential plan was followed by Dayanik et al. (2008a) and Sezer (2010) to solve sequential change detection problems, each admitting a one-sided optimal stopping region with only one decision boundary. The multisource Bayesian sequential binary hypothesis testing problem, however, is far more challenging with a two-sided optimal stopping region and two critical decision boundaries, which should be determined simultaneously. Optimal stopping problems for jump-diffusions, which admit two-sided optimal stopping rules, appear also in finance and real-option theory for pricing American-type financial contracts, which, we believe, can be tackled very effectively with the same method of this paper. We refer the reader to Salminen (1985), Beibel and Lerche (1997, 2001), Christensen and Irle (2011), Cissé et al. (2012) for examples of and additional remarks on problems with two-sided stopping regions. For the general theory of optimal stopping for (jump) diffusions, the books by Shiryaev (1978), Peskir and Shiryaev (2006), Øksendal and Sulem (2007) and the references cited therein can be consulted.

In Sect. 5, we show that the general multisource sequential testing problem can be reduced to a simple one where the observations consist of those of only one Brownian motion and only one compound Poisson process (i.e., $d = m = 1$). Therefore, in the remainder except Sect. 5, we assume that $d = m = 1$ and drop the superscripts used to identify local characteristics, arrival times, and marks with different Brownian motions and compound Poisson processes.

In Sect. 2, we start with the precise description of the problem and the derivation of an auxiliary optimal stopping problem. Section 3 introduces a key jump operator and successive approximations to the value function of the auxiliary optimal stopping problem, whose solution is explicitly identified, along with the Bayes-optimal decision rule for the Bayesian sequential binary hypothesis testing problem, in Sect. 4. Section 5 shows how to reduce the multisource problem to that of one Brownian motion and one compound Poisson process, the solution of which was already given in Sect. 4. Section 6 concludes with a numerical algorithm to find Bayes ε-optimal decision rules and its illustrations on several numerical examples. Some of the proofs are deferred to the Appendix.

2 Problem description and a model

Let X be a Brownian motion with constant drift μ, and let $(T_n, Z_n)_{n\geq 1}$ be a compound Poisson process with arrival times $(T_n)_{n\geq 1}$, marks $(Z_n)_{n\geq 1}$ on some measurable space $(E, \mathcal{E})$, arrival rate λ, and mark distribution $\nu(\cdot)$ on $(E,\mathcal{E})$, independent of Brownian motion X. Denote by $\mathbb{F}^X=(\mathcal{F}_t^X)_{t\ge 0}$, $\mathbb{F}^p=(\mathcal{F}_t^p)_{t\ge 0}$, and $\mathbb{F}=(\mathcal{F}_t)_{t\ge 0}$ the Brownian, compound Poisson, and observation filtrations, respectively, enlarged suitably to satisfy the usual conditions of right-continuity and completion with $\mathbb{P}$-negligible sets.

Suppose that the drift μ and arrival rate and mark distribution (λ,ν) are unknown, but exactly one of the following two simple hypotheses,

$$H_0:\quad (\mu,\lambda,\nu)=(\mu_0,\lambda_0,\nu_0)\quad \text{versus}\quad H_1:\quad (\mu,\lambda,\nu)=(\mu_1,\lambda_1,\nu_1), \tag{2.1}$$

is correct for some known μ_0, μ_1, (λ_0,ν_0), and (λ_1,ν_1), where $\mu_0\neq\mu_1$, $\lambda_0<\lambda_1$, and probability measures ν_0 and ν_1 on $(E,\mathcal{E})$ are equivalent. The unknown index of the correct hypothesis is denoted by Θ, which we assume is a random variable with prior distribution

$$\mathbb{P}\{\Theta=1\}=1-\mathbb{P}\{\Theta=0\}=\pi\quad \text{for some known } \pi\in(0,1).$$

One is allowed to observe processes X and $(T_n,Z_n)_{n\ge 1}$ as long as desired before making a final choice between hypotheses H_0 and H_1. Each time-unit before a decision is made costs one, and choosing the wrong hypothesis costs a or b monetary units, respectively, if the choice is H_0 or H_1. The objective is to minimize the expected total costs of sampling time and a wrong terminal decision.

Hence, every acceptable decision rule (τ,d) consists of an $\mathbb{F}$-stopping time τ and a $\{0,1\}$-valued $\mathcal{F}_\tau$-measurable random variable d and is associated with the Bayes risk

$$R_{\tau,d}(\pi):=\mathbb{E}\big[\tau+1_{\{\tau<\infty\}}(a1_{\{d=0,\Theta=1\}}+b1_{\{d=1,\Theta=0\}})\big]. \tag{2.2}$$

The problem is to (i) calculate the minimum Bayes risk

$$U(\pi):=\inf_{(\tau,d)\in\Delta} R_{(\tau,d)}(\pi),\quad \pi\in(0,1) \tag{2.3}$$

over the collection Δ of all acceptable decision rules (τ,d), and (ii) to find an acceptable decision rule that attains the minimum, if such a rule exists.

2.1 Model

We will now construct a model for the problem just described. Let $(\Omega,\mathcal{F},\mathbb{P}_0)$ be a probability space hosting the following independent stochastic elements: (i) X is a Brownian motion with drift rate μ_0, (ii) $(T_n,Z_n)_{n\ge 1}$ is a compound Poisson process with arrival rate λ_0 and mark distribution ν_0 on $(E,\mathcal{E})$, and (iii) Θ is a Bernoulli random variable with success probability $\pi\in(0,1)$.

We denote by $\mathbb{G}=(\mathcal{G}_t)_{t\ge 0}$ the filtration obtained by enlarging the observation filtration $\mathbb{F}$ with the information about Θ; i.e., $\mathcal{G}_t:=\mathcal{F}_t\vee\sigma(\Theta)$ for every $t\ge 0$, and introduce the likelihood-ratio process

$$\begin{aligned} L_t=\exp\bigg\{&(\mu_1-\mu_0)(X_t-X_0)-\left[\frac{\mu_1^2-\mu_0^2}{2}+\lambda_1-\lambda_0\right]t\\ &+\sum_{0<T_n\le t}\log\left(\frac{\lambda_1}{\lambda_0}\frac{d\nu_1}{d\nu_0}(Z_n)\right)\bigg\},\quad t\ge 0. \end{aligned} \tag{2.4}$$

Let $\mathbb{P}$ be a new probability measure on $(\Omega, \mathcal{G}_\infty)$, whose restriction to each $\mathcal{G}_t$, $t \geq 0$ is defined in terms of the Radon–Nikodym derivative

$$\left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_0}\right|_{\mathcal{G}_t} = \xi_t := 1_{\{\Theta=0\}} + 1_{\{\Theta=1\}} L_t, \quad t \geq 0. \tag{2.5}$$

An application of Girsanov theorem shows that under $\mathbb{P}$, the process $X_t - [(1-\Theta)\mu_0 + \Theta\mu_1]t$ is a standard $(\mathbb{P}, \mathbb{G})$-Brownian motion, and $(T_n, Z_n)_{n\geq 1}$ is a marked point process with $(\mathbb{P}, \mathbb{G})$-compensating measure $(1-\Theta)\nu_0(\mathrm{d}z)\,\lambda_0\,\mathrm{d}t + \Theta\nu_1(\mathrm{d}z)\lambda_1\,\mathrm{d}t$. Moreover, because $\Theta \in \mathcal{G}_0$ and $L_0 = 1$, the distributions of Θ under $\mathbb{P}_0$ and $\mathbb{P}$ are the same. Under probability measure $\mathbb{P}$ defined by (2.5), we have the same setup as in the problem description. Therefore, in the remainder we will work with the model constructed here.

Starting from any arbitrary but fixed initial state $\phi \in \mathbb{R}_+$, let us define the process

$$\Phi_0 = \phi \quad \text{and} \quad \Phi_t = \Phi_0 L_t, \quad t \geq 0. \tag{2.6}$$

The Bayes theorem implies that

$$\begin{aligned}\frac{\mathbb{P}\{\Theta = 1 \mid \mathcal{F}_t\}}{\mathbb{P}\{\Theta = 0 \mid \mathcal{F}_t\}} &= \frac{\mathbb{E}_0[\xi_t 1_{\{\Theta=1\}} \mid \mathcal{F}_t]}{\mathbb{E}_0[\xi_t 1_{\{\Theta=0\}} \mid \mathcal{F}_t]} = \frac{L_t\mathbb{P}_0\{\Theta = 1 \mid \mathcal{F}_t\}}{\mathbb{P}_0\{\Theta = 0 \mid \mathcal{F}_t\}} \\ &= \frac{\pi}{1-\pi} L_t, \quad t \geq 0 \quad \mathbb{P}^{\frac{\pi}{1-\pi}}\text{-a.s.}\end{aligned}$$

because L_t is $\mathcal{F}_t$-measurable, and Θ and $\mathcal{F}_t$ are independent under $\mathbb{P}_0$. Namely, Φ_t is the conditional odds of the event $\{\Theta = 1\}$ given the observations of X and $(T_n, Z_n)_{n\geq 1}$ until time $t \geq 0$.

The next proposition, the proof of which is very similar to that of Proposition 2.1 of Dayanik and Sezer (2006), shows that the sequential hypothesis testing problem can be reduced to an optimal stopping problem for the conditional odds-ratio process Φ in (2.6).

Proposition 2.1 *The Bayes risk $R_{\tau,d}(\pi)$ in (2.2) can be written as*

$$\begin{aligned}R_{\tau,d}(\pi) &= b(1-\pi)\mathbb{P}_0^{\frac{\pi}{1-\pi}}\{\tau < \infty\} \\ &\quad + (1-\pi)\mathbb{E}_0^{\frac{\pi}{1-\pi}}\left[\int_0^\tau (1+\Phi_t)\,\mathrm{d}t + (a\Phi_\tau - b)1_{\{d=0,\tau<\infty\}}\right],\end{aligned} \tag{2.7}$$

where $\mathbb{P}_0^\phi$ is the probability $\mathbb{P}_0$ with $\Phi_0 = \phi$, and $\mathbb{E}_0^\phi$ is the expectation with respect to $\mathbb{P}_0^\phi$ for every $\phi \in \mathbb{R}_+$. If we define

$$d(t) := 1_{(b/a,\infty)}(\Phi_t), \quad t \geq 0, \tag{2.8}$$

then the pair $(\tau, d(\tau))$ belongs to Δ. We have $R_{\tau,d}(\pi) \geq R_{\tau,d(\tau)}(\pi)$ for every $(\tau, d) \in \Delta$ and $\pi \in (0,1)$, and the minimum Bayes risk $U(\pi)$ of (2.3) can be written as

$$U(\pi) \equiv \inf_{(\tau,d)\in\Delta} R_{\tau,d}(\pi) = b(1-\pi) + (1-\pi)V\left(\frac{\pi}{1-\pi}\right), \quad \pi \in (0,1) \tag{2.9}$$

in terms of the value function $V(\cdot)$ of the auxiliary optimal stopping problem

$$V(\phi) := \inf_{\tau\in\mathbb{F}} \mathbb{E}_0^\phi\left[\int_0^\tau g(\Phi_t)\,\mathrm{d}t + 1_{\{\tau<\infty\}}h(\Phi_\tau)\right], \quad \phi \geq 0, \tag{2.10}$$

where the running cost function $g:\mathbb{R}_+\mapsto\mathbb{R}$ and the terminal cost function $h:\mathbb{R}_+\mapsto\mathbb{R}$ are defined by

$$g(\phi):=1+\phi \quad \textit{and} \quad h(\phi):=-(a\phi-b)^-. \tag{2.11}$$

Remark 2.1 If $\mathbb{E}_0^\phi\tau=+\infty$ for some $\tau\in\mathbb{F}$ and $\phi\in\mathbb{R}_+$, then $\mathbb{E}_0^\phi[\int_0^\tau g(\Phi_t)\,\mathrm{d}t+1_{\{\tau<\infty\}}h(\Phi_\tau)]\geq\mathbb{E}_0^\phi\tau-b=+\infty$ because $g(\phi)\geq 1$ and $h(\phi)\geq -b$. Therefore, in (2.10), the infimum can be restricted, without any loss, to those $\tau\in\mathbb{F}$ for which $\mathbb{E}_0^\phi\tau<\infty$.

Let us denote the point process generated by $(T_n,Z_n)_{n\geq 1}$ with

$$p\big((0,t]\times B\big)=\sum_{n=1}^\infty 1_{(0,t]\times B}(T_n,Z_n), \quad t\geq 0.$$

Then the likelihood-ratio process L in (2.4) can be written as

$$\begin{aligned} L_t=\exp\bigg\{&(\mu_1-\mu_0)(X_t-X_0)-\bigg[\frac{\mu_1^2-\mu_0^2}{2}+\lambda_1-\lambda_0\bigg]t \\ &+\int_{(0,t]}\int_E\log\bigg(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(z)\bigg)p(\mathrm{d}s\times\mathrm{d}z)\bigg\}, \quad t\geq 0, \end{aligned}$$

and an application of Itô's rule to (2.6) gives the dynamics of process Φ as

$$\begin{aligned} \Phi_0=\phi \quad \text{and} \quad \mathrm{d}\Phi_t&=(\mu_1-\mu_0)\Phi_t(\mathrm{d}X_t-\mu_0\,\mathrm{d}t) \\ &+\Phi_{t-}\int_E\bigg(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(z)-1\bigg)\big[p(\mathrm{d}t\times\mathrm{d}z)-\nu_0(\mathrm{d}z)\lambda_0\,\mathrm{d}t\big], \quad t\geq 0, \end{aligned}$$

and for every sufficiently smooth function $w:\mathbb{R}_+\mapsto\mathbb{R}$, we have

$$\begin{aligned} \mathrm{d}w(\Phi_t)&=(Aw)(\Phi_t)\,\mathrm{d}t+(\mu_1-\mu_0)\Phi_{t-}w'(\Phi_t)(\mathrm{d}X_t-\mu_0\,\mathrm{d}t) \\ &+\int_E\bigg[w\bigg(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(z)\Phi_{t-}\bigg)-w(\Phi_{t-})\bigg]\big[p(\mathrm{d}t\times\mathrm{d}z)-\nu_0(\mathrm{d}z)\lambda_0\,\mathrm{d}t\big], \quad t\geq 0, \end{aligned} \tag{2.12}$$

where A is the $(\mathbb{P}_0,\mathbb{F})$-infinitesimal generator of Φ given by

$$\begin{aligned} (Aw)(\phi)&=(\lambda_0-\lambda_1)\phi w'(\phi)+\frac{(\mu_1-\mu_0)^2}{2}\phi^2 w''(\phi) \\ &+\lambda_0\int_E\bigg[w\bigg(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(z)\phi\bigg)-w(\phi)\bigg]\nu_0(\mathrm{d}z). \end{aligned} \tag{2.13}$$

Note that, for every $k\geq 0$ and $t\geq 0$, (2.4) implies that

$$\begin{aligned} \Phi_{T_k+t}&=\Phi_0 L_{T_k+t}=\Phi_0 L_{T_k}\frac{L_{T_k+t}}{L_{T_k}} \\ &=\Phi_{T_k}\exp\bigg\{(\mu_1-\mu_0)(X_{T_k+t}-X_{T_k})-\bigg[\frac{\mu_1^2-\mu_0^2}{2}+\lambda_1-\lambda_0\bigg]t \end{aligned}$$

$$+\sum_{T_k<T_n\le T_k+t}\log\left(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(Z_n)\right)\Bigg\}. \tag{2.14}$$

Let us set $T_0\equiv 0$ and introduce

$$X_t^{(k)}:=X_{T_k+t},\quad t\ge 0,\ k\ge 0,$$
$$\big(T_\ell^{(k)},Z_\ell^{(k)}\big):=(T_{k+\ell}-T_k,Z_{k+\ell}),\quad \ell\ge 1,\ k\ge 0,$$
$$\mathbb{F}^{(k)}:=\big(\mathcal{F}_t^{(k)}\big)_{t\ge 0}\quad\text{with } \mathcal{F}_0^{(k)}:=\mathcal{F}_{T_k},\ k\ge 0,\quad\text{and}$$
$$\mathcal{F}_t^{(k)}:=\mathcal{F}_0^{(k)}\vee\sigma\big\{X_u^{(k)};\ 0\le u\le t\big\}\vee\sigma\big\{\big(T_\ell^{(k)},Z_\ell^{(k)}\big);\ 0\le T_\ell^{(k)}\le t,\ \ell\ge 1\big\}.$$

Then, for every $k\ge 0$, $(X_t^{(k)})_{t\ge 0}$ is a $(\mathbb{P}_0,\mathbb{F}^{(k)})$-Brownian motion with drift μ_0, and $(T_\ell^{(k)},Z_\ell^{(k)})_{\ell\ge 1}$ is a $(\mathbb{P}_0,\mathbb{F}^{(k)})$-compound Poisson process with arrival rate λ_0 and mark distribution ν_0 on $(E,\mathcal{E})$. If we define

$$Y_t^{k,\phi}:=\phi\exp\left\{(\mu_1-\mu_0)\big(X_t^{(k)}-X_0^{(k)}\big)-\left[\frac{\mu_1^2-\mu_0^2}{2}+\lambda_1-\lambda_0\right]t\right\},$$
$$t\ge 0,\ k\ge 0,\ \phi\ge 0, \tag{2.15}$$

then the sample paths of the conditional odds-ratio process Φ in (2.6) and (2.14) can be decomposed into diffusion and jump parts as in

$$\Phi_t=\begin{cases} Y_{t-T_k}^{k,\Phi_{T_k}}, & \text{if } t\in[T_k,T_{k+1}) \text{ for some } k\ge 0,\\ \dfrac{\lambda_1}{\lambda_0}\dfrac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(Z_{k+1})Y_{T_{k+1}-T_k}^{k,\Phi_{T_k}}, & \text{if } t=T_{k+1} \text{ for some } k\ge 0.\end{cases} \tag{2.16}$$

The process $Y^{k,\phi}$ is a diffusion with dynamics

$$Y_0^{k,\phi}=\phi\quad\text{and}\quad \mathrm{d}Y_t^{k,\phi}=(\mu_1-\mu_0)Y_t^{k,\phi}\big(\mathrm{d}X_t^{(k)}-\mu_0\,\mathrm{d}t\big)+(\lambda_0-\lambda_1)Y_t^{k,\phi}\,\mathrm{d}t,\quad t\ge 0.$$

In the remainder, we will take advantage of the decomposition in (2.16) of the process Φ to solve the auxiliary optimal stopping problem in (2.10).

3 Jump operator and successive approximations

We will denote Y^{0,Φ_0} by Y^{Φ_0}, which is a diffusion with dynamics

$$Y_0^{\Phi_0}=\Phi_0,\quad \mathrm{d}Y_t^{\Phi_0}=(\lambda_0-\lambda_1)Y_t^{\Phi_0}\,\mathrm{d}t+(\mu_1-\mu_0)Y_t^{\Phi_0}(\mathrm{d}X_t-\mu_0\,\mathrm{d}t)\quad t\ge 0 \tag{3.1}$$

and $(\mathbb{P}_0,\mathbb{F})$-infinitesimal generator

$$(A_0w)(\phi)=(\lambda_0-\lambda_1)\phi w'(\phi)+\frac{1}{2}(\mu_1-\mu_0)^2\phi^2w''(\phi) \tag{3.2}$$

acting on twice-continuously differentiable functions $w:\mathbb{R}_+\mapsto\mathbb{R}$. For every bounded Borel function $w:\mathbb{R}_+\mapsto\mathbb{R}$, let us define

$$(Kw)(\phi):=\int_E w\left(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(z)\phi\right)\nu_0(\mathrm{d}z),\quad \phi\in\mathbb{R}_+, \tag{3.3}$$

and the jump operator

$$(Jw)(\phi) := \inf_{\tau\in\mathbb{F}^X} \mathbb{E}_0^{\phi}\left[\int_0^{\tau} e^{-\lambda_0 t}\big[g+\lambda_0(Kw)\big]\big(Y_t^{\Phi_0}\big)\mathrm{d}t + e^{-\lambda_0\tau}h\big(Y_\tau^{\Phi_0}\big)\right], \quad \phi\in\mathbb{R}_+, \tag{3.4}$$

which is itself a discounted optimal stopping problem for the diffusion Y^{Φ_0} in (3.1), with discount rate λ_0, running cost function $g(\cdot)+\lambda_0(Kw)(\cdot)$ and terminal cost function $h(\cdot)$.

As a possible solution to (2.10), consider the following strategy. For an arbitrary but fixed stopping time τ of process X, i.e., $\tau\in\mathbb{F}^X$, suppose that we decided to stop the process Φ at τ on $\{\tau<T_1\}$ and continue with an optimal stopping rule from T_1 onwards on $\{\tau\ge T_1\}$. Because of the decomposition in (2.16) of the sample paths of Φ, the expected total cost of this strategy should be equal to $\mathbb{E}_0^{\phi}[\int_0^{\tau\wedge T_1} g(\Phi_t)\,\mathrm{d}t + 1_{\{\tau<T_1\}}h(\Phi_\tau) + 1_{\{\tau\ge T_1\}}V(\Phi_{T_1})]$, which can be written as

$$\begin{aligned}
&\mathbb{E}_0^{\phi}\left[\int_0^{\infty} 1_{\{T_1>t\}}g\big(Y_t^{\Phi_0}\big)1_{\{\tau>t\}}\mathrm{d}t + 1_{\{\tau<T_1\}}h\big(Y_\tau^{\Phi_0}\big) + 1_{\{\tau\ge T_1\}}V\left(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(Z_1)Y_{T_1}^{\Phi_0}\right)\right]\\
&= \int_0^{\infty} e^{-\lambda_0 t}\mathbb{E}_0^{\phi_0}\big[g\big(Y_t^{\Phi_0}\big)1_{\{\tau>t\}}\big]\,\mathrm{d}t + \mathbb{E}_0^{\phi}\big[e^{-\lambda_0\tau}h\big(Y_\tau^{\Phi_0}\big)\big]\\
&\quad + \mathbb{E}_0^{\phi}\left[\int_0^{\tau} \lambda_0 e^{-\lambda_0 t}\left(\int_E V\left(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(z)Y_t^{\Phi_0}\right)\nu_0(\mathrm{d}z)\right)\mathrm{d}t\right]\\
&= \mathbb{E}_0^{\phi}\left[\int_0^{\tau} e^{-\lambda_0 t}\big[g\big(Y_t^{\Phi_0}\big)+\lambda_0(KV)\big(Y_t^{\Phi_0}\big)\big]\,\mathrm{d}t + e^{-\lambda_0 t}h\big(Y_\tau^{\Phi_0}\big)\right]
\end{aligned}$$

since τ and Y^{Φ_0} are functionals of process X, and X and (T_1, Z_1) are $\mathbb{P}_0^{\phi}$-independent. One also expects that $V(\phi)$ is the *smallest* expected total cost over all such strategies and solves

$$\begin{aligned}
V(\phi) &= \inf_{\tau\in\mathbb{F}^X} \mathbb{E}_0^{\phi}\left[\int_0^{\tau} e^{-\lambda_0 t}\big[g+\lambda_0(KV)\big]\big(Y_t^{\Phi_0}\big)\,\mathrm{d}t + e^{-\lambda_0\tau}h\big(Y_\tau^{\Phi_0}\big)\right]\\
&\equiv (JV)(\phi), \quad \phi\ge 0.
\end{aligned}\tag{3.5}$$

Later, we will prove that this conjecture is indeed true.

Lemma 3.1 *If $w_1(\cdot)\le w_2(\cdot)$ are bounded, then $(Jw_1)(\cdot)\le(Jw_2)(\cdot)$. If $-b\le w(\cdot)\le 0$, then $-b\le(Jw)(\cdot)\le h(\cdot)$. If $w(\cdot)$ is increasing and concave, then so is $(Jw)(\cdot)$.*

Proof The monotonicity of $w\mapsto Jw$ is obvious. We have $(Jw)(\cdot)\le h(\cdot)$ because $\tau\equiv 0$ is one of the acceptable stopping times. If $w(\cdot)\ge -b$, then $(Kw)(\cdot)\ge -b$ and

$$\begin{aligned}
(Jw)(\phi) &= \inf_{\tau\in\mathbb{F}^X} \mathbb{E}_0^{\phi}\left[\int_0^{\tau} e^{-\lambda_0 t}\big[g\big(Y_t^{\Phi_0}\big)+\lambda_0(KV)\big(Y_t^{\Phi_0}\big)\big]\,\mathrm{d}t + e^{-\lambda_0\tau}h\big(Y_\tau^{\Phi_0}\big)\right]\\
&\ge \inf_{\tau\in\mathbb{F}^X} \mathbb{E}_0^{\phi}\left[\int_0^{\tau} \lambda_0 e^{-\lambda_0 t}(-b)\,\mathrm{d}t + e^{-\lambda_0\tau}(-b)\right]\\
&= \inf_{\tau\in\mathbb{F}^X}(-b)\mathbb{E}_0^{\phi_0}\big[1-e^{-\lambda_0\tau}+e^{-\lambda_0\tau}\big] = -b.
\end{aligned}$$

To establish the last claim, note first that the definition of the process $Y^{\Phi_0}\equiv Y^{0,\Phi_0}$ in (2.15) implies that $Y_0^{\phi_1}\le Y_0^{\phi_2}$ for every $t\ge 0$ if $0\le\phi_1\le\phi_2$. The functional $(Kw)(\cdot)$ in (3.3) is nondecreasing if $w(\cdot)$ is nondecreasing. Therefore, if $0 \le \phi_1 \le \phi_2$, then $(Kw)(Y_t^{\phi_1}) \le (Kw)(Y_t^{\phi_2})$ for every $t \ge 0$. Because the function $h(\cdot)$ is also nondecreasing, we conclude that $(Jw)(\phi_1) \le (Jw)(\phi_2)$ if $0 \le \phi_1 \le \phi_2$.

Finally, the terminal reward function $h(\cdot)$ is concave, and if $w(\cdot)$ is concave, then the function $(Kw)(\cdot)$ is also concave. Because Y^ϕ in (2.15) linear in ϕ, the integrand and expectation in (3.4) are concave in ϕ for every fixed stopping time $\tau \in \mathbb{F}^X$. Because $(Jw)(\cdot)$ is the infimum of a class of concave functions, it is itself concave. □

In (3.5), we anticipated that the value function $V(\cdot)$ in (2.10) coincides with a fixed point of operator J, which can be found by means of successive approximations. Let us define

$$v_0(\cdot) := h(\cdot) \quad \text{and} \quad v_n(\cdot) := (Jv_{n-1})(\cdot), \quad n \ge 1. \tag{3.6}$$

Lemma 3.2 *The sequence $(v_n(\cdot))_{n\ge 0}$ is decreasing, and $v_\infty(\cdot) := \lim_{n\to\infty} v_n(\cdot) \equiv \inf_{n\ge 0} v_n(\cdot)$ exists; $v_n(\cdot)$, $n \ge 0$ and $v_\infty(\cdot)$ are nondecreasing, concave, and bounded between $-b$ and $h(\cdot)$.*

Proof Note that $\phi \mapsto v_0(\phi) = h(\phi) = -(a\phi - b)^-$ is nondecreasing, concave, and bounded between $-b$ and $h(\phi)$. Suppose that $v_n(\cdot)$ for some $n \ge 0$ has the same properties. Then by Lemma 3.1, $v_{n+1}(\cdot) = (Jv_n)(\cdot)$ is nondecreasing, concave, and bounded between $-b$ and $h(\cdot)$. We have $v_1(\cdot) \le (Jv_0)(\cdot) = (Jh)(\cdot) \le h(\cdot) = v_0(\cdot)$, and if $v_n(\cdot) \le v_{n-1}(\cdot)$ for some $n \ge 1$, then Lemma 3.1 also implies that $v_{n+1}(\cdot) = (Jv_n)(\cdot) \le (Jv_{n-1})(\cdot) = v_n(\cdot)$. Finally, $v_\infty(\cdot)$ is nondecreasing, concave, and bounded between $-b$ and $h(\cdot)$ because it is the pointwise limit of the $v_n(\cdot)$'s, each of which has the same properties. □

Lemma 3.3 *The function $v_\infty(\cdot)$ is the largest solution of the equation $v(\cdot) = (Jv)(\cdot)$ less than or equal to $h(\cdot)$.*

Proof Because $-b \le v_n(\cdot) \le h(\cdot)$ for every $n \ge 0$, the bounded convergence theorem implies

$$\begin{aligned}
v_\infty(\phi) &= \inf_{n\ge 0} v_{n+1}(\phi) = \inf_{\tau\in\mathbb{F}^X} \inf_{n\ge 0} \mathbb{E}_0^\phi \left[\int_0^\tau e^{-\lambda_0 t} \left[g\left(Y_t^{\Phi_0}\right) + \lambda_0 (Kv_n)\left(Y_t^{\Phi_0}\right) \right] \mathrm{d}t + e^{-\lambda_0 \tau} h\left(Y_\tau^{\Phi_0}\right) \right] \\
&= \inf_{\tau\in\mathbb{F}^X} \lim_{n\to\infty} \mathbb{E}_0^\phi \left[\int_0^\tau e^{-\lambda_0 t} \left[g\left(Y_t^{\Phi_0}\right) + \lambda_0 (Kv_n)\left(Y_t^{\Phi_0}\right) \right] \mathrm{d}t + e^{-\lambda_0 \tau} h\left(Y_\tau^{\Phi_0}\right) \right] \\
&= \inf_{\tau\in\mathbb{F}^X} \mathbb{E}_0^\phi \left[\int_0^\tau e^{-\lambda_0 t} \left[g\left(Y_t^{\Phi_0}\right) + \lambda_0 \left(K\left(\lim_{n\to\infty} v_n\right)\right)\left(Y_t^{\Phi_0}\right) \right] \mathrm{d}t + e^{-\lambda_0 \tau} h\left(Y_\tau^{\Phi_0}\right) \right] \\
&= (Jv_\infty)(\phi).
\end{aligned}$$

Let $v(\cdot)$ be any solution of $v(\cdot) = (Jv)(\cdot)$ such that $v(\cdot) \le h(\cdot)$. Then $v(\cdot) = (Jv)(\cdot) \le (Jh)(\cdot) = v_0(\cdot)$. Suppose that $v(\cdot) \le v_n(\cdot)$ for some $n \ge 0$. Then $v(\cdot) = (Jv)(\cdot) \le (Jv_n)(\cdot) = v_{n+1}(\cdot)$. Hence, $v(\cdot) \le v_n(\cdot)$ for every $n \ge 0$. Therefore, $v(\cdot) \le \inf_{n\ge 0} v_n(\cdot) = v_\infty(\cdot)$. □

4 The solution

Firstly, we will identify explicitly the solution of the optimal stopping problem in (3.4), namely the value function $(Jw)(\cdot)$ and an optimal stopping time $\tau \in \mathbb{F}^X$ that attains the infimum in (3.4), for every fixed Borel function $w(\cdot)$ satisfying the following assumption.

Assumption *Let $w : \mathbb{R}_+ \mapsto \mathbb{R}$ be increasing, concave, bounded between $-b$ and h of* (2.11).

Let $\psi(\cdot)$ and $\eta(\cdot)$ be increasing and decreasing solutions, respectively, of the ODE

$$\begin{aligned} 0 &= (A_0 f - \lambda_0 f)(\phi) \\ &= \frac{(\mu_1-\mu_0)^2}{2}\phi^2 f''(\phi) + (\lambda_0-\lambda_1)\phi f'(\phi) - \lambda_0 f(\phi), \quad \phi \in (0,\infty), \end{aligned} \tag{4.1}$$

where A_0 is the $(\mathbb{P}_0, \mathbb{F}^X)$-infinitesimal generator in (3.2) of Y^{Φ_0}. More precisely,

$$\psi(\phi) = \phi^{\alpha_1} \quad \text{and} \quad \eta(\phi) = \phi^{\alpha_0}, \quad \phi > 0,$$

and $\alpha_0 < 0 < 1 < \alpha_1$ are the solutions of the quadratic equation

$$0 = \frac{(\mu_1-\mu_0)^2}{2}\alpha(\alpha-1) + (\lambda_0-\lambda_1)\alpha - \lambda_0$$

or

$$0 = \alpha^2 + \left[\frac{2(\lambda_0-\lambda_1)}{(\mu_1-\mu_0)^2} - 1\right]\alpha - \frac{2\lambda_0}{(\mu_1-\mu_0)^2}.$$

With the convention $\inf\varnothing = \infty$ (which will be also be followed in the remainder of the text), let us define the hitting and exit times

$$\tau_\ell := \inf\big\{t \ge 0 \colon Y_t^{\Phi_0} = \ell\big\} \quad \text{and} \quad \tau_{\ell,r} = \inf\big\{t \ge 0;\ Y_t^{\Phi_0} \notin (\ell, r)\big\}, \quad 0 < \ell < r < \infty$$

of process Y^{Φ_0}, and the functions

$$\begin{aligned} \psi_\ell(\phi) &:= \psi(\phi) - \eta(\phi)\frac{\eta(\ell)}{\psi(\ell)} \quad \text{and} \\ \eta_r(\phi) &:= \eta(\phi) - \psi(\phi)\frac{\eta(r)}{\psi(r)}, \quad 0 < \ell < r < \infty,\ \phi > 0, \end{aligned}$$

which are the increasing and decreasing solutions, respectively, of (4.1) subject to the conditions $f(\ell) = 0$ and $f(r) = 0$, respectively. The next lemma can be proven by an application of Itô formula; see also Borodin and Salminen (1996) and Karlin and Taylor (1981).

Lemma 4.1 *For every $0 < \ell < \phi < r < \infty$, we have*

(i) $$\mathbb{E}_0^\phi\big[e^{-\lambda_0 \tau_\ell} 1_{\{\tau_\ell < \tau_r\}}\big] = \frac{\psi(\phi)\eta(r) - \psi(r)\eta(\phi)}{\psi(\ell)\eta(r) - \psi(r)\eta(\phi)} = \frac{\eta_r(\phi)}{\eta_r(\ell)},$$

(ii) $$\mathbb{E}_0^\phi\big[e^{-\lambda_0 \tau_r} 1_{\{\tau_\ell > \tau_r\}}\big] = \frac{\psi(\ell)\eta(\phi) - \psi(\phi)\eta(\ell)}{\psi(\ell)\eta(r) - \psi(r)\eta(\ell)} = \frac{\psi_\ell(\phi)}{\psi_\ell(r)},$$

(iii) $$\mathbb{E}_0^\phi\big[e^{-\lambda_0 \tau_{\ell,r}} h\big(Y_{\tau_{\ell,r}}^{\Phi_0}\big)\big] = h(\ell)\frac{\eta_r(\phi)}{\eta_r(\ell)} + h(r)\frac{\psi_\ell(\phi)}{\psi_\ell(r)}.$$

All three expectations are twice-continuously differentiable in ϕ and unique solutions of the ODE $A_0 f - \lambda_0 f = 0$, $\ell < \phi < r$ with boundary conditions (i) $f(\ell) = 1$, $f(r) = 0$, (ii) $f(\ell) = 0$, $f(r) = 1$, *and* (iii) $f(\ell) = h(\ell)$, $f(r) = h(r)$, *respectively.*

We define the drift rate $q(\cdot)$ and diffusion rate $p^2(\cdot)$ of Y^{Φ_0} in (3.1), the Wronskian $W(\cdot)$ of $\psi(\cdot)$ and $\eta(\cdot)$, and the Wronskian $W_{\ell,r}(\cdot)$ of $\psi_r(\cdot)$ and $\eta_\ell(\cdot)$ for every $0<\ell<r<\infty$ by

$$q(\phi)=(\lambda_0-\lambda_1)\phi, \qquad p^2(\phi)=(\mu_1-\mu_0)^2\phi^2,$$
$$W(\phi)=\psi'(\phi)\eta(\phi)-\psi(\phi)\eta'(\phi)=(\alpha_1-\alpha_0)\phi^{\alpha_0+\alpha_1-1},$$
$$W_{\ell,r}(\phi)=\psi_\ell'(\phi)\eta_r(\phi)-\psi_\ell(\psi)\eta_r'(\phi)=W(\phi)\left[1-\frac{\eta(r)}{\eta(\ell)}\frac{\psi(\ell)}{\psi(r)}\right],$$

and for every function $w:\mathbb{R}_+\mapsto\mathbb{R}$ and $0<\ell<r<\infty$ and $0<\ell<r<\infty$, the operators

$$(H_{\ell,r}w)(\phi):=\mathbb{E}_0^\phi\left[\int_0^{\tau_{\ell,r}} e^{-\lambda_0 t}\left[g\left(Y_t^{\Phi_0}\right)+\lambda_0(Kw)\left(Y_t^{\Phi_0}\right)\right]\mathrm{d}t+e^{-\lambda_0\tau_{\ell,r}}h\left(Y_{\tau_{\ell,r}}^{\Phi_0}\right)\right], \quad \phi>0, \tag{4.2}$$

$$(Hw)(\phi):=\mathbb{E}_0^\phi\left[\int_0^\infty e^{-\lambda_0 t}\left[g\left(Y_t^{\Phi_0}\right)+\lambda_0(Kw)\left(Y_t^{\Phi_0}\right)\right]\mathrm{d}t\right], \quad \phi>0. \tag{4.3}$$

Since $\mathbb{E}_0^\phi[Y_t^{\Phi_0}]=\mathbb{E}_0^\phi[\Phi_0\exp\{(\mu_1-\mu_0)(X_t-\mu_0 t)-[\frac{(\mu_1-\mu_0)^2}{2}+\lambda_1-\lambda_0]t\}]=\phi e^{-(\lambda_1-\lambda_0)t}$, we have

$$\begin{aligned}\left|(Hw)(\phi)\right| &\le \mathbb{E}_0^\phi\left[\int_0^\infty e^{-\lambda_0 t}\left(1+Y_t^{\Phi_0}+\lambda_0 b\right)\mathrm{d}t\right]=\frac{1+\lambda_0 b}{\lambda_0}+\int_0^\infty e^{-\lambda_0 t}\mathbb{E}_0^\phi\left[Y_t^{\Phi_0}\right]\mathrm{d}t\\ &=\frac{1+\lambda_0 b}{\lambda_0}+\phi\int_0^\infty e^{-\lambda_0 t}e^{-(\lambda_1-\lambda_0)t}\,\mathrm{d}t=\frac{1+\lambda_0 b}{\lambda_0}+\frac{\phi}{\lambda_1}<\infty, \quad \phi>0.\end{aligned} \tag{4.4}$$

Also $(H_{\ell,r}w)(\phi)<\infty$ and $(Jw)(\phi)<\infty$ for $\phi>0$, because for every $\mathbb{F}^X$-stopping time τ

$$\begin{aligned}&\left|\mathbb{E}_0^\phi\left[\int_0^\tau e^{-\lambda_0 t}\left[g+\lambda_0(Kw)\right]\left(Y_t^{\Phi_0}\right)\mathrm{d}t+e^{-\lambda_0\tau}h\left(Y_\tau^{\Phi_0}\right)\right]\right|\\ &\quad\le\mathbb{E}_0^\phi\left[\int_0^\tau e^{-\lambda_0 t}\left(1+Y_t^{\Phi_0}+\lambda_0 b\right)\mathrm{d}t+e^{-\lambda_0\tau}b\right]\\ &\quad\le\mathbb{E}_0^\phi\left[\int_0^\infty e^{-\lambda_0 t}\left(1+Y_t^{\Phi_0}\right)\mathrm{d}t+b\left(1-e^{-\lambda_0\tau}+e^{-\lambda_0\tau}\right)\right]=\frac{1}{\lambda_0}+\frac{\phi}{\lambda_1}+b<\infty.\end{aligned}$$

Lemma 4.2 *Let $k:\mathbb{R}_+\mapsto\mathbb{R}$ be a Borel function such that $|k(\phi)|\le c(1+\phi)$ for every $\phi\in\mathbb{R}_+$ for some constant $c>0$. Then $\mathbb{E}_0^\phi[\int_0^{\tau_{\ell,r}}e^{-\lambda_0 t}k(Y_t^{\Phi_0})\,\mathrm{d}t]$ equals*

$$\eta_r(\phi)\int_\ell^\phi\frac{2\psi_\ell(\xi)}{p^2(\xi)W_{\ell,r}(\xi)}k(\xi)\,\mathrm{d}\xi+\psi_\ell(\phi)\int_\phi^r\frac{2\eta_r(\xi)}{p^2(\xi)W_{\ell,r}(\xi)}k(\xi)\,\mathrm{d}\xi \tag{4.5}$$

for every $\ell<\phi<r$, which is twice-continuously differentiable on (ℓ,r), continuous on $[\ell,r]$ and unique solution of boundary value problem $(A_0 f)(\phi)-\lambda_0 f(\phi)+k(\phi)=0$ for all $\ell<\phi<r$ with $f(\ell)=f(r)=0$. Moreover,

$$\begin{aligned}&\mathbb{E}_0^\phi\left[\int_0^\infty e^{-\lambda_0 t}k\left(Y_t^{\Phi_0}\right)\mathrm{d}t\right]\\ &\quad=\lim_{\ell\downarrow 0,r\uparrow\infty}\mathbb{E}_0^\phi\left[\int_0^{\tau_{\ell,r}}e^{-\lambda_0 t}k\left(Y_t^{\Phi_0}\right)\mathrm{d}t\right]\end{aligned}$$

$$= \eta(\phi) \int_0^\phi \frac{2\psi(\xi)}{p^2(\xi)W(\xi)} k(\xi)\,\mathrm{d}\xi + \psi(\phi) \int_\phi^\infty \frac{2\eta(\xi)}{p^2(\xi)W(\xi)} k(\xi)\,\mathrm{d}\xi, \quad \phi > 0, \tag{4.6}$$

which is twice-continuously differentiable on $(0,\infty)$, *and satisfies the ODE* $(A_0 f)(\phi) - \lambda_0 f(\phi) + k(\phi)(\phi) = 0$ *for every* $\phi \in (0,\infty)$. *If the limit* $k(0+) = \lim_{\phi\downarrow 0} k(\phi)$ *exists, then*

$$\lim_{\phi\downarrow 0} \mathbb{E}_0^\phi\left[\int_0^\infty e^{-\lambda_0 t} k\big(Y_t^{\Phi_0}\big)\,\mathrm{d}t\right] \quad \textit{exists and equals } \frac{k(0+)}{\lambda_0}.$$

The proof of Lemma 4.2 is omitted here and given in the Appendix at the end. Since $|g(\phi) + \lambda_0(Kw)(\phi)| \le (1 + \phi + \lambda_0 b) \le (1 + \lambda_0 b)(1 + \phi)$ for every $\phi > 0$, we can apply Lemmas 4.1 and 4.2 to $k(\phi) = g(\phi) + \lambda_0(Kw)(\phi)$ to reach the following corollaries.

Corollary 4.1 *For every* $0 < \ell < r < \infty$, *we have* $(H_{\ell,r}w)(\phi) = h(\phi)$ *if* $\phi \notin (\ell, r)$, *and*

$$\begin{aligned}(H_{\ell,r}w)(\phi) &= \eta_r(\phi) \int_\ell^\phi \frac{2\psi_\ell(\xi)}{p^2(\xi)W_{\ell,r}(\xi)} \big[g(\xi) + \lambda_0(Kw)(\xi)\big]\,\mathrm{d}\xi \\ &\quad + \psi_\ell(\phi) \int_\phi^r \frac{2\eta_r(\xi)}{p^2(\xi)W_{\ell,r}(\xi)} \big[g(\xi) + \lambda_0(Kw)(\xi)\big]\,\mathrm{d}\xi \\ &\quad + \frac{\eta_r(\phi)}{\eta_r(\ell)} h(\ell) + \frac{\psi_\ell(\phi)}{\psi_\ell(r)} h(r) \quad \textit{if } \phi \in (\ell, r).\end{aligned} \tag{4.7}$$

The function $\phi \mapsto (H_{\ell,r}w)(\phi)$ *is the unique twice-continuously differentiable function on* $[\ell, r]$ *which solves the boundary-value problem*

$$\begin{gathered}(A_0 f)(\phi) - \lambda_0 f(\phi) + g(\phi) + \lambda_0(Kf)(\phi) = 0, \quad \phi \in (\ell, r), \\ f(\ell) = h(\ell) \quad \textit{and} \quad f(r) = h(r).\end{gathered}$$

Corollary 4.2 *We have* $(Hw)(\phi) = \lim_{\ell\downarrow 0, r\uparrow\infty} (H_{\ell,r}w)(\phi)$ *for every* $\phi > 0$, *and*

$$\begin{aligned}(Hw)(\phi) &= \eta(\phi) \int_0^\phi \frac{2\psi(\xi)}{p^2(\xi)W(\xi)} \big[g(\xi) + \lambda_0(Kw)(\xi)\big]\,\mathrm{d}\xi \\ &\quad + \psi(\phi) \int_\phi^\infty \frac{2\eta(\xi)}{p^2(\xi)W(\xi)} \big[g(\xi) + \lambda_0(Kw)(\xi)\big]\,\mathrm{d}\xi, \quad \phi > 0,\end{aligned} \tag{4.8}$$

which satisfies the ODE

$$\big(A_0(Hw)\big)(\phi) - \lambda_0(Hw)(\phi) + g(\phi) + \lambda_0\big(K(Hw)\big)(\phi) = 0, \quad \phi > 0. \tag{4.9}$$

Since $w(0+) = \lim_{\phi\downarrow 0} w(\phi)$ *exists,* $(Hw)(0+) = \lim_{\phi\downarrow 0}(Hw)(\phi)$ *exists and equals* $w(0+)/\lambda_0$.

The strong Markov property of process Y^{Φ_0} at every $\mathbb{F}^X$ stopping time τ implies that

$$(Hw)(\phi) = \mathbb{E}_0^\phi\left[\int_0^\tau e^{-\lambda_0 t}\big[g\big(Y_t^{\Phi_0}\big) + \lambda_0(Kw)\big(Y_t^{\Phi_0}\big)\big]\,\mathrm{d}t\right] + \mathbb{E}_0^\phi\big[e^{-\lambda_0\tau}(Hw)\big(Y_\tau^{\Phi_0}\big)\big], \quad \phi > 0.$$

Because $(Hw)(\phi) < \infty$ by (4.4), we have $\mathbb{E}_0^\phi[\int_0^\tau e^{-\lambda_0 t}[g(Y_t^{\Phi_0}) + \lambda_0 (Kw)(Y_t^{\Phi_0})]\,\mathrm{d}t] = (Hw)(\phi) - [e^{-\lambda_0 \tau}(Hw)(Y_\tau^{\Phi_0})]$ for every $\phi > 0$ and $\tau \in \mathbb{F}^X$, which we can substitute in $(Jw)(\phi)$ to get

$$\begin{aligned}(Jw)(\phi) &= \inf_{\tau\in\mathbb{F}^X} \mathbb{E}_0^\phi\left[\int_0^\tau e^{-\lambda_0 t}\big[g\big(Y_t^{\Phi_0}\big) + \lambda_0 (Kw)\big(Y_t^{\Phi_0}\big)\big]\,\mathrm{d}t + e^{-\lambda_0\tau} h\big(Y_\tau^{\Phi_0}\big)\right] \\ &= (Hw)(\phi) - \sup_{\tau\in\mathbb{F}^X} \mathbb{E}_0^\phi\big[e^{-\lambda_0\tau}(Hw-h)\big(Y_\tau^{\Phi_0}\big)\big], \quad \phi > 0. \end{aligned} \tag{4.10}$$

The correspondence between two optimal stopping problems in (4.10) can also be established by means of the recently published results of Cissé et al. [2012 see, e.g., Lemma 3.4]. Thus to solve $(Jw)(\cdot)$ we shall solve the optimal stopping problem

$$(Gw)(\phi) := (Hw)(\phi) - (Jw)(\phi) = \sup_{\tau\in\mathbb{F}^X} \mathbb{E}_0^\phi\big[e^{-\lambda_0\tau}(Hw-h)\big(Y_\tau^{\Phi_0}\big)\big], \quad \phi > 0 \tag{4.11}$$

by potential-theoretic direct methods of Dayanik and Karatzas (2003) and Dayanik (2008). Since

$$\lim_{\ell\downarrow 0}\psi(\ell) = 0, \qquad \lim_{\ell\downarrow 0}\eta(\ell) = \infty \quad \text{and} \quad \lim_{r\uparrow\infty}\eta(r) = 0, \qquad \lim_{r\uparrow\infty}\psi(r) = \infty, \tag{4.12}$$

both 0 and ∞ are natural boundaries for Y^{Φ_0}. Since $\alpha_0 < 0$ and $\alpha_1 > 1$, (4.4) implies that

$$\begin{aligned} 0 &\le \frac{(Hw-h)^+(\phi)}{\eta(\phi)} \le \frac{|(Hw)(\phi)| + |h(\phi)|}{\eta(\phi)} \le \left(\frac{1+\lambda_0 b}{\lambda_0} + \frac{\phi}{\lambda_1} + b\right)\phi^{-\alpha_0} \xrightarrow{\phi\downarrow 0} 0, \\ 0 &\le \frac{(Hw-h)^+(\phi)}{\psi(\phi)} \le \frac{|(Hw)(\phi)| + |h(\phi)|}{\psi(\phi)} \le \left(\frac{1+\lambda_0 b}{\lambda_0} + \frac{\phi}{\lambda_1} + b\right)\phi^{-\alpha_1} \xrightarrow{\phi\uparrow\infty} 0, \end{aligned}$$

and we have $\overline{\lim}_{\phi\downarrow 0}\frac{(Hw-h)^+(\phi)}{\eta(\phi)} = \overline{\lim}_{\phi\uparrow\infty}\frac{(Hw-h)^+(\phi)}{\psi(\phi)} = 0$. Therefore, the value function $(Gw)(\phi)$ of the optimal stopping problem in (4.11) is finite by Proposition 5.10 of Dayanik and Karatzas (2003). Because $(Hw-h)(\cdot)$ is also continuous by Corollary 4.2, Proposition 5.13 of Dayanik and Karatzas (2003) guarantees that

$$\Gamma[w] := \big\{\phi > 0;\ (Gw)(\phi) = (Hw)(\phi) - h(\phi)\big\} \equiv \big\{\phi > 0;\ (Jw)(\phi) = h(\phi)\big\} \tag{4.13}$$

is the optimal stopping region and

$$\tau[w] := \inf\big\{t \ge 0;\ Y_t^{\Phi_0} \in \Gamma[w]\big\} \tag{4.14}$$

is an optimal stopping time for the problem in (4.11)—*and* for the problem in (3.4) as well, because of the correspondence between the problems in (4.10).

We can also identify explicitly the structure of the optimal stopping region $\Gamma[w]$ of (4.13). Let us define increasing function $F : (0,\infty) \mapsto \mathbb{R}$ and operator $(Lw)(\cdot)$ on $\mathbb{R}_+$ by

$$\begin{aligned} F(\phi) &:= \frac{\psi(\phi)}{\eta(\phi)} = \phi^{\alpha_1-\alpha_0}, \quad \phi > 0 \quad \text{and} \\ (Lw)(\zeta) &:= \begin{cases} \left(\frac{Hw-h}{\eta}\right)\circ F^{-1}(\zeta), & 0 < \zeta < \infty, \\ 0, & \zeta = 0, \end{cases} \end{aligned} \tag{4.15}$$

and denote by $(Mw)(\cdot)$ the smallest nonnegative concave majorant of $(Lw)(\cdot)$ on $\mathbb{R}_+$. Then by Proposition 5.12 and Remark 5.2 of Dayanik and Karatzas (2003) and for every $\phi > 0$

$$(Gw)(\phi) = \eta(\phi)(Mw)\big(F(\phi)\big) \quad \text{and} \quad \Gamma[w] = F^{-1}\big(\big\{\zeta > 0;\ (Mw)(\zeta) = (Lw)(\zeta)\big\}\big). \tag{4.16}$$

The explicit expressions in (4.8) for $(Hw)(\cdot)$ and in (2.11) for $h(\cdot)$ reveal that

$$\begin{aligned}(Hw-h)(\phi) &= \frac{1}{\lambda_0} + \frac{\phi}{\lambda_1} + \frac{(-\alpha_0)\alpha_1}{\alpha_1-\alpha_0}\phi^{\alpha_0}\int_0^\phi \xi^{-1-\alpha_0}(Kw)(\xi)\,d\xi \\ &\quad + \frac{(-\alpha_0)\alpha_1}{\alpha_1-\alpha_0}\phi^{\alpha_1}\int_\phi^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\,d\xi + (a\phi-b)^-, \quad \phi > 0.\end{aligned}$$

Lemmas 4.3 and 4.4 below will help us identify the shape of function $(Lw)(\cdot)$, which will later allow us to describe explicitly its smallest nonnegative concave majorant $(Mw)(\cdot)$ and optimal stopping region $\Gamma[w]$, reexpressed in (4.16) in terms of $(Mw)(\cdot)$ and $(Lw)(\cdot)$.

Lemma 4.3 *If* $w(\cdot) \not\equiv 0$, *then*

(i) $$\lim_{\phi\downarrow 0}\int_0^\phi \xi^{-1-\alpha_0}(Kw)(\xi)\,d\xi = 0,$$

(ii) $$\lim_{\phi\downarrow 0}\phi^{\alpha_0}\int_0^\phi \xi^{-1-\alpha_0}(Kw)(\xi)\,d\xi = \frac{w(0+)}{(-\alpha_0)},$$

(iii) $$\lim_{\phi\downarrow 0}\int_\phi^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\,d\xi = -\infty,$$

(iv) $$\lim_{\phi\downarrow 0}\phi^{\alpha_1}\int_\phi^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\,d\xi = \frac{w(0+)}{\alpha_1},$$

(v) $$\lim_{\phi\uparrow\infty}\phi^{\alpha_0}\int_0^\phi \xi^{-1-\alpha_0}(Kw)(\xi)\,d\xi = \frac{w(\infty)}{(-\alpha_0)},$$

(vi) $$\lim_{\phi\uparrow\infty}\int_\phi^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\,d\xi = 0,$$

(vii) $$\lim_{\phi\uparrow\infty}\phi^{\alpha_1}\int_\phi^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\,d\xi = \frac{w(\infty)}{\alpha_1}.$$

The proof of the lemma can be found in the Appendix. By (i) and (iv) of Lemma 4.3, $\lim_{\phi\downarrow 0}(Hw-h)(\phi) = \frac{1}{\lambda_0} + \frac{(-\alpha_0)\alpha_1}{\alpha_1-\alpha_0}\frac{w(0+)}{\alpha_1} + b$ is finite. Because $\alpha_0 < 0$, we get

$$\lim_{\zeta\downarrow 0}(Lw)(\zeta) = \lim_{\phi\downarrow 0}\frac{(Hw)(\phi)-h(\phi)}{\eta(\phi)} = \lim_{\phi\downarrow 0}\big[(Hw)(\phi)-h(\phi)\big]\phi^{-\alpha_0} = 0.$$

On the other hand, according to Lemma 4.3 (v) and (vii), for every $\varepsilon > 0$, there is some $\phi_0 > 0$ such that for every $\phi \ge \phi_0$

$$(Hw-h)(\phi) \ge \frac{1}{\lambda_0} + \frac{\phi}{\lambda_1} + \frac{(-\alpha_0)\alpha_1}{\alpha_1-\alpha_0}\left[\frac{w(+\infty)}{(-\alpha_0)} - \frac{\varepsilon}{2}\right] + \frac{(-\alpha_0)\alpha_1}{\alpha_1-\alpha_0}\left[\frac{w(+\infty)}{\alpha_1} - \frac{\varepsilon}{2}\right]$$

$$= \frac{1}{\lambda_0} + \frac{\phi}{\lambda_1} + w(+\infty) - \varepsilon \frac{(-\alpha_0)\alpha_1}{\alpha_1 - \alpha_0} \xrightarrow{\phi \uparrow \infty} \infty.$$

Since $\alpha_0 < 0$, $\lim_{\zeta \uparrow \infty}(Lw)(\zeta) = \lim_{\phi \uparrow \infty} \frac{(Hw)(\phi) - h(\phi)}{\eta(\phi)} = \lim_{\phi \uparrow \infty}[(Hw)(\phi) - h(\phi)]\phi^{-\alpha_0} = \infty$, and

$$\begin{aligned}
(Lw)'\big(F(\phi)\big) &= \left(\frac{(Hw)(\phi) - h(\phi)}{\eta(\phi)}\right)' \frac{1}{F'(\phi)} = \frac{(-\alpha_0)}{\lambda_0(\alpha_1 - \alpha_0)}\phi^{-\alpha_1} + \frac{1 - \alpha_0}{\lambda_1(\alpha_1 - \alpha_0)}\phi^{1-\alpha_1} \\
&\quad + \frac{(-\alpha_0)\alpha_1}{\alpha_1 - \alpha_0} \int_\phi^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\,\mathrm{d}\xi \\
&\quad + \left[\frac{(-\alpha_0)b}{\alpha_1 - \alpha_0}\phi^{-\alpha_1} - \frac{(1-\alpha_0)a}{\alpha_1 - \alpha_0}\phi^{1-\alpha_1}\right] 1_{(0<b/a)}(\phi).
\end{aligned}$$

Because $\alpha_1 > 1$, Lemma 4.3(vi) implies that $\lim_{\zeta \uparrow \infty}(Lw)'(\zeta) = \lim_{\phi \uparrow \infty}(Lw)'(F(\phi)) = 0$. On the other hand, Lemma 4.3(iv) implies that for every sufficiently small $\varepsilon > 0$

$$\begin{aligned}
\lim_{\zeta \downarrow 0}(Lw)'(\zeta) &= \lim_{\phi \downarrow 0}(Lw)'\big(F(\phi)\big) \\
&= \lim_{\phi \downarrow 0} \phi^{-\alpha_1}\left[\frac{(-\alpha_0)}{\lambda_0(\alpha_1 - \alpha_0)} + \frac{(-\alpha_0)\alpha_1}{\alpha_1 - \alpha_0}\phi^{\alpha_1} \int_\phi^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\,\mathrm{d}\xi \right. \\
&\quad \left. + \frac{(-\alpha_0)b}{\alpha_1 - \alpha_0} + \frac{1 - \alpha_0}{\alpha_1 - \alpha_0}\left(\frac{1}{\lambda_1} - a\right)\phi\right] \\
&\geq \lim_{\phi \downarrow 0} \phi^{-\alpha_1}\left[\frac{(-\alpha_0)}{\lambda_0(\alpha_1 - \alpha_0)} + \frac{(-\alpha_0)\alpha_1}{\alpha_1 - \alpha_0}\left(\frac{w(0+)}{\alpha_1} - \varepsilon\right) + \frac{(-\alpha_0)b}{\alpha_1 - \alpha_0} - \varepsilon\right] \\
&\geq \lim_{\phi \downarrow 0} \phi^{-\alpha_1}\left[\frac{(-\alpha_0)}{2\lambda_0(\alpha_1 - \alpha_0)} + \frac{(-\alpha_0)}{\alpha_1 - \alpha_0}\big(w(0+) + b\big)\right] = +\infty
\end{aligned}$$

because $\alpha_0 < 0$, $\alpha_1 > 1$, and $w(0+) + b \geq 0$. Note also that

$$\begin{aligned}
&(Lw)'\left(F\left(\frac{b}{a}+\right)\right) - (Lw)'\left(F\left(\frac{b}{a}-\right)\right) \\
&= -\left[\frac{(-\alpha_0)b}{\alpha_1 - \alpha_0}\left(\frac{b}{a}\right)^{-\alpha_1} - \frac{(1-\alpha_0)a}{\alpha_1 - \alpha_0}\left(\frac{b}{a}\right)^{1-\alpha_1}\right] \\
&= -\left[\frac{(-\alpha_0)b}{\alpha_1 - \alpha_0}\left(\frac{b}{a}\right)^{-\alpha_1} - \frac{(1-\alpha_0)b}{\alpha_1 - \alpha_0}\left(\frac{b}{a}\right)^{-\alpha_1}\right] = \frac{b}{\alpha_1 - \alpha_0}\left(\frac{b}{a}\right)^{-\alpha_1} > 0,
\end{aligned}$$

which completes the proof of the next lemma.

Lemma 4.4 *We have*

(i) $\lim_{\zeta \downarrow 0}(Lw)(\zeta) = 0$, (ii) $\lim_{\zeta \uparrow \infty}(Lw)(\zeta) = +\infty$,

(iii) $\lim_{\zeta \downarrow 0}(Lw)'(\zeta) = +\infty$, (iv) $\lim_{\zeta \uparrow \infty}(Lw)'(\zeta) = 0$,

(v) $\lim_{\zeta \uparrow F(b/a)}(Lw)'(\zeta) < \lim_{\zeta \downarrow F(b/a)}(Lw)'(\zeta)$.

Let us also study the sign of the second derivative $(Lw)''(\cdot)$ of $(Lw)(\cdot)$. Dayanik and Karatzas [(2003), p. 192] showed that

$$\begin{aligned}
&(Lw)''\big(F(\phi)\big) = \frac{2\eta(\phi)}{p^2(\phi)W(\phi)F'(\phi)}\big((A_0-\lambda_0)(Hw-h)\big)(\phi), \quad && \phi \in \mathbb{R}_+ \setminus \{b/a\},\\
&\operatorname{sgn}\big[(Lw)''\big(F(\phi)\big)\big]\operatorname{sgn}\big[(A_0-\lambda_0)(Hw-h)(\phi)\big], && \phi \in \mathbb{R}_+ \setminus \{b/a\},
\end{aligned} \tag{4.17}$$

because $\eta(\cdot)$, $p^2(\cdot)$, $W(\cdot)$, and $F'(\cdot)$ are positive. By Corollary 4.2 and (2.11), we have

$$(A_0-\lambda_0)(Hw-h)(\phi) = \begin{cases} -1-\lambda_0 b-(1-a\lambda_1)\phi-\lambda_0(Kw)(\phi), & \text{if } 0<\phi<b/a,\\ -1-\phi-\lambda_0(Kw)(\phi), & \text{if } \phi>b/a, \end{cases}$$

which is convex on $(0, b/a)$ and $(b/a,\infty)$. It is also decreasing on $(b/a,\infty)$ and negative for every large enough ϕ. It has jump discontinuity at $\phi = b/a$ and $(A_0-\lambda_0)(Hw-h)(b/a+) - (A_0-\lambda_0)(Hw-h)(b/a-) = (\lambda_1-\lambda_0)b > 0$. Moreover, because $w(\cdot) \ge -b$, we have $(A_0-\lambda_0)(Hw-h)(0+) = -1-\lambda_0 w(0+)-\lambda_0 b = -1-\lambda_0[w(0+)+b] \le -1$. Therefore, $(A_0-\lambda_0)(Hw-h)(\phi)$ is negative in a nonempty open neighborhood of both $\phi = 0$ and $\phi = +\infty$ (after one-point compactification of $\mathbb{R}_+$) and changes its sign at most once in each of the intervals $(0, b/a)$ and $(b/a,\infty)$. Then (4.17) implies that $(Lw)(\zeta)$ is always strictly concave in some open nonempty neighborhood of $\zeta = 0$ and $\zeta = \infty$ and strictly convex on some bounded closed interval containing $\zeta = F(b/a)$. The function $(Lw)(\zeta)$ is strictly increasing for every $\zeta \in [F(b/a),\infty)$, because $(Lw)(\zeta) = [(Hw)/\eta](F^{-1}(\zeta))$ for every $\zeta > F(b/a)$ and $(Hw)(\cdot)$ is increasing. Since $(Lw)(0+) = 0$ and $(Lw)'(0+) = \infty$ by Lemma 4.4 (ii) and (iv), and since $(Lw)(\zeta)$ is strictly concave in some open nonempty neighborhood of $\zeta = 0$, $Lw(\zeta)$ is positive and strictly increasing in some open nonempty neighborhood of $\zeta = 0$. Finally, (v) of Lemma 4.4 implies that there are two unique numbers $0 < \zeta_1[w] < F(b/a) < \zeta_2[w] < \infty$ such that

$$(Lw)'\big(\zeta_1[w]\big) = \frac{(Lw)(\zeta_2[w]) - (Lw)(\zeta_1[w])}{\zeta_2[w]-\zeta_1[w]} = (Lw)'\big(\zeta_2[w]\big),$$

and the smallest nonnegative concave majorant $(Mw)(\cdot)$ of $(Lw)(\cdot)$ coincides with $(Lw)(\cdot)$ on $[0,\zeta_1[w]] \cup [\zeta_2[w],\infty)$ and with the straight line tangent to $(Lw)(\cdot)$ at $\zeta = \zeta_1[w]$ and $\zeta = \zeta_2[w]$ on the interval $[\zeta_1[w],\zeta_2[w]]$. Namely,

$$(Mw)(\zeta) = \begin{cases} (Lw)(\zeta), \quad \text{if } \zeta \in [0,\zeta_1[w]] \cup [\zeta_2[w],\infty),\\ \frac{\zeta_2[w]-\zeta}{\zeta_2[w]-\zeta_1[w]}(Lw)(\zeta_1[w]) + \frac{\zeta-\zeta_1[w]}{\zeta_2[w]-\zeta_1[w]}(Lw)(\zeta_2[w]),\\ \quad \text{if } \zeta \in (\zeta_1[w],\zeta_2[w]); \end{cases}$$

see Fig. 1 for the illustrations of $(Lw)(\cdot)$ and its smallest nonnegative concave majorants $(Mw)(\cdot)$. If we define

$$\begin{aligned}
\phi_1[w] &:= F^{-1}\big(\zeta_1[w]\big) = \big(\zeta_1[w]\big)^{1/(\alpha_1-\alpha_0)},\\
\phi_2[w] &:= F^{-1}\big(\zeta_2[w]\big) = \big(\xi_2[w]\big)^{1/(\alpha_1-\alpha_0)},
\end{aligned} \tag{4.18}$$

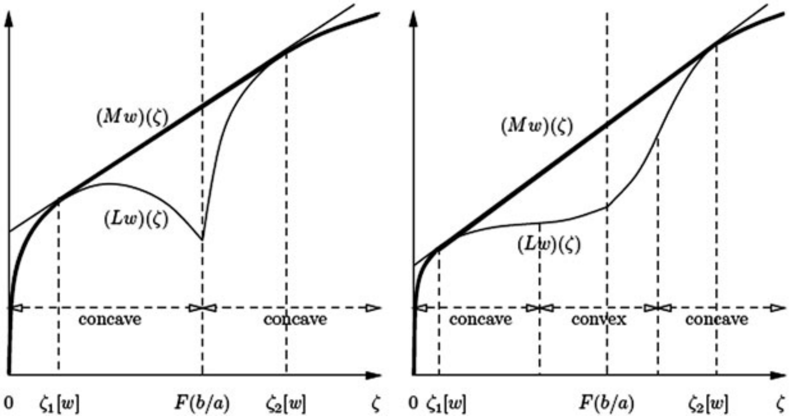


Fig. 1 Two possible forms of function $(Lw)(\cdot)$ and their smallest nonnegative concave majorants on $\mathbb{R}_+$

then (4.16) identifies value function $(Gw)(\cdot)$ of the optimal stopping problem in (4.11) by

$$(Gw)(\phi) = \begin{cases} (Hw)(\phi) - h(\phi), \quad \phi \in (0, \phi_1[w]] \cup [\phi_2[w], \infty), \\ \frac{(\phi_2[w])^{\alpha_1-\alpha_0} - \phi^{\alpha_1-\alpha_0}}{(\phi_2[w])^{\alpha_1-\alpha_0} - (\phi_1[w])^{\alpha_1-\alpha_0}} (Hw-h)(\phi_1[w]) \\ \quad + \frac{\phi^{\alpha_1-\alpha_0} - (\phi_1[w])^{\alpha_1-\alpha_0}}{(\phi_2[w])^{\alpha_1-\alpha_0} - (\phi_1[w])^{\alpha_1-\alpha_0}} (Hw-h)(\phi_2[w]), \\ \quad \phi \in (\phi_1[w], \phi_2[w]) \end{cases} \tag{4.19}$$

and the optimal stopping region $\Gamma[w]$ by

$$\Gamma[w] = \big\{\phi > 0;\ (Gw)(\phi) = (Hw)(\phi) - h(\phi)\big\} = \big(0, \phi_1[w]\big] \cup \big[\phi_2[w], \infty\big).$$

Therefore, the stopping time $\tau[w]$ in (4.14), which is optimal for the problems in both (4.11) and (3.4), is given by

$$\tau[w] = \inf\big\{t \geq 0;\ Y_t^{\Phi_0} \in \big(0, \phi_1[w]\big] \cup \big[\phi_2[w], \infty\big)\big\} \equiv \tau_{\phi_1[w], \phi_2[w]}, \tag{4.20}$$

which completes the proof of the next proposition.

Proposition 4.1 *An optimal stopping time for the problem in* (3.4) *is given by* $\tau[w] = \tau_{\phi_1[w],\phi_2[w]}$ *in* (4.20), *where* $0 < \phi_1[w] < b/a < \phi_2[w] < \infty$ *are defined by* (4.18). *For every* $\phi > 0$, *we have* $(Jw)(\phi) = (H_{\phi_1[w],\phi_2[w]}w)(\phi)$, *which can be calculated explicitly with* (4.7).

Remark 4.1 Since $\zeta \mapsto [(Hw-h)/\eta] \circ F^{-1}(\zeta) \equiv (Lw)(\zeta) = (Mw)(\zeta)$ on $(0, \zeta_1[w]] \cup [\zeta_2[w], \infty)$ is strictly concave, we have $0 > \frac{d^2}{d\zeta^2}(\frac{Hw-h}{\eta}) \circ F^{-1}(\zeta)$ for $\zeta \in (0, \zeta_1[w]] \cup [\zeta_2[w], \infty)$, and (4.17) implies that $(A_0 - \lambda_0)(Hw-h)(\phi) < 0$ for every $\phi \in (0, \phi_1[w]] \cup [\phi_2[w], \infty)$.

Remark 4.2 The value function $(Gw)(\cdot)$ of the optimal stopping problem in (4.11) is continuously differentiable on $\mathbb{R}_+$, twice continuously-differentiable on $\mathbb{R}_+ \setminus \{\phi_1[w], \phi_2[w]\}$ and satisfies the variational inequalities

$$\begin{aligned}
&\text{(i)} && (A_0-\lambda_0)(Gw)(\phi)=0, && \phi\in\big(\phi_1[w],\phi_2[w]\big),\\
&\text{(ii)} && (Gw)(\phi)>(Hw)(\phi)-h(\phi), && \phi\in\big(\phi_1[w],\phi_2[w]\big),\\
&\text{(iii)} && (A_0-\lambda_0)(Gw)(\phi)<0, && \phi\in\big(0,\phi_1[w]\big)\cup\big(\phi_2[w],\infty\big),\\
&\text{(iv)} && (Gw)(\phi)=(Hw)(\phi)-h(\phi), && \phi\in\big(0,\phi_1[w]\big]\cup\big[\phi_2[w],\infty\big),
\end{aligned}$$

where (i) and (iv) follow from (4.19), (iii) from (iv) and Remark 4.1, and (ii) from strict concavity/convexity of $(Lw)(\zeta)$ on $(\zeta_1[w],\zeta_2[w])$ and from that $(Mw)(\zeta)$ coincides with straight line which is tangent at $\zeta=\zeta_1[w]$ and $\zeta=\zeta_2[w]$ to $(Lw)(\zeta)$ and majorizes it everywhere.

Because $(Jw)(\phi)=(Hw)(\phi)-(Gw)(\phi)$ for every $\phi>0$ by (4.11) and twice continuously-differentiable function $(Hw)(\cdot)$ satisfies (4.9) by Corollary 4.2, $(Jw)(\cdot)$ is continuously differentiable on $\mathbb{R}_+$, twice continuously-differentiable on $\mathbb{R}_+\setminus\{\phi_1[w],\phi_2[w]\}$ and

$$\begin{aligned}
&(A_0-\lambda_0)(Jw)(\phi)\\
&\quad=(A_0-\lambda_0)(Hw)(\phi)-(A_0-\lambda_0)(Gw)(\phi)\\
&\quad=-\big(1+\phi+\lambda_0(Kw)(\phi)\big)-(A_0-\lambda_0)(Gw)(\phi),\quad \phi\in\mathbb{R}_+\setminus\big\{\phi_1[w],\phi_2[w]\big\}.
\end{aligned}$$

Now it immediately follows from the above variational inequalities that $(Gw)(\cdot)$ solves that $(Jw)(\cdot)$ satisfies the variational inequalities

$$\begin{aligned}
&\text{(i)} && (A_0-\lambda_0)(Jw)(\phi)+1+\phi+\lambda_0(Kw)(\phi)=0, && \phi\in\big(\phi_1[w],\phi_2[w]\big),\\
&\text{(ii)} && (Jw)(\phi)<h(\phi), && \phi\in\big(\phi_1[w],\phi_2[w]\big),\\
&\text{(iii)} && (A_0-\lambda_0)(Jw)(\phi)+1+\phi+\lambda_0(Kw)(\phi)>0, && \phi\in\big(0,\phi_1[w]\big)\cup\big(\phi_2[w],\infty\big),\\
&\text{(iv)} && (Jw)(\phi)=h(\phi), && \phi\in\big(0,\phi_1[w]\big]\cup\big[\phi_2[w],\infty\big).
\end{aligned}\tag{4.21}$$

Recall now from Lemma 3.2 that the limit $v_\infty(\cdot)=\lim_{n\to\infty}v_n(\cdot)\equiv\inf_{n\geq0}v_n(\cdot)$ of successive approximations $(v_n(\cdot))_{n\geq0}$ in (3.6) is nondecreasing, concave, and bounded between $-b$ and $h(\cdot)$; hence, it satisfies the assumption on page 109. Moreover, it is a fixed point of jump operator J by Lemma 3.3. Then by Remark 4.2 and for $w=v_\infty$, the function $(Jv_\infty)(\cdot)\equiv v_\infty(\cdot)$ is continuously differentiable on $\mathbb{R}_+$, twice continuously-differentiable on $\mathbb{R}_+\setminus\{\phi_1[w],\phi_2[w]\}$, and satisfies the variational inequalities in (4.21). More precisely,

$$\begin{aligned}
&\text{(i)} && (A_0-\lambda_0)v_\infty(\phi)+1+\phi+\lambda_0(Kv_\infty)(\phi)=0, && \phi\in\big(\phi_1[v_\infty],\phi_2[v_\infty]\big)\\
&\text{(ii)} && v_\infty(\phi)<h(\phi), && \phi\in\big(\phi_1[v_\infty],\phi_2[v_\infty]\big),\\
&\text{(iii)} && (A_0-\lambda_0)v_\infty(\phi)+1+\phi+\lambda_0(Kv_\infty)(\phi)>0, && \phi\in\big(0,\phi_1[v_\infty]\big)\cup\big(\phi_2[v_\infty],\infty\big)\\
&\text{(iv)} && v_\infty(\phi)=h(\phi), && \phi\in\big(0,\phi_1[v_\infty]\big]\cup\big[\phi_2[v_\infty],\infty\big).
\end{aligned}$$

Since (2.13) and (3.2) imply for $\phi\in\mathbb{R}_+\setminus\{\phi_1[w],\phi_2[w]\}$, $(A_0-\lambda_0)v_\infty(\phi)+\lambda_0(Kv_\infty)(\phi)$ equals

$$(\lambda_0-\lambda_1)\phi v_\infty'(\phi)+\frac{(\mu_1-\mu_0)^2}{2}\phi^2v_\infty''(\phi)+\lambda_0\left[\int_E v_\infty\left(\frac{\lambda_1}{\lambda_0}\frac{d\nu_1}{d\nu_0}(z)\phi\right)-v_\infty(\phi)\right]\nu_0(dz),$$

which is the $(\mathbb{P}_0, \mathbb{F})$-infinitesimal generator $(Av_\infty)(\phi)$ of jump-diffusion conditional odds-ratio process Φ of (2.6), the variational inequalities satisfied by $v_\infty(\cdot)$ become

$$\begin{aligned}
&\text{(i)} && Av_\infty(\phi) + 1 + \phi = 0, && \phi \in \big(\phi_1[v_\infty], \phi_2[v_\infty]\big),\\
&\text{(ii)} && v_\infty(\phi) < h(\phi), && \phi \in \big(\phi_1[v_\infty], \phi_2[v_\infty]\big),\\
&\text{(iii)} && Av_\infty(\phi) + 1 + \phi > 0, && \phi \in \big(0, \phi_1[v_\infty]\big) \cup \big(\phi_2[v_\infty], \infty\big),\\
&\text{(iv)} && v_\infty(\phi) = h(\phi), && \phi \in \big(0, \phi_1[v_\infty]\big] \cup \big[\phi_2[v_\infty], \infty\big).
\end{aligned} \tag{4.22}$$

Let us denote the first exit time of Φ from the interval (ℓ, r) by

$$\begin{aligned}
&\widetilde{\tau}_{\ell,r} := \inf\big\{t \ge 0;\ \Phi_t \in \big(0, \ell\big] \cup [r, \infty)\big\}, \quad 0 \le \ell < r \le \infty, \quad \text{and,}\\
&\widetilde{\tau}[w] := \widetilde{\tau}_{\phi_1[w],\phi_2[w]} \quad \text{for every } w : \mathbb{R}_+ \mapsto \mathbb{R} \text{ satisfying the assumption on page 109.}
\end{aligned}$$

Applying Itô's rule to $w(\cdot) = v_\infty(\cdot)$ as in (2.13) gives

$$\begin{aligned}
&v_\infty(\Phi_{t\wedge\tau\wedge\widetilde{\tau}_{\ell,r}})\\
&\quad = v_\infty(\phi) + \int_0^{t\wedge\tau\wedge\widetilde{\tau}_{\ell,r}} (Av_\infty)(\Phi_s)\,\mathrm{d}s + \int_0^{t\wedge\tau\wedge\widetilde{\tau}_{\ell,r}} (\mu_1 - \mu_0)\Phi_s v'_\infty(\Phi_s)(\mathrm{d}X_s - \mu_0\,\mathrm{d}t)\\
&\qquad + \int_0^{t\wedge\tau\wedge\widetilde{\tau}_{\ell,r}} \int_E \left[v_\infty\left(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(z)\Phi_{s-}\right) - v_\infty(\Phi_{s-})\right]\big[p(\mathrm{d}s \times \mathrm{d}z) - \lambda_0\nu_0(\mathrm{d}z)\,\mathrm{d}s\big]
\end{aligned}$$

for every $t \ge 0$ and $\mathbb{F}$-stopping time τ, where both stochastic integrals are square-integrable $(\mathbb{P}_0, \mathbb{F})$-martingales because

$$\begin{aligned}
&\mathbb{E}_0^\phi\left[\int_0^{t\wedge\tau\wedge\widetilde{\tau}_{\ell,r}} (\mu_1 - \mu_0)^2\Phi_s^2\big(v'_\infty(\Phi_s)\big)^2\,\mathrm{d}s\right] \le t(\mu_1 - \mu_0)^2 r^2 \max_{\phi\in[\ell,r]}\big(v'_\infty(\phi)\big)^2 < \infty, \quad t \ge 0,\\
&\mathbb{E}_0^\phi\left[\int_0^{t\wedge\tau\wedge\widetilde{\tau}_{\ell,r}} \int_E \left|v_\infty\left(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(z)\Phi_{s-}\right) - v_\infty(\Phi_{s-})\right|^2 \lambda_0\nu_0(\mathrm{d}z)\,\mathrm{d}s\right] \le 4b^2\lambda_0 t < \infty, \quad t \ge 0.
\end{aligned}$$

Therefore, taking expectations of both sides leads to

$$\begin{aligned}
\mathbb{E}_0^\phi\big[h(\Phi_{t\wedge\tau\wedge\widetilde{\tau}_{\ell,r}})1_{\{\tau<\infty\}}\big] &\ge \mathbb{E}_0^\phi\big[v_\infty(\Phi_{t\wedge\tau\wedge\widetilde{\tau}_{\ell,r}})\big]\\
&= v_\infty(\phi) + \mathbb{E}_0^\phi\left[\int_0^{t\wedge\tau\wedge\widetilde{\tau}_{\ell,r}} (Av_\infty)(\Phi_s)\,\mathrm{d}s\right]\\
&\ge v_\infty(\infty) - \mathbb{E}_0^\phi\left[\int_0^{t\wedge\tau\wedge\widetilde{\tau}_{\ell,r}} (1 + \Phi_s)\,\mathrm{d}s\right]
\end{aligned} \tag{4.23}$$

for every $t \ge 0$, $\mathbb{F}$-stopping time τ, and $0 < \ell < r < \infty$, where the inequalities follow from (4.22). Note that $\widetilde{\tau}_{\ell,r} \to \infty$ almost surely as $\ell \downarrow 0$ and $r \uparrow \infty$, $h(\cdot)$ in (2.11) is continuous and bounded, and $1 + \Phi_s \ge 0$ for every $s \ge 0$. Firstly taking limits in (4.23) as $\ell \downarrow 0$, $r \uparrow \infty$, and $t \uparrow \infty$, and then the bounded convergence theorem on the left-hand side and the monotone convergence theorem on the right-hand side give $\mathbb{E}_0^\phi[h(\Phi_\tau)1_{\{\tau<\infty\}}] \ge v_\infty(\phi) - \mathbb{E}_0^\phi[\int_0^\tau (1 + \Phi_s)\,\mathrm{d}s]$ for every $\mathbb{F}$-stopping time τ. Rearranging the terms and taking infimum over all $\mathbb{F}$-stopping times yield $v_\infty(\phi) \le \inf_{\tau\in\mathbb{F}} \mathbb{E}_0^\phi[\int_0^\tau (1 + \Phi_s)\,\mathrm{d}s + 1_{\{\tau<\infty\}}h(\Phi_\tau)] \equiv V(\phi)$ for every $\phi > 0$.

Lemma 4.5 *For every* $0 < \ell < r < \infty$ *and integer* $k > 0$, *we have* $\sup_{\phi\in[\ell,r]} \mathbb{E}_0^{\phi} \widetilde{\tau}_{\ell,r}^k < \infty$.

The lemma, the proof of which can be found in the Appendix, implies that $\mathbb{P}_0^{\phi}\{\widetilde{\tau}[v_\infty] < \infty\} = 1$ for every $\phi > 0$. If we replace τ and $\widetilde{\tau}_{\ell,r}$ in the equality in (4.23) with $\widetilde{\tau}[v_\infty] \equiv \widetilde{\tau}_{\phi_1[v_\infty],\phi_2[v_\infty]}$, the equality becomes $\mathbb{E}_0^{\phi}[v_\infty(\Phi_{t\wedge\widetilde{\tau}[v_\infty]})] = v_\infty(\phi) + \mathbb{E}_0^{\phi}[\int_0^{t\wedge\widetilde{\tau}[v_\infty]}(Av_\infty)(\Phi_s)\,\mathrm{d}s]$ $= v_\infty(\phi) - \mathbb{E}_0^{\phi}[\int_0^{t\wedge\widetilde{\tau}[v_\infty]}(1+\Phi_s)\,\mathrm{d}s]$ for every $t \ge 0$. After taking the limits of both sides as $t \to \infty$, the bounded and monotone convergence theorems give that $\mathbb{E}_0^{\phi}[v_\infty(\Phi_{\widetilde{\tau}[v_\infty]})] = v_\infty(\phi) - \mathbb{E}_0^{\phi}[\int_0^{\widetilde{\tau}[v_\infty]}(1+\Phi_s)\,\mathrm{d}s]$, and rearranging the terms leads to $v_\infty(\phi) = \mathbb{E}_0^{\phi}[\int_0^{\widetilde{\tau}[v_\infty]}(1+\Phi_s)\,\mathrm{d}s + 1_{\{\widetilde{\tau}[v_\infty]<\infty\}} h_\infty(\Phi_{\widetilde{\tau}[v_\infty]})] \ge V(\phi)$ for all $\phi > 0$, since on $\{\widetilde{\tau}[v_\infty] < \infty\}$ we have $\Phi_{\widetilde{\tau}[v_\infty]} \in (0, \phi_1[v_\infty]] \cup [\phi_2[v_\infty], \infty)$, where $v_\infty(\cdot)$ coincides with $h(\cdot)$ by (4.22). The reverse inequality shown before Lemma 4.5 and this complete the proof of the following main result.

Proposition 4.2 *For every* $\phi > 0$, *we have* $v_\infty(\phi) = V(\phi)$, *and* $\widetilde{\tau}[v_\infty] = \widetilde{\tau}_{\phi_1[v_\infty],\phi_2[v_\infty]} = \inf\{t \ge 0;\ \Phi_t \in (0, \phi_1[v_\infty]] \cup [\phi_2[v_\infty], \infty)\}$ *is optimal for the problem in* (2.10).

We will next show that $v_\infty(\cdot) \equiv V(\cdot)$ is the uniform limit of successive approximations $(v_n(\cdot))_{n\ge 0}$ in (3.6) with an explicit error bound, which leads to an efficient numerical algorithm. Let us denote by $\underline{w} : \mathbb{R}_+ \mapsto \mathbb{R}$ the constant function

$$\underline{w}(\phi) = -b \quad \text{for every } \phi > 0.$$

The function $\underline{w}(\cdot)$ is concave, nondecreasing, and bounded between $-b$ and $h(\cdot)$; namely, it satisfies the assumption on page 109. Therefore, by Proposition 4.1 there are numbers $0 < \phi_1[\,\underline{w}\,] < b/a < \phi_2[\,\underline{w}\,] < \infty$ such that

$$(J\underline{w})(\phi) = \mathbb{E}_0^{\phi}\left[\int_0^{\tau[\,\underline{w}\,]} e^{-\lambda_0 t}\big(1 + Y_t^{\Phi_0} + \lambda_0 (K\underline{w})\big(Y_t^{\Phi_0}\big)\big)\,\mathrm{d}t + e^{-\lambda_0 \tau[\,\underline{w}\,]} h\big(Y_{\tau[\,\underline{w}\,]}^{\Phi_0}\big)\right], \quad \phi > 0,$$

where $\tau[\,\underline{w}\,] = \tau_{\phi_1[\,\underline{w}\,],\phi_2[\,\underline{w}\,]} = \inf\{t \ge 0;\ Y_t^{\Phi_0} \in (0, \phi_1[\,\underline{w}\,]] \cup [\phi_2[\,\underline{w}\,], \infty)\}$ is an optimal stopping rule for $(J\underline{w})(\cdot)$. We denote by $\|f\|$ the sup-norm $\sup_{\phi\in\mathbb{R}_+} |f(\phi)|$ of any function $f : \mathbb{R}_+ \mapsto \mathbb{R}$.

Proposition 4.3 *Let* $w_1(\cdot) \le w_2(\cdot)$ *be any two functions satisfying the assumption on page* 109. *Then we have* $0 < \phi_1[\,\underline{w}\,] \le \phi_1[w_1] \le \phi_1[w_2] \le b/a \le \phi_2[w_2] \le \phi_2[w_1] \le \phi_2[\,\underline{w}\,] < \infty$, *and*

$$\|Jw_1 - Jw_2\| \le \beta\|w_1 - w_2\|, \quad \textit{where} \quad \beta := 1 - \left(\frac{\phi_1[\,\underline{w}\,]}{\phi_2[\,\underline{w}\,]}\right)^{(-\alpha_0)\wedge\alpha_1} \in (0,1).$$

Hence, J *is a contraction mapping acting on functions satisfying the assumption on page* 109.

Proof Lemma 3.1 implies that $(Jw_1)(\cdot) \le (Jw_2)(\cdot)$, and $(0, \phi_1[w_1]] \cup [\phi_2[w_1], \infty) = \Gamma[w_1] = \{\phi > 0;\ (Jw_1)(\phi) \ge h(\phi)\} \subseteq \{\phi > 0;\ (Jw_2)(\phi) \ge h(\phi)\} = \Gamma[w_2] = (0, \phi_1[w_2]]$ $\cup\, [\phi_2[w_2], \infty)$. Therefore, $0 < \phi_1[w_1] \le \phi_1[w_2] \le b/a \le \phi_2[w_2] \le \phi_2[w_1] < \infty$. Because we also have $\underline{w}(\cdot) \le w_1(\cdot)$ and $\underline{w}(\cdot) \le w_2(\cdot)$, we obtain by the same reasoning that $0 < \phi_1[\,\underline{w}\,] \le \phi_1[w_1] \le \phi_1[w_2] \le b/a \le \phi_2[w_2] \le \phi_2[w_1] \le \phi_2[\,\underline{w}\,] < \infty$, which implies that the optimal stopping times $\tau[\,\underline{w}\,]$, $\tau[w_1]$, and $\tau[w_2]$, respectively, for the problems $(J\underline{w})(\phi)$,

$(Jw_1)(\phi)$, and $(Jw_2)(\phi)$ are ordered as $\tau[\underline{w}] \geq \tau[w_1] \geq \tau[w_2]$ almost surely. For every $\phi > 0$ observe that

$$\begin{aligned}
&(Jw_1)(\phi) - (Jw_2)(\phi) \\
&\quad\leq \mathbb{E}_0^\phi\left[\int_0^{\tau[w_2]} e^{-\lambda_0 t}\left[g\left(Y_t^{\Phi_0}\right) + \lambda_0 (Kw_1)\left(Y_t^{\Phi_0}\right)\right] \mathrm{d}t + e^{-\lambda_0 \tau[w_2]} h\left(Y_{\tau[w_2]}^{\Phi_0}\right)\right] \\
&\qquad - \mathbb{E}_0^\phi\left[\int_0^{\tau[w_2]} e^{-\lambda_0 t}\left[g\left(Y_t^{\Phi_0}\right) + \lambda_0 (Kw_2)\left(Y_t^{\Phi_0}\right)\right] \mathrm{d}t + e^{-\lambda_0 \tau[w_2]} h\left(Y_{\tau[w_2]}^{\Phi_0}\right)\right] \\
&\quad= \mathbb{E}_0^\phi\left[\int_0^{\tau[w_2]} e^{-\lambda_0 t}\left[g\left(Y_t^{\Phi_0}\right) + \lambda_0 \left(K(w_1 - w_2)\right)\left(Y_t^{\Phi_0}\right)\right] \mathrm{d}t\right] \\
&\quad= \|w_1 - w_2\|\, \mathbb{E}_0^\phi\left[\int_0^{\tau[w_2]} \lambda_0 e^{-\lambda_0 t}\, \mathrm{d}t\right] = \|w_1 - w_2\| \left(1 - \mathbb{E}_0^\phi\left[e^{-\lambda_0 \tau[w_2]}\right]\right) \\
&\quad\leq \|w_1 - w_2\|\left(1 - \mathbb{E}_0^\phi\left[e^{-\lambda_0 \tau[\underline{w}]}\right]\right).
\end{aligned} \tag{4.24}$$

Because for every $\phi \in (\phi_1[\underline{w}], \phi_2[\underline{w}])$ Lemma 4.1 implies that

$$\begin{aligned}
\mathbb{E}_0^\phi\left[e^{-\lambda_0 \tau[\underline{w}]}\right] &\geq \mathbb{E}_0^\phi\left[e^{-\lambda_0 \tau_{\phi_1[\underline{w}]}}\right] \geq \mathbb{E}_0^{\phi_2[\underline{w}]}\left[e^{-\lambda_0 \tau_{\phi_1[\underline{w}]}}\right] = \frac{\eta(\phi_2[\underline{w}])}{\eta(\phi_1[\underline{w}])} = \left(\frac{\phi_2[\underline{w}]}{\phi_1[\underline{w}]}\right)^{\alpha_0}, \\
\mathbb{E}_0^\phi\left[e^{-\lambda_0 \tau[\underline{w}]}\right] &\geq \mathbb{E}_0^\phi\left[e^{-\lambda_0 \tau_{\phi_2[\underline{w}]}}\right] \geq \mathbb{E}_0^{\phi_1[\underline{w}]}\left[e^{-\lambda_0 \tau_{\phi_2[\underline{w}]}}\right] = \frac{\psi(\phi_1[\underline{w}])}{\psi(\phi_2[\underline{w}])} = \left(\frac{\phi_1[\underline{w}]}{\phi_2[\underline{w}]}\right)^{\alpha_1},
\end{aligned}$$

and $\mathbb{E}_0^\phi[e^{-\lambda_0 \tau[\underline{w}]}] = 1$ for every $\phi \in (0, \phi_1[\underline{w}]] \cup [\phi_2[\underline{w}], \infty)$, we get

$$\mathbb{E}_0^\phi\left[e^{-\lambda_0 \tau[\underline{w}]}\right] \geq \max\left\{\left(\frac{\phi_2[\underline{w}]}{\phi_1[\underline{w}]}\right)^{\alpha_0}, \left(\frac{\phi_1[\underline{w}]}{\phi_2[\underline{w}]}\right)^{\alpha_1}\right\} = \left(\frac{\phi_1[\underline{w}]}{\phi_2[\underline{w}]}\right)^{(-\alpha_0)\wedge\alpha_1}.$$

Therefore, it follows from (4.24) that $(Jw_1)(\phi) - (Jw_2)(\phi) \leq \beta\|w_1 - w_2\|$ for every $\phi > 0$. Interchanging $w_1(\cdot)$ and $w_2(\cdot)$ gives the opposite inequality, which completes the proof. □

Corollary 4.3 *The successive approximation $v_n(\phi)$ in (3.6) converges to $v_\infty(\phi)$ as $n \to \infty$ uniformly in $\phi > 0$. More precisely, $0 \leq v_n(\phi) - v_\infty(\phi) \leq \beta^n b$ for every $\phi > 0$ and $n \geq 0$, where $0 < \beta < 1$ is defined as in Proposition 4.3.*

Proof By Lemma 3.2, $v_n(\cdot)$ for every $n \geq 0$ and $v_\infty(\cdot)$ satisfy the assumption on page 109, and $0 \leq v_n(\phi) - v_\infty(\phi)$ for every $\phi > 0$ and $n \geq 0$. Because $(Jv_{n-1})(\cdot) = v_n(\cdot)$ by definition and $(Jv_\infty)(\cdot) = v_\infty(\cdot)$ by Lemma 3.3, Proposition 4.3 implies that $\|v_n - v_\infty\| = \|Jv_{n-1} - Jv_\infty\| \leq \beta\|v_{n-1} - v_\infty\| \leq \cdots \leq \beta^n\|v_0 - v_\infty\| \leq \beta^n b$. □

Corollary 4.4 *The optimal stopping regions $\Gamma[v_n] = \{\phi > 0;\ (Jv_n)(\phi) \geq h(\phi)\} = (0, \phi_1[v_n]] \cup [\phi_2[v_n], \infty)$, $n \in \{0, 1, \ldots\} \cup \{\infty\}$ are decreasing: $\Gamma[v_0] \supseteq \Gamma[v_1] \supseteq \cdots \supseteq \Gamma[v_\infty]$, and $0 < \phi_1[\underline{w}] \leq \phi_1[v_\infty] \leq \cdots \leq \phi_1[v_1] \leq \phi_1[v_0] \leq b/a \leq \phi_2[v_0] \leq \phi_2[v_1] \leq \cdots \leq \phi_2[v_\infty] \leq \phi_2[\underline{w}] < \infty$. Moreover, $\phi_1[v_\infty] = \lim_{n\to\infty} \downarrow \phi_1[v_n]$ and $\phi_2[v_\infty] = \lim_{n\to\infty} \uparrow \phi_2[v_n]$.*

Proof Because $\underline{w}(\cdot) \leq v_\infty(\cdot) \leq \cdots \leq v_1(\cdot) \leq v_0(\cdot)$, the monotonicity of the optimal stopping regions and optimal decision boundaries follow from Proposition 4.3. Because $(\phi_1[v_n])_{n\geq 0}$ is decreasing, and $(\phi_2[v_n])_{n\ge 0}$ is increasing, the limits $\lim_{n\to\infty}\phi_1[v_n]$ and $\lim_{n\to\infty}\phi_2[v_n]$ exist, and $\phi_1[v_\infty]\le \lim_{n\to\infty}\phi_1[v_n]$ and $\phi_2[v_\infty]\ge \lim_{n\to\infty}\phi_2[v_n]$. For the proof of the reverse inequalities, note that Corollary 4.3 implies that

$$\begin{aligned}(Jv_\infty)\Big(\lim_{n\to\infty}\phi_i[v_n]\Big) &= v_\infty\Big(\lim_{n\to\infty}\phi_i[v_n]\Big)\ge v_{k+1}\Big(\lim_{n\to\infty}\phi_i[v_n]\Big)-b\beta^{k+1}\\ &=(Jv_k)\Big(\lim_{n\to\infty}\phi_i[v_n]\Big)-b\beta^{k+1}\\ &=h\Big(\lim_{n\to\infty}\phi_i[v_n]\Big)-b\beta^{k+1}\quad \text{for every } k\ge 0 \text{ and } i=1,2,\end{aligned}$$

since $\lim_{n\to\infty}\phi_1[v_n]\le\phi_1[v_k]$, $\lim_{n\to\infty}\phi_2[v_n]\ge\phi_2[v_k]$, and $\lim_{n\to\infty}\phi_i[v_n]\in\Gamma[v_k]=(0,\phi_1[v_k]]\cup[\phi_2[v_k],\infty)=\{\phi>0;\ (Jv_k)(\phi)=h(\phi)\}$ for $i=1,2$. Because $0<\beta<1$, taking limits of both sides as $k\to\infty$ leads to $(Jv_\infty)(\lim_{n\to\infty}\phi_i[v_n])\ge h(\lim_{n\to\infty}\phi_i[v_n])$ for $i=1,2$. Hence, $\lim_{n\to\infty}\phi_i[v_n]$ belongs to $\Gamma[v_\infty]=(0,\phi_1[v_\infty]]\cup[\phi_2[v_\infty],\infty)$ for every $i=1,2$, which implies $\phi_1[v_\infty]\ge\lim_{n\to\infty}\phi_1[v_n]$ and $\phi_2[v_\infty]\le\lim_{n\to\infty}\phi_2[v_n]$, since $\phi_1[v_\infty]\le b/a\le\phi_2[v_\infty]$ and $\lim_{n\to\infty}\phi_1[v_n]\le b/a\le\lim_{n\to\infty}\phi_2[v_n]$ because of the first part of Corollary 4.4. □

Proposition 4.4 *For every $n\ge 1$ and $\phi>0$, we have $V(\phi)\le\mathbb{E}_0^\phi[\int_0^{\widetilde\tau[v_n]}(1+\Phi_t)\,dt+1_{\{\widetilde\tau[v_n]<\infty\}}h(\Phi_{\widetilde\tau[v_n]})]\le V(\phi)+b\beta^{n+1}$. Therefore, for every $\varepsilon>0$ and integer $n\ge 1$ such that $b\beta^{n+1}\le\varepsilon$, the stopping time $\widetilde\tau[v_n]$ is ε-optimal for the auxiliary problem in (2.10).*

Proof The first inequality follows from the definition of $V(\cdot)$ in (2.10). For the proof of the second inequality, recall from Proposition 4.2 that $V(\cdot)\equiv v_\infty(\cdot)$. If we replace τ and $\widetilde\tau_{\ell,r}$ in the equality in (4.23) with $\widetilde\tau[v_n]=\widetilde\tau_{\phi_1[v_n],\phi_2[v_n]}$, then the equality becomes $\mathbb{E}_0^\phi[v_\infty(\Phi_{t\wedge\widetilde\tau[v_n]})]=v_\infty(\phi)+\mathbb{E}_0^\phi[\int_0^{t\wedge\widetilde\tau[v_n]}(Av_\infty)(\Phi_s)\,ds]=v_\infty(\phi)-\mathbb{E}_0^\phi[\int_0^{t\wedge\widetilde\tau[v_n]}(1+\Phi_s)\,ds]$ for every $t\ge 0$, where the second equality follows from that $\Gamma[v_n]\supseteq\Gamma[v_\infty]$ and $\widetilde\tau[v_n]\le\widetilde\tau[v_\infty]$ a.s. by Corollary 4.4, and that $\Phi_t\in(\phi_1[v_\infty],\phi_2[v_\infty])$ and $(Av_\infty)(\Phi_t)=-(1+\Phi_t)$ for every $0\le t\le\widetilde\tau[v_n]$ by (i) of (4.22). Since $\widetilde\tau[v_n]=\widetilde\tau_{\phi_1[v_n],\phi_2[v_n]}$ is finite a.s. by Lemma 4.5, after taking limits of both sides as $t\to\infty$, the bounded and monotone convergence theorems give that $\mathbb{E}_0^\phi[v_\infty(\Phi_{\widetilde\tau[v_n]})]=v_\infty(\phi)-\mathbb{E}_0^\phi[\int_0^{\widetilde\tau[v_n]}(1+\Phi_s)\,ds]$, and Corollary 4.3 implies $v_\infty(\phi)\ge\mathbb{E}_0^\phi[\int_0^{\widetilde\tau[v_n]}(1+\Phi_s)\,ds+1_{\{\widetilde\tau[v_n]<\infty\}}h(\Phi_{\widetilde\tau[v_n]})]-b\beta^{n+1}$ for every $\phi>0$ since $\Phi_{\widetilde\tau[v_n]}$ belongs almost surely to $\Gamma[v_n]$, on which $v_{n+1}(\cdot)\equiv(Jv_n)(\cdot)=h(\cdot)$ by (4.13). □

Corollary 4.5 *The decision rule $(\widetilde\tau[v_\infty],d(\widetilde\tau[v_\infty]))$, with $d(\cdot)$ as in (2.8), is Bayes optimal for the Bayesian sequential binary hypothesis testing problem in (2.1); namely, $U(\pi)=R_{\widetilde\tau[v_\infty],d(\widetilde\tau[v_\infty])}(\pi)$ for every $\pi\in(0,1)$. We also have $U(\pi)\le R_{\widetilde\tau[v_n],d(\widetilde\tau[v_n])}(\pi)\le U(\pi)+b\beta^{n+1}$ for every $\pi\in(0,1)$ and $n\ge 1$. Therefore, for every $\varepsilon>0$ and integer $n\ge 1$ such that $b\beta^{n+1}<\varepsilon$, the decision rule $(\widetilde\tau[v_n],d(\widetilde\tau[v_n]))$ is Bayes ε-optimal for the problem in (2.1).*

Proof By (2.9) and Proposition 4.2, $U(\pi)=b(1-\pi)+(1-\pi)v_\infty(\frac{\pi}{1-\pi})$ equals $b(1-\pi)+(1-\pi)\,\mathbb{E}_0^{\frac{\pi}{1-\pi}}[\int_0^{\widetilde\tau[v_\infty]}(1+\Phi_t)\,dt+1_{\{\widetilde\tau[v_\infty]<\infty\}}h(\Phi_{\widetilde\tau[v_\infty]})]=R_{\widetilde\tau[v_\infty],d(\widetilde\tau[v_\infty])}(\pi)$. On the other hand, $(\widetilde\tau[v_n],d(\widetilde\tau[v_n]))$ is admissible, and Proposition 2.1 implies $U(\pi)\le R_{\widetilde\tau[v_n],d(\widetilde\tau[v_n])}=b(1-\pi)+(1-\pi)\,\mathbb{E}_0^{\frac{\pi}{1-\pi}}[\int_0^{\widetilde\tau[v_n]}(1+\Phi_t)\,dt+1_{\{\widetilde\tau[v_n]<\infty\}}h(\Phi_{\widetilde\tau[v_n]})]\le b(1-\pi)+(1-\pi)[v_\infty(\frac{\pi}{1-\pi})+b\beta^{n+1}]\le U(\pi)+b\beta^{n+1}$ for every $\pi\in(0,1)$, where the first inequality follows from Proposition 4.4. □

5 Bayesian sequential binary hypothesis testing for several independent Brownian motions and compound Poisson processes

Let us now turn to the multisource Bayesian sequential binary hypothesis testing problem in (1.1) described in the introduction. The problem is to find a decision rule (τ, d), consisting of a stopping rule τ of the observation filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ with

$$\begin{aligned}\mathcal{F}_t &:= \sigma\big\{X_s^{(i)};\ 0 \leq s \leq t,\ 1 \leq i \leq d\big\}\\ &\quad \vee \sigma\big\{\big(T_n^{(j)}, Z_n^{(j)}\big);\ 0 < T_n^{(j)} \leq t,\ n \geq 1,\ 1 \leq j \leq m\big\}, \quad t \geq 0,\end{aligned} \tag{5.1}$$

and a $\{0, 1\}$-valued $\mathcal{F}_\tau$-measurable random variable d, with the smallest Bayes risk $R_{\tau,d}(\pi)$ in (1.2). As in the model on page 103, we start on $(\Omega, \mathcal{F}, \mathbb{P}_0)$ with some auxiliary probability measure $\mathbb{P}_0$, under which the following stochastic elements are independent: (i) $X^{(i)}$, $1 \leq i \leq d$ are independent Brownian motions with drifts rate $\mu_0^{(i)}$, $1 \leq i \leq d$; (ii) $(T_n^{(j)}, Z_n^{(j)})_{n\geq 1}$, $1 \leq j \leq m$ are independent compound Poisson processes with arrival rates $\lambda_0^{(j)}$, $1 \leq j \leq m$ and mark distributions $\nu_0^{(j)}$, $1 \leq j \leq m$; (iii) Θ is a Bernoulli random variable with success probability $\pi \in (0, 1)$. The filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ obtained by enlarging the observation filtration $\mathbb{F}$ with the information about Θ; i.e., $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\Theta)$ for every $t \geq 0$, remains the same, except that $\mathcal{F}_t$ is now given by (5.1). Finally, true probability measure $\mathbb{P}$ is obtained from $\mathbb{P}_0$ after a change of measure with the same Radon–Nikodym derivative ξ_t, $t \geq 0$ in (2.5), except that the likelihood-ratio process in (2.4) becomes

$$\begin{aligned}L_t = \exp&\left\{\sum_{i=1}^{d}\big(\mu_1^{(i)} - \mu_0^{(i)}\big)\big(X_t^{(i)} - X_0^{(i)} - \mu_0^{(i)} t\big) - \frac{t}{2}\sum_{i=1}^{d}\big(\mu_1^{(i)} - \mu_0^{(i)}\big)^2\right\}\\ &\times \exp\left\{-\left(\sum_{j=1}^{m}\lambda_1^{(j)} - \sum_{j=1}^{m}\lambda_0^{(j)}\right)t + \sum_{j=1}^{m}\sum_{0<T_n^{(j)}\leq t}\log\left(\frac{\lambda_1^{(j)}}{\lambda_0^{(j)}}\frac{d\nu_1^{(j)}}{d\nu_0^{(j)}}\big(Z_n^{(j)}\big)\right)\right\}.\end{aligned}$$

Let us define for every $t \geq 0$,

$$X_t := \frac{\sum_{i=1}^{d}(\mu_1^{(i)} - \mu_0^{(i)})X_t^{(i)}}{\sqrt{\sum_{i=1}^{d}(\mu_1^{(i)} - \mu_0^{(i)})^2}}, \qquad \mu_0 := \frac{\sum_{i=1}^{d}(\mu_1^{(i)} - \mu_0^{(i)})\mu_0^{(i)}}{\sqrt{\sum_{i=1}^{d}(\mu_1^{(i)} - \mu_0^{(i)})^2}},$$

$$\mu_1 := \frac{\sum_{i=1}^{d}(\mu_1^{(i)} - \mu_0^{(i)})\mu_1^{(i)}}{\sqrt{\sum_{i=1}^{d}(\mu_1^{(i)} - \mu_0^{(i)})^2}}.$$

Then $\mu_1 - \mu_0 = \sqrt{\sum_{i=1}^{d}(\mu_1^{(i)} - \mu_0^{(i)})^2}$, and $\mu_0 \neq \mu_1$ if and only if $\mu_0^{(i)} \neq \mu_1^{(i)}$ for some $1 \leq i \leq d$. Moreover, X is a Brownian motion whose drift rate μ equals μ_0 under H_0 and μ_1 under H_1.

In the meantime, let $(T_n, Z_n)_{n\geq 1}$ be the new point process on the extended mark space $\widetilde{E} := \{1, \ldots, m\} \times E$ obtained by the superposition of point processes $(T_n^{(j)}, Z_n^{(j)})_{n\geq 1}$, $1 \leq j \leq m$. More precisely, $(T_n)_{n\geq 1}$ is obtained by relabeling the superposition of $(T_n^{(j)})_{n\geq 1}$, $1 \leq j \leq m$ in the increasing order, and

$$Z_n := \big(j, Z_k^{(j)}\big) \quad \text{if } T_n = T_k^{(j)} \text{ for some } 1 \leq j \leq m \text{ and } k \geq 1.$$

Then $(T_n, Z_n)_{n\geq 1}$ is a compound Poisson process with arrival rate $\lambda = \sum_{j=1}^m \lambda^{(j)}$ and mark distribution $\nu(\{j\} \times A) = [\lambda^{(j)}/\lambda]\nu^{(j)}(A)$ for $1 \leq j \leq m$ and $A \in \mathcal{E}$, which equal

$$\lambda_0 := \sum_{j=1}^m \lambda_0^{(j)} \quad \text{and} \quad \nu_0\big(\{j\} \times A\big) := \frac{\lambda_0^{(j)}}{\lambda_0}\nu_0^{(j)}(A) \quad \text{under hypothesis } H_0,$$

$$\lambda_1 := \sum_{j=1}^m \lambda_1^{(j)} \quad \text{and} \quad \nu_1\big(\{j\} \times A\big) := \frac{\lambda_1^{(j)}}{\lambda_1}\nu_1^{(j)}(A) \quad \text{under hypothesis } H_1,$$

respectively. Then the likelihood-ratio process L can be rewritten as

$$L_t = \exp\left\{(\mu_1 - \mu_0)(X_t - X_0) - \left[\frac{\mu_1^2 - \mu_0^2}{2} + \lambda_1 - \lambda_0\right]t + \sum_{T_n \leq t} \log\left(\frac{\lambda_1}{\lambda_0}\frac{d\nu_1}{d\nu_0}(Z_n)\right)\right\},$$

which has the same form as (2.4). Hence, the multisource Bayesian sequential binary hypothesis testing problem becomes exactly the same as the problem formulated and solved in Sects. 2–4 for the new processes X and $(T_n, Z_n)_{n\geq 1}$ with new drift rates μ_0, μ_1 and new arrival rates and mark distributions (λ_0, ν_0), (λ_1, ν_1) under hypotheses H_0 and H_1.

6 Numerical algorithm and examples

In Fig. 2, we describe a numerical algorithm to calculate the successive approximations $v_n(\cdot)$, $n \geq 0$ in (3.6) of the value function $V(\cdot) \equiv v_\infty(\cdot)$ of the auxiliary optimal stopping problem in (2.10) and Bayes ε-optimal decision rules for the Bayesian sequential binary hypothesis testing problem in (2.1). In the examples described below and illustrated in Fig. 3, that algorithm is used to calculate the approximations $v_n(\cdot)$, $n \geq 0$ until the maximum absolute difference between successive approximations is reduced below an acceptable level. Corollary 4.3 guarantees the termination of the algorithm after finite number of iterations with any specified positive error bound. Corollary 4.3 also provides an upper bound on the number of iterations necessary to attain the desired error bound.

Nine panels in Fig. 3 display the approximate value functions and minimum Bayes risks corresponding to nine examples. In each example, the observation process consists of a Brownian motion X with drift μ and a simple Poisson process $(T_n)_{n\geq 1}$ (i.e., marks Z_n, $n \geq 1$ are known and equal to one almost surely) with arrival rate λ. Under the null hypothesis H_0, we assume that the unknown drift and arrival rates are equal to $\mu_0 = 0$ and $\lambda_0 = 1$, respectively. We also assume that the costs of wrongly choosing H_0 and H_1 are the same and equal to $a = b = 0.5$. However, drift rate μ_1 and arrival rate λ_1 under alternative hypothesis H_1 are different in nine examples; drift rate μ_1 takes values 2, 3, 4 along three columns, respectively, and arrival rate λ_1 takes values 7, 9, 11 along three rows, respectively.

Each panel is divided in two parts. The upper part shows the optimal Bayes risk $U(\cdot)$ of (2.3) on [0, 1] displayed on the upper horizontal axis, and the lower part shows the value function $V(\cdot)$ of the auxiliary optimal stopping problem in (2.10) on $\mathbb{R}_+$ displayed on the lower horizontal axis. Both $U(\cdot)$ and $V(\cdot)$ are plotted with solid curves. These functions are compared with $U_p(\cdot)$, $V_p(\cdot)$, $U_X(\cdot)$, and $V_X(\cdot)$, where $U_p(\cdot)$ and $U_X(\cdot)$ are obtained by taking the infimum in (2.3) over the stopping times of smaller natural filtrations $\mathbb{F}^p$ and $\mathbb{F}^X$ of Poisson process and Brownian motion, respectively. On the other hand, $V_p(\cdot)$ and $V_X(\cdot)$

Initialization: Set $v_0(\phi)=h(\phi)$ and $\underline{w}(\phi)=-b$ for every $\phi>0$. Calculate $F(\phi)=\psi(\phi)/\eta(\phi)=\phi^{\alpha_1-\alpha_0}$ for every $\phi>0$. Set $n=1$.
Step 1: Calculate

$$(Lv_n)(\zeta)=\left(\frac{Hv_n-h}{\eta}\right)\circ F^{-1}(\zeta),\quad \zeta\geq 0.$$

Step 2: Calculate critical boundaries $\zeta_1[v_n]$ and $\zeta_2[v_n]$, which are unique solutions of

$$(Lv_n)'\big(\zeta_1[v_n]\big)=\frac{(Lv_n)(\zeta_2[v_n])-(Lv_n)(\zeta_1[v_n])}{\zeta_2[v_n]-\zeta_1[v_n]}=(Lv_n)'\big(\zeta_2[v_n]\big).$$

Recall that $0<\zeta_1[\underline{w}]\leq\zeta[v_n]\leq F(b/a)\leq\zeta_2[v_n]\leq\zeta_2[\underline{w}]<\infty$, and the lower bound $\zeta_1[\underline{w}]$ and upper bound $\zeta_2[\underline{w}]$ on the critical boundaries $\zeta_1[v_n]$ and $\zeta_2[v_n]$ for $n\in\{1,2,\dots\}\cup\{\infty\}$ are useful to control the computer memory. Calculate the smallest nonnegative concave majorant $(Mv_n)(\cdot)$ of $(Lv_n)(\cdot)$ on $\mathbb{R}_+$ by

$$(Mv_n)(\zeta)=\begin{cases}(Lv_n)(\zeta), & \text{if}\quad \zeta\in[0,\zeta_1[v_n]]\cup[\zeta_2[v_n],\infty),\\ \frac{\zeta_2[v_n]-\zeta}{\zeta_2[v_n]-\zeta_1[v_n]}(Lv_n)(\zeta_1[v_n]) & \\ \quad+\frac{\zeta-\zeta_1[v_n]}{\zeta_2[v_n]-\zeta_1[v_n]}(Lv_n)(\zeta_2[v_n]), & \text{if}\quad \zeta\in(\zeta_1[v_n],\zeta_2[v_n]).\end{cases}$$

Step 3: Calculate $\phi_1[v_n]=F^{-1}(\zeta_1[v_n])$ and $\phi_2[v_n]=F^{-1}(\zeta_2[v_n])$ and

$$(Gv_n)(\phi)=\begin{cases}(Hv_n)(\phi)-h(\phi),\quad \phi\in(0,\phi_1[v_n]]\cup[\phi_2[v_n],\infty),\\ \frac{(\phi_2[v_n])^{\alpha_1-\alpha_0}-\phi^{\alpha_1-\alpha_0}}{(\phi_2[v_n])^{\alpha_1-\alpha_0}-(\phi_1[v_n])^{\alpha_1-\alpha_0}}(Hv_n-h)(\phi_1[v_n])\\ \quad+\frac{\phi^{\alpha_1-\alpha_0}-(\phi_1[v_n])^{\alpha_1-\alpha_0}}{(\phi_2[v_n])^{\alpha_1-\alpha_0}-(\phi_1[v_n])^{\alpha_1-\alpha_0}}(Hv_n-h)(\phi_2[v_n]),\\ \quad\phi\in(\phi_1[v_n],\phi_2[v_n]).\end{cases}$$

Step 4: Calculate $v_{n+1}(\phi)=(Jv_n)(\phi)=(Hv_n)(\phi)-(Gv_n)(\phi)$ for every $\phi>0$.
Step 5: If some stopping criterion has not yet been satisfied (e.g., the uniform bound $b\beta^n$ on $\|v_\infty-v_n\|$ has not yet been reduced below some desired error level), then set n to $n+1$ and go to **Step 1**, otherwise stop.
Outcome: After the algorithm terminates with $v_{n+1}(\cdot)$, $\phi_1[v_n]$, and $\phi_2[v_n]$,

(i) we have $v_{n+1}(\phi)-b\beta^{n+1}\leq V(\phi)\leq v_{n+1}(\phi)$ for every $\phi>0$,
(ii) the stopping time $\widetilde{\tau}[v_n]=\inf\{t\geq 0;\ \Phi_t\notin(\phi_1[v_n],\phi_2[v_n])\}$ is ε-optimal for every $\varepsilon>b\beta^{n+1}$ for the auxiliary optimal stopping problem in (2.10); i.e.,

$$V(\phi)\leq\mathbb{E}_0^\phi\left[\int_0^{\widetilde{\tau}[v_n]}(1+\Phi_t)\,\mathrm{d}t+h(\Phi_{\widetilde{\tau}[v_n]})\right]\leq V(\phi)+b\beta^{n+1},\quad \phi>0,$$

(iii) the decision rule $(\widetilde{\tau}[v_n],d(\widetilde{\tau}[v_n]))$ is Bayes ε-optimal for every $\varepsilon>b\beta^{n+1}$ for the Bayesian sequential binary hypothesis testing problem in (2.1); i.e.,

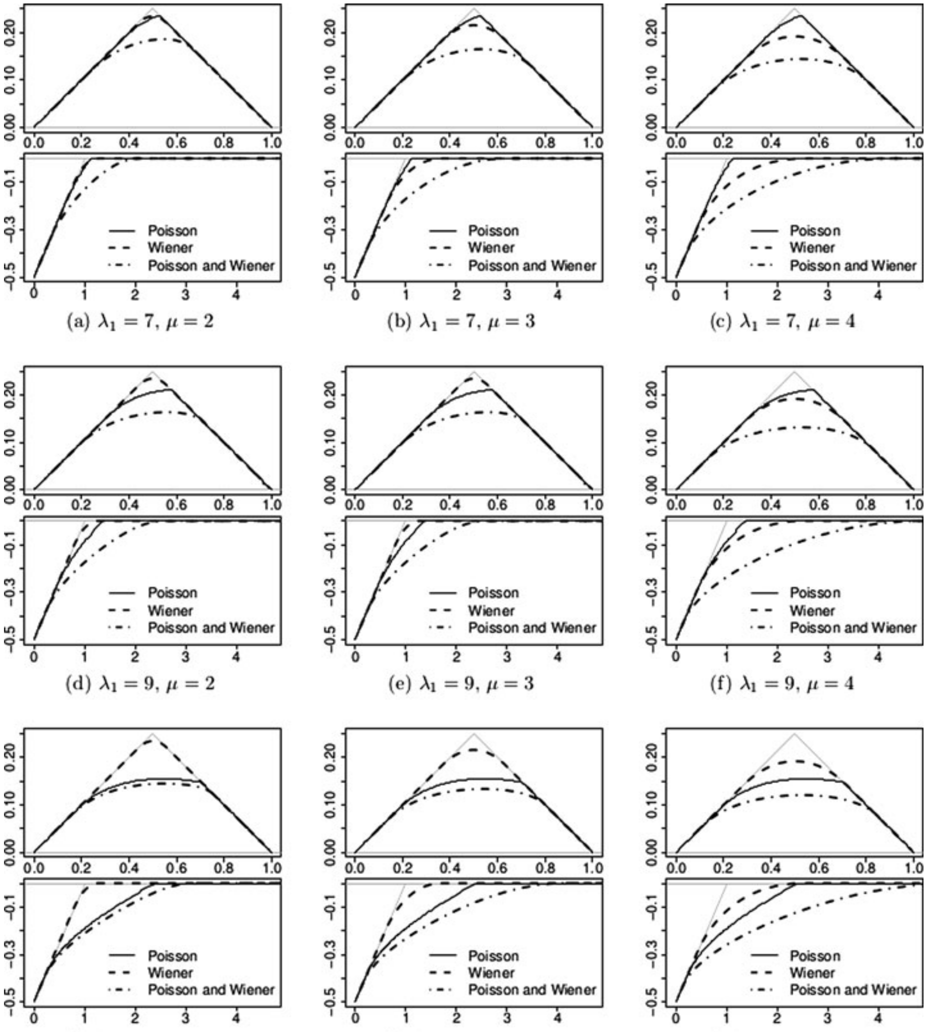
$$U(\pi)\leq R_{\widetilde{\tau}[v_n],d(\widetilde{\tau}[v_n])}(\pi)\leq U(\pi)+b\beta^{n+1},\quad \pi\in(0,1).$$

Fig. 2 Numerical algorithm to solve the Bayesian sequential binary hypothesis testing problem in (2.1)

are the value functions of the optimal stopping problems analogous to (2.10); i.e.,

$$V_p(\phi):=\inf_{\tau\in\mathbb{F}^p}\mathbb{E}_0^\phi\left[\int_0^\tau g\big(\Phi_t^{(p)}\big)\,\mathrm{d}t+1_{\{\tau<\infty\}}h\big(\Phi_\tau^{(p)}\big)\right],\quad \phi\geq 0,$$

$$V_X(\phi):=\inf_{\tau\in\mathbb{F}^X}\mathbb{E}_0^\phi\left[\int_0^\tau g\big(\Phi_t^{(X)}\big)\,\mathrm{d}t+1_{\{\tau<\infty\}}h\big(\Phi_\tau^{(X)}\big)\right],\quad \phi\geq 0,$$



(a) $\lambda_1 = 7$, $\mu = 2$ (b) $\lambda_1 = 7$, $\mu = 3$ (c) $\lambda_1 = 7$, $\mu = 4$

(d) $\lambda_1 = 9$, $\mu = 2$ (e) $\lambda_1 = 9$, $\mu = 3$ (f) $\lambda_1 = 9$, $\mu = 4$

(g) $\lambda_1 = 11$, $\mu = 2$ (h) $\lambda_1 = 11$, $\mu = 3$ (i) $\lambda_1 = 11$, $\mu = 4$

Fig. 3 In all of nine examples above, $\lambda_0 = 1$, $\mu_0 = 0$, and the cost parameters are $a = b = 0.5$. Jumps are of unit size under both hypotheses

where

$$\Phi_t^{(p)} := \frac{\mathbb{P}\{\Theta \le t \mid \mathcal{F}_t^p\}}{\mathbb{P}\{\Theta > t \mid \mathcal{F}_t^p\}} \quad \text{and} \quad \Phi_t^{(X)} := \frac{\mathbb{P}\{\Theta \le t \mid \mathcal{F}_t^X\}}{\mathbb{P}\{\Theta > t \mid \mathcal{F}_t^X\}} \quad \text{for every } t \ge 0.$$

The functions $U_p(\cdot)$, $V_p(\cdot)$ and $U_X(\cdot)$, $V_X(\cdot)$ are related each other in the same way as $U(\cdot)$, $V(\cdot)$ are in (2.9).

The differences in the Bayes risks $U_p(\cdot)$, $U_X(\cdot)$, and $U(\cdot)$ are due to the contributions of observing the processes X and $(T_n)_{n\ge1}$ separately or simultaneously to the efforts to identify the correct hypothesis about the drift rate X and arrival rate of $(T_n)_{n\ge1}$. Poisson process observations provide more information than Brownian motion observations in Figs. 3 (d), (e), (g), (h), and (i). Brownian motion observations provide more information than Poisson process observations in Figs. 3 (a), (b), and (f). In every case, however, observing both Poisson process and Brownian motion provides more information, which is often significantly more than two processes provide separately, as in Figs. 3 (a), (b), (c), (d), (e), (f), and (i).

Intuitively, we expect the contributions of both information sources, observed separately or concurrently, to increase as μ_1 and λ_1 get farther away from $\mu_0 = 0$ and $\lambda_0 = 1$, respectively, and the results reported in Fig. 3 support these expectations: the Bayes risks $U_p(\cdot)$ and $U(\cdot)$ shift downward along the rows, and $U_X(\cdot)$ and $U(\cdot)$ do the same along the columns.

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Appendix

A.1 The proof of Lemma 4.2

Let $G_{\ell,r}(\phi,\xi) = \frac{\psi_\ell(\phi\wedge\xi)\eta_r(\phi\vee\xi)}{p^2(\xi)W_{\ell,r}(\xi)}$, $0<\phi,\xi<\infty$ be the Green's function of the boundary value problem $(A_0 f)(\phi) - \lambda_0 f(\phi) = -k(\phi)$ for every $\ell<\phi<r$ and $f(\ell)=f(r)=0$. Then $f(\phi) = \int_\ell^r G_{\ell,r}(\phi,\xi)k(\xi)\,\mathrm{d}\xi = \eta_r(\phi)\int_\ell^\phi \frac{2\psi_\ell(\xi)}{p^2(\xi)W_{\ell,r}(\xi)}k(\xi)\,\mathrm{d}\xi + \psi_\ell(\phi)\int_\phi^r \frac{2\eta_r(\xi)}{p^2(\xi)W_{\ell,r}(\xi)}k(\xi)\,\mathrm{d}\xi$, which is also the right-hand side of (4.5), is twice continuously differentiable and solves uniquely the boundary value problem above. Then $e^{-\lambda_0(\tau_{\ell,r}\wedge t)}f(Y^{\Phi_0}_{\tau_{\ell,r}\wedge t}) = f(Y_0^{\Phi_0}) + \int_0^{\tau_{\ell,r}\wedge t} e^{-\lambda_0 s}(A_0 f - \lambda_0 f)(Y_s^{\Phi_0})\,\mathrm{d}s + \int_0^{\tau_{\ell,r}\wedge t} e^{-\lambda_0 s} f'(Y_s^{\Phi_0})p(Y_s^{\Phi_0})(\mathrm{d}X_s - \mu_0\,\mathrm{d}s)$ for every $t\ge 0$ by Itô rule. Because $f'(\cdot)$ and $p(\cdot)$ are continuous on $[\ell,r]\subset(0,\infty)$, they are bounded, and the stochastic integral on the right-hand side is a square-integrable $(\mathbb{P}_0,\mathbb{F}^X)$-martingale. Taking firstly the expectations of both sides and then their limits as $t\to\infty$, and finally rearranging the terms lead to $f(\phi) = \mathbb{E}_0^\phi[\int_0^{\tau_{\ell,r}} e^{-\lambda_0 s}k(Y_s^{\Phi_0})\,\mathrm{d}s] + \mathbb{E}_0^{\Phi_0}[e^{-\lambda_0\tau_{\ell,r}} f(Y^{\Phi}_{\tau_{\ell,r}})] = \mathbb{E}_0^\phi[\int_0^{\tau_{\ell,r}} e^{-\lambda_0 s}k(Y_s^{\Phi_0})\,\mathrm{d}s]$, because $f(\ell)=f(r)=0$. For the proof of (4.6), note $\lim_{\ell\downarrow 0, r\uparrow\infty}\tau_{\ell,r}=\infty$ $\mathbb{P}_0^\phi$-a.s. for all $\phi\in(0,\infty)$ since 0 and ∞ are natural boundaries of Y^{Φ_0}. Moreover, $\int_0^{\tau_{\ell,r}} e^{-\lambda_0 t}|k(Y_t^{\Phi_0})|\,\mathrm{d}t \le c\int_0^{\tau_{\ell,r}} e^{-\lambda_0 t}(1+Y_t^{\Phi_0})\,\mathrm{d}t \le c\int_0^\infty e^{-\lambda_0 t}(1+Y_t^{\Phi_0})\,\mathrm{d}t$, and since $\mathbb{E}_0^\phi[Y_t^{\Phi_0}] = \phi e^{-(\lambda_1-\lambda_0)t}$ as in (4.4), we have $\mathbb{E}_0^\phi[\int_0^\infty e^{-\lambda_0 t}(1+Y_t^{\Phi_0})\,\mathrm{d}t] = \frac{1}{\lambda_0} + \int_0^\infty e^{-\lambda_0 t}\mathbb{E}_0^\phi[Y_t^{\Phi_0}]\,\mathrm{d}t = \frac{1}{\lambda_0} + \frac{\phi}{\lambda_1} < \infty$. Therefore, $\lim_{\ell\downarrow 0,r\uparrow\infty}\int_0^{\tau_{\ell,r}} e^{-\lambda_0 t}k(Y_t^{\Phi_0})\,\mathrm{d}t = \int_0^\infty e^{-\lambda_0 t}k(Y_t^{\Phi_0})\,\mathrm{d}t$-a.s. and the dominated convergence theorem implies $\mathbb{E}_0^\phi[\int_0^\infty e^{-\lambda_0 t}\times k(Y_t^{\Phi_0})\,\mathrm{d}t] = \lim_{\ell\downarrow 0,r\uparrow\infty}\mathbb{E}_0^\phi[\int_0^{\tau_{\ell,r}} e^{-\lambda_0 t}k(Y_t^{\Phi_0})\,\mathrm{d}t]$ equals

$$\begin{aligned}
&\lim_{\ell\downarrow 0,r\uparrow\infty}\left[\eta_r(\phi)\int_\ell^\phi \frac{2\psi_\ell(\xi)}{p^2(\xi)W_{\ell,r}(\xi)}k(\xi)\,\mathrm{d}\xi + \psi_\ell(\phi)\int_\phi^r \frac{2\eta_r(\xi)}{p^2(\xi)W_{\ell,r}(\xi)}k(\xi)\,\mathrm{d}\xi\right]\\
&\quad= \lim_{\ell\downarrow 0,r\uparrow\infty}\left[\frac{\eta_r(\phi)}{1-\frac{\eta(r)}{\eta(\ell)}\frac{\psi(\ell)}{\psi(r)}}\int_\ell^\phi \frac{2\psi_\ell(\xi)}{p^2(\xi)W(\xi)}k(\xi)\,\mathrm{d}\xi\right.\\
&\qquad\left.+\frac{\psi_\ell(\phi)}{1-\frac{\eta(r)}{\eta(\ell)}\frac{\psi(\ell)}{\psi(r)}}\int_\phi^r \frac{2\eta_r(\xi)}{p^2(\xi)W(\xi)}k(\xi)\,\mathrm{d}\xi\right].
\end{aligned} \tag{A.1}$$

We have $\lim_{\ell\downarrow 0}\uparrow\psi_\ell(\xi)=\psi(\xi)$ and $\lim_{r\uparrow\infty}\uparrow\eta_r(\xi)=\eta(\xi)$ for every $\xi>0$. Because $\alpha_0<0$, $\int_\ell^\phi \frac{2\psi_\ell(\xi)}{p^2(\xi)W(\xi)}|k(\xi)|\,\mathrm{d}\xi \le c\int_0^\phi \frac{2\psi(\xi)}{p^2(\xi)W(\xi)}(1+\xi)\,\mathrm{d}\xi = c\int_0^\phi \xi^{-1-\alpha_0}(1+\xi)\,\mathrm{d}\xi<\infty$, and the dominated convergence theorem implies that $\lim_{\ell\downarrow 0}\int_\ell^\phi \frac{2\psi_\ell(\xi)}{p^2(\xi)W(\xi)}k(\xi)\,\mathrm{d}\xi =$

$\int_0^\phi \frac{2\psi(\xi)}{p^2(\xi)W(\xi)} k(\xi)\,\mathrm{d}\xi$. Similarly, because $\alpha_1 > 0$, we have $\int_\phi^r \frac{2\eta_r(\xi)}{p^2(\xi)W(\xi)} |k(\xi)|\,\mathrm{d}\xi \le c\int_\phi^\infty \frac{2\eta(\xi)}{p^2(\xi)W(\xi)}(1+\xi)\,\mathrm{d}\xi = c\int_\alpha^\infty \xi^{-1-\alpha_1}(1+\xi)\,\mathrm{d}\xi < \infty$, and by the dominated convergence, $\lim_{r\uparrow\infty}\int_\phi^r \frac{2\eta_r(\xi)}{p^2(\xi)W(\xi)}k(\xi)\,\mathrm{d}\xi = \int_\phi^\infty \frac{2\eta(\xi)}{p^2(\xi)W(\xi)}k(\xi)\,\mathrm{d}\xi$. Taking the limits on the right-hand side of (A.1) and using (4.12) complete the proof of (4.6), which can be directly shown to satisfy $(A_0 f - \lambda_0 f)(\phi) + k(\phi) = 0$ for every $\phi > 0$.

Finally, suppose that the limit $k(0+) = \lim_{\phi\downarrow 0} k(\phi)$ exists. For every $0 < \phi \le 1$, we have $Y_t^\phi \le Y_t^1$ for every $t \ge 0$ $\mathbb{P}_0^1$-a.s. Note that $\mathbb{E}_0^\phi[\int_0^\infty e^{-\lambda_0 t} k(Y_t^{\Phi_0})\,\mathrm{d}t] = \mathbb{E}_0^1[\int_0^\infty e^{-\lambda_0 t} k(Y_t^\phi)\,\mathrm{d}t]$ for every $\phi > 0$, where $|k(Y_t^\phi)| \le c(1+Y_t^\phi) \le c(1+Y_t^1)$, and $\mathbb{E}_0^1[\int_0^\infty e^{-\lambda_0 t}(1+Y_t^1)\,\mathrm{d}t] = \int_0^\infty e^{-\lambda_0 t}(1+\mathbb{E}_0^1 Y_t^1)\,\mathrm{d}t = \int_0^\infty e^{-\lambda_0 t}(1+e^{-(\lambda_1-\lambda_0)t})\,\mathrm{d}t = \int_0^\infty e^{-\lambda_0 t}\,\mathrm{d}t + \int_0^\infty e^{-\lambda_1 t}\,\mathrm{d}t = \frac{1}{\lambda_0} + \frac{1}{\lambda_1} < \infty$. Because $\lim_{\phi\downarrow 0} Y_t^\phi = 0$ and $\lim_{\phi\downarrow 0} k(Y_t^\phi) = k(0+)$ for every $t \ge 0$ $\mathbb{P}^1$-a.s., the dominated convergence implies that $\lim_{\phi\downarrow 0}\mathbb{E}_0^\phi[\int_0^\infty e^{-\lambda_0 t} k(Y_t^{\Phi_0})\,\mathrm{d}t] = \lim_{\phi\downarrow 0}\mathbb{E}_0^1[\int_0^\infty e^{-\lambda_0 t} \times k(Y_t^\phi)\,\mathrm{d}t] = \mathbb{E}_0^1[\int_0^\infty e^{-\lambda_0 t}(\lim_{\phi\downarrow 0} k(Y_t^\phi))\,\mathrm{d}t] = k(0+)\int_0^\infty e^{-\lambda_0 t}\,\mathrm{d}t = \frac{k(0+)}{\lambda_0}$, which completes the proof.

A.2 The proof of Lemma 4.3

Recall $w : \mathbb{R}_+ \mapsto \mathbb{R}$ is increasing and $-b \le w(\phi) \le h(\phi)$, $\phi \in \mathbb{R}_+$. Then $w(0+) < 0$, since otherwise $w(\cdot) \equiv 0$. Also $(Kw)(\phi) = \int_E w(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(z)\phi)\nu_0(\mathrm{d}z)$ is increasing and $-b \le (Kw)(\phi) \le 0$, $\phi \in \mathbb{R}_+$. Then $\lim_{\phi\downarrow 0}\int_0^\phi \xi^{-1-\alpha_0}|(Kw)(\xi)|\,\mathrm{d}\xi \le b\lim_{\phi\downarrow 0}\int_0^\phi \xi^{-1-\alpha_0}\,\mathrm{d}\xi = b\lim_{\phi\downarrow 0}\frac{\phi^{-\alpha_0}}{(-\alpha_0)}$, since $\alpha_0 < 0$, and (i) follows.

Next notice that $\lim_{\phi\downarrow 0}(Kw)(\phi) = (Kw)(0+) = w(0+)$ by the bounded convergence theorem. For every fixed $\varepsilon > 0$, there exists some $\delta > 0$ such that $\phi \in (0,\delta)$ implies that $w(0+) \le (Kw)(\phi) \le w(0+)(1-\varepsilon)$. Then for every $\phi \in (0,\delta)$ $w(0+)\int_0^\phi \xi^{-1-\alpha_0}\,\mathrm{d}\xi \le \int_0^\phi \xi^{-1-\alpha_0}(Kw)(\xi)\,\mathrm{d}\xi \le w(0+)(1-\varepsilon)\int_0^\phi \xi^{-1-\alpha_0}\,\mathrm{d}\xi$ or $\frac{w(0+)}{(-\alpha_0)} \le \phi^{\alpha_0}\int_0^\phi \xi^{-1-\alpha_0}(Kw)(\xi)\,\mathrm{d}\xi \le \frac{w(0+)}{(-\alpha_0)}(1-\varepsilon)$, which proves (ii) after taking limits as $\phi \downarrow 0$, since $\varepsilon > 0$ was arbitrary.

Because $w(0+) \equiv (Kw)(0+) < 0$, there exists some δ such that $\phi \in (0,\delta)$ implies that $(Kw)(\phi) < (1/2)(Kw)(0+)$. Then $\int_\phi^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\,\mathrm{d}\xi \le \int_\phi^\delta \xi^{-1-\alpha_1}(Kw)(\xi)\,\mathrm{d}\xi \le \frac{w(0+)}{2}\int_\phi^\delta \xi^{-1-\alpha_1}\,\mathrm{d}\xi = \frac{w(0+)(\delta^{-\alpha_1}-\phi^{-\alpha_1})}{2(-\alpha_1)}$ for every $\phi \in (0,\delta)$, and because $w(0+) < 0$ and $\alpha_1 > 1$, we have $\overline{\lim}_{\phi\downarrow 0}\int_\phi^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\,\mathrm{d}\xi \le -\infty$, which completes the proof of (iii).

For every fixed $\varepsilon > 0$, there exists some $\delta > 0$ such that $\phi \in (0,\delta)$ implies that $w(0+) \equiv (Kw)(0+) \le (Kw)(\phi) \le w(0+)(1-\varepsilon)$. Therefore, for every $\phi \in (0,\delta)$ we have $\frac{w(0+)}{\alpha_1}(\phi^{-\alpha_1} - \delta^{-\alpha_1}) \le \int_\phi^\delta \xi^{-1-\alpha_1}(Kw)(\xi)\,\mathrm{d}\xi \le \frac{w(0+)}{\alpha_1}(1-\varepsilon)(\phi^{-\alpha_1}-\delta^{-\alpha_1})$. Adding $\int_\delta^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\,\mathrm{d}\xi$, which is finite, and multiplying by ϕ^{α_1} all three sides give

$$\begin{aligned}
&\phi^{\alpha_1}\int_\delta^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\,\mathrm{d}\xi + \frac{w(0+)}{\alpha_1}\big[1-(\phi/\delta)^{\alpha_1}\big] \\
&\quad\le \phi^{\alpha_1}\int_\phi^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\,\mathrm{d}\xi \\
&\quad\le \phi^{\alpha_1}\int_\delta^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\,\mathrm{d}\xi + \frac{w(0+)}{\alpha_1}(1-\varepsilon)\big[1-(\phi/\delta)^{\alpha_1}\big].
\end{aligned}$$

Since $\alpha_1 > 0$, $\frac{w(0+)}{\alpha_1} \le \underline{\lim}_{\phi\downarrow 0}\phi^{\alpha_1}\int_\phi^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\,\mathrm{d}\xi \le \overline{\lim}_{\phi\downarrow 0}\phi^{\alpha_1}\int_\phi^\infty \xi^{-1-\alpha_1}(Kw)(\xi) \times \mathrm{d}\xi \le \frac{w(0+)}{\alpha_1}(1-\varepsilon)$, which completes the proof of (iv) because $\varepsilon > 0$ is arbitrary.

For the proof of (v), firstly note that the monotonicity of $w(\cdot)$ and the bounded convergence theorem implies that $(Kw)(\infty) = \lim_{\phi\uparrow\infty}(Kw)(\phi) = \lim_{\phi\uparrow\infty}\int w(\frac{\lambda_1}{\lambda_0}\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_0}(z)\phi)\nu_0(\mathrm{d}z)$

exists and equals $w(\infty)$. Then for every $\varepsilon > 0$ there exists some $M > 0$ such that $\phi > M$ implies that $w(\infty) - \varepsilon \le (Kw)(\phi) \le w(\infty)$. Therefore, for every $\phi > M$

$$\big[w(\infty) - \varepsilon\big] \int_M^\phi \xi^{-1-\alpha_0}\, \mathrm{d}\xi \le \int_M^\phi \xi^{-1-\alpha_0}(Kw)(\xi)\, \mathrm{d}\xi \le w(\infty) \int_M^\infty \xi^{-1-\alpha_0}\, \mathrm{d}\xi,$$

$$\frac{w(\infty) - \varepsilon}{(-\alpha_0)}\big(\phi^{-\alpha_0} - M^{-\alpha_0}\big) \le \int_M^\phi \xi^{-1-\alpha_0}(Kw)(\xi)\, \mathrm{d}\xi \le \frac{w(\infty)}{(-\alpha_0)}\big(\phi^{-\alpha_0} - M^{-\alpha_0}\big).$$

Adding $\int_0^M \xi^{-1-\alpha_0}(Kw)(\xi)\, \mathrm{d}\xi$, which is finite, and multiplying by ϕ^{α_0} all three sides give

$$\phi^{\alpha_0} \int_0^M \xi^{-1-\alpha_0}(Kw)(\xi)\, \mathrm{d}\xi + \frac{w(\infty) - \varepsilon}{(-\alpha_0)}\big[1 - (\phi/M)^{\alpha_0}\big]$$
$$\le \phi^{\alpha_0} \int_M^\phi \xi^{-1-\alpha_0}(Kw)(\xi)\, \mathrm{d}\xi \le \phi^{\alpha_0} \int_0^M \xi^{-1-\alpha_0}(Kw)(\xi)\, \mathrm{d}\xi + \frac{w(\infty)}{(-\alpha_0)}\big[1 - (\phi/M)^{\alpha_0}\big].$$

Letting $\phi \uparrow \infty$ and recalling that $\alpha_0 < 0$ gives $\frac{w(\infty)-\varepsilon}{(-\alpha_0)} \le \underline{\lim}_{\phi\uparrow\infty} \phi^{\alpha_0} \int_M^\phi \xi^{-1-\alpha_0}(Kw)(\xi)\, \mathrm{d}\xi \le \overline{\lim}_{\phi\uparrow\infty} \phi^{\alpha_0} \int_M^\phi \xi^{-1-\alpha_0}(Kw)(\xi)\, \mathrm{d}\xi \le \frac{w(\infty)}{(-\alpha_0)}$. Because $\varepsilon > 0$ is arbitrary, this proves (v). And (vi) follows from $\alpha_1 > 0$ and that $\lim_{\phi\uparrow\infty} \int_\phi^\infty \xi^{-1-\alpha_1}|(Kw)(\xi)|\, \mathrm{d}\xi \le b \cdot \lim_{\phi\uparrow\infty} \int_\phi^\infty \xi^{-1-\alpha_1}\, \mathrm{d}\xi$ $= b \cdot \lim_{\phi\uparrow\infty} \frac{\phi^{-\alpha_1}}{\alpha_1} = 0$. To prove (vii), let ε and M be as in the proof of (v). Then for $\phi > M$

$$\big(w(\infty) - \varepsilon\big) \int_\phi^\infty \xi^{-1-\alpha_1}\, \mathrm{d}\xi \le \int_\phi^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\, \mathrm{d}\xi \le w(\infty) \int_\phi^\infty \xi^{-1-\alpha_1}\, \mathrm{d}\xi,$$

$$\frac{w(\infty) - \varepsilon}{\alpha_1}\phi^{-\alpha_1} \le \int_\phi^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\, \mathrm{d}\xi \le \frac{w(\infty)}{\alpha_1}\phi^{-\alpha_1}.$$

Multiplying all sides by ϕ^{α_1} gives $\frac{w(\infty)-\varepsilon}{\alpha_1} \le \phi^{\alpha_1} \int_\phi^\infty \xi^{-1-\alpha_1}(Kw)(\xi)\, \mathrm{d}\xi \le \frac{w(\infty)}{\alpha_1}$ for all $\phi > M$, which proves (vii) and Lemma 4.3 after taking limit as $\phi \uparrow \infty$ because $\varepsilon > 0$ is arbitrary.

A.3 The proof of Lemma 4.5

Because Y^{Φ_0} is a regular diffusion on $\mathbb{R}_+$, $\mathbb{P}_0^r\{\tau_\ell < \infty\} > 0$ and there exists some $0 < t < \infty$ such that $\mathbb{P}_0^r\{\tau_\ell < t\} > 0$. On the other hand, the sample-path decomposition in (2.16) of jump-diffusion process Φ into diffusion part Y^{Φ_0} and jump part, which are $\mathbb{P}_0$-independent, implies that $\delta := \mathbb{P}_0^r\{\widetilde{\tau}_{\ell,\infty} \le t\} \ge \mathbb{P}_0^r\{\widetilde{\tau}_{\ell,\infty} \le t, T_1 > t\} = \mathbb{P}_0^r\{\tau_\ell \le t, T_1 > t\} = \mathbb{P}_0^r\{\tau_\ell \le t\}\mathbb{P}_0^r\{T_1 > t\} = \mathbb{P}_0^r\{\tau_\ell \le t\}e^{-\lambda_0 t} > 0$. Next for every $\phi \in (\ell, r)$,

$$\begin{aligned}
\mathbb{P}_0^\phi\{\widetilde{\tau}_{\ell,r} > t\} &\le \mathbb{P}_0^\phi\{\widetilde{\tau}_{\ell,\infty} > t\} \\
&= \mathbb{P}_0^\phi\big\{\Phi_u \ge \ell \text{ for every } u \in [0,t]\big\} \\
&= \mathbb{P}_0^\phi\big\{L_u \ge \ell/\phi \text{ for every } u \in [0,t]\big\} \le \mathbb{P}_0^\phi\big\{L_u \ge \ell/r \text{ for every } u \in [0,t]\big\} \\
&= \mathbb{P}_0^r\big\{\Phi_u \ge \ell \text{ for every } u \in [0,t]\big\} = \mathbb{P}_0^r\{\widetilde{\tau}_{\ell,\infty} > t\} = 1 - \delta.
\end{aligned}$$

Hence, $\sup_{\phi\in[\ell,r]}\mathbb{P}_0^\phi\{\widetilde{\tau}_{\ell,r}>t\}\le 1-\delta<1$, and $\mathbb{E}_0^\phi\widetilde{\tau}_{\ell,r}^k=\sum_{m=0}^\infty \mathbb{E}_0^\phi[\widetilde{\tau}_{\ell,r}^k 1_{\{mt<\widetilde{\tau}_{\ell,r}\le(m+1)t\}}]\le t^k\sum_{m=0}^\infty(m+1)^k\mathbb{P}_0^\phi\{\widetilde{\tau}_{\ell,r}>mt\}$ for all $k>0$. Since Φ is a strong $(\mathbb{P}_0,\mathbb{F})$-Markov process,

$$\begin{aligned}\mathbb{P}_0^\phi\{\widetilde{\tau}_{\ell,r}>mt\}&=\mathbb{P}_0^\phi\big\{\widetilde{\tau}_{\ell,r}>(m-1)t,\widetilde{\tau}_{\ell,r}>mt\big\}=\mathbb{E}_0^\phi\big[1_{\{\widetilde{\tau}_{\ell,r}>(m-1)t\}}\mathbb{P}_0^{\Phi(m-1)t}\{\widetilde{\tau}_{\ell,r}>t\}\big]\\&\le(1-\delta)\,\mathbb{P}_0^\phi\big\{\widetilde{\tau}_{\ell,r}>(m-1)t\big\}\le\cdots\le(1-\delta)^m\quad\text{for every } m\ge 1,\end{aligned}$$

and $\mathbb{E}_0^\phi\widetilde{\tau}_{\ell,r}^k\le t^k\sum_{m=0}^\infty(m+1)^k(1-\delta)^m<\infty$ for every $\phi\in[\ell,r]$, which proves Lemma 4.5.

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