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An induction theorem for the unit groups of Burnside rings of 2-groups

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Abstract

Let G be a 2-group and $B(G)^\times$ denote the group of units of the Burnside ring of G. For each subquotient H/K of G, there is a generalized induction map from $B(H/K)^\times$ to $B(G)^\times$ defined as the composition of inflation and multiplicative induction maps. We prove that the product of generalized induction maps $\prod B(H/K)^\times \to B(G)^\times$ is surjective when the product is taken over the set of all subquotients that are isomorphic to the trivial group or a dihedral 2-group of order 2^n with $n \geqslant 4$. As an application, we give an algebraic proof for a theorem by Tornehave [The unit group for the Burnside ring of a 2-group, Aarhus Universitet Preprint series 1983/84 41, May 1984] which states that tom Dieck's exponential map from the real representation ring of G to $B(G)^\times$ is surjective. We also give a sufficient condition for the surjectivity of the exponential map from the Burnside ring of G to $B(G)^\times$.

Keywords: Units of Burnside ring; Real representation ring

1. Introduction

The Burnside ring of a finite group G is defined to be the Grothendieck ring of the semi-ring generated by isomorphism classes of finite (left) G-sets where the addition and

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multiplication are given by disjoint unions and cartesian products. We denote the Burnside ring of G by $B(G)$, and its unit group by $B(G)^\times$. The Burnside ring of G can be imbedded, as a subring, into the ring of superclass functions $C(G) = \mathbb{Z}^{Cl(G)}$ where $Cl(G)$ denotes the set of conjugacy classes of subgroups of G, and $\mathbb{Z}^{Cl(G)}$ denotes the ring of functions from $Cl(G)$ to $\mathbb{Z}$. So, the unit group of $B(G)$, being isomorphic to a subgroup of $C(G)^\times = \{\pm 1\}^{Cl(G)}$, is an elementary abelian 2-group. Our ultimate goal is to relate the 2-rank of $B(G)^\times$ to other well known group theoretical invariants.

Throughout the paper we assume G is a 2-group. The reasons for restricting ourselves to 2-groups are as follows: First, the unit group $B(G)^\times$ is quite difficult to understand for a composite group. For example, the assertion that $B(G)^\times \cong \mathbb{Z}/2$ when G is an odd order group is equivalent to the odd order theorem. On the other hand, when G is a p-group with $p > 2$, it is easy to show that $B(G)^\times = \{\pm 1\}$, and so there is nothing to study. We also believe that the unit group functor $B(-)^\times$ over 2-groups is an interesting object in the category of biset functors over 2-groups.

We use mainly two ingredients for studying $B(G)^\times$. One is a complete characterization of $B(G)^\times$ as a subgroup of $C(G)^\times$ given by Yoshida [12]. We explain this characterization in detail at the end of Section 2. The other ingredient is the structure of $B(G)^\times$ as a Mackey functor together with appropriate restriction, induction and conjugation maps. There are also inflation and deflation maps defined in a suitable sense. These maps are defined and studied in detail in [12] and we give an overview in Section 3.

The induction map is particularly interesting since we are using a multiplicative induction map instead of the usual induction map on the Burnside ring. Given a subgroup $H \leqslant G$, the multiplicative induction map $\operatorname{jnd}_H^G : B(H)^\times \to B(G)^\times$ is defined on the Burnside ring as the polynomial extension of the assignment $X \to \operatorname{Map}_H(G, X)$ where X is an H-set, and $\operatorname{Map}_H(G, X)$ is the set of H-maps $f : G \to X$. Note that given a normal subgroup $K \trianglelefteq H$, we have a homomorphism, called the inflation map, $\operatorname{inf}_{H/K}^H : B(H/K)^\times \to B(H)^\times$ defined by considering a H/K-set as an H-set through the quotient map $H \to H/K$. We call the composition $\operatorname{jnd}_H^G \operatorname{inf}_{H/K}^H$ the generalized induction map from subquotient H/K to G. The main result of the paper is the following induction theorem:

Theorem 1.1. *Let G be a 2-group, and let $\mathcal{H}$ denote the collection of all subquotients of G which are isomorphic to the trivial group or a dihedral group of order 2^n with $n \geqslant 4$. Then, the product of generalized induction maps*

$$\prod_{H/K \in \mathcal{H}} \operatorname{jnd}_H^G \operatorname{inf}_{H/K}^H : \prod_{H/K \in \mathcal{H}} B(H/K)^\times \to B(G)^\times$$

is surjective.

One of the ways to obtain units in the Burnside ring of G is to construct exponential maps from the Burnside ring $B(G)$ or from the real representation ring $R(G, \mathbb{R})$ to the unit group of superclass functions $C(G)^\times$, and then show that they actually lie in $B(G)^\times$. For example, given a real representation V of G, we can define a unit superclass function $H \to \operatorname{sgn}(\dim V^H)$ for all $H \leqslant G$ where $\operatorname{sgn}(n) = (-1)^n$ for $n \in \mathbb{Z}$. Tom Dieck showed that these superclass functions lay in the Burnside ring, so one gets an exponential map from the real representation ring $R(G,\mathbb{R})$ to $B(G)^\times$ which is now referred to as tom Dieck's homomorphism (see [8, p. 242] for details). As a corollary of Theorem 1.1, we obtain an algebraic proof for the following result:

Corollary 1.2 (Tornehave [11]). *Let G be a 2-group. Then, tom Dieck's homomorphism*

$$\Theta : R(G,\mathbb{R}) \to B(G)^\times$$

is surjective.

There is a similar exponential map from the Burnside ring $B(G)$ to its unit group $B(G)^\times$. Given a G-set X, consider the superclass function $f_X : H \to \mathrm{sgn}(|X/H|)$ for all $H \leqslant G$. The exponential map $\exp : B(G) \to B(G)^\times$ is defined as the linear extension of the assignment $X \to f_X$ for G-sets. This map is closely related to the $B(G)$-module structure on $B(G)^\times$ which has been studied extensively by Matsuda in [9,10]. The connection comes from the fact that the exponential map can be defined also as $\exp(x) = (-1) \uparrow x$ where $(-1) \uparrow x$ denotes the action of $x \in B(G)$ on $-1 \in B(G)^\times$ (see Section 7 for more details). We prove

Corollary 1.3. *If G is a 2-group which has no subquotients isomorphic to the dihedral group of order 16, then the exponential map*

$$\exp : B(G) \to B(G)^\times$$

is surjective. In this case, $B(G)^\times$ is generated by -1 as a module over $B(G)$.

Corollary 1.3 applies, in particular, to all 2-groups of exponent 4. This includes all 2-groups which can be expressed as an extension of an elementary abelian 2-group by an elementary abelian 2-group. Also, it is well known that the exponential map is not surjective when G is a dihedral 2-group of order at least 16 (see Matsuda [10]). So, the corollary cannot be improved further using the induction theorem. On the other hand, the converse of the corollary does not hold either: There are 2-groups where the exponential map is surjective although they have a dihedral section of order 16. So, Corollary 1.3 provides a sufficient condition for surjectivity of exponential map, which is not a necessary condition.

2. Superclass functions and idempotent basis

Let G be a finite group. The Burnside ring $B(G)$ is defined as the Grothendieck ring of the semi-ring generated by G-isomorphism classes of finite (left) G-sets where the addition and multiplication are given by disjoint unions and cartesian products. So, as an abelian group $B(G)$ is generated by isomorphism classes of (left) G-sets, and isomorphism classes of transitive G-sets form a basis for $B(G)$. A transitive G-set is isomorphic to $G/H := \{gH \mid g \in G\}$ as a G-set, and any two such G-sets G/H and G/K are isomorphic if and only if H and K are conjugate to each other. Therefore, $B(G)$ is a free abelian group with basis $\{[G/H] \mid [H] \in Cl(G)\}$, where $Cl(G)$ is the set of conjugacy classes $[H]$ of subgroups $H \leqslant G$. In other words, $B(G)$ decomposes as the direct sum of cyclic $\mathbb{Z}$-modules

$$B(G) = \bigoplus_{[H]\in Cl(G)} \mathbb{Z}[G/H].$$

The multiplicative structure can be explained in terms of the basis by the following double coset formula:

$$[G/H][G/K] = \sum_{HgK\in H\backslash G/K} \big[G/(H\cap {}^gK)\big]$$

where ${}^gK = gKg^{-1}$.

A *superclass function* is a map from the set of subgroups of G to $\mathbb{Z}$ which is constant on conjugacy classes of subgroups. We will denote the set of superclass functions by $C(G) := \mathbb{Z}^{Cl(G)}$. It is easy to see that $C(G)$ is a ring under the usual addition and multiplication of functions. For each $H \leqslant G$, consider the map $s_H : B(G) \to \mathbb{Z}$ defined as the linear extension of the assignment $s_H(X) = |X^H|$ where $|X^H|$ denotes the number of points in X fixed by H. It is easy to see that $s_H(X \times Y) = s_H(X)s_H(Y)$, hence s_H is a ring homomorphism. It is well known that the ring homomorphisms s_H and s_K are equal if and only if H and K are conjugate. Therefore, for each element $x \in B(G)$, one can define a superclass function $f_x \in C(G)$ by setting $f_x(H) = s_H(x)$. This defines a ring homomorphism

$$\varphi : B(G) \to C(G) := \mathbb{Z}^{Cl(G)}$$

which is injective. The injectivity follows from the fact that if $|X^H| = |Y^H|$ for each $H \leqslant G$, then X and Y are isomorphic as G-sets. We sometimes identify $B(G)$ with its image in $C(G)$, and write $x(H) = s_H(x)$ for $x \in B(G)$.

The image of φ is characterized by the following theorem:

Theorem 2.1 (tom Dieck [7, Section 1.3]). *Let G be a finite group. For each $H \leqslant G$, let $W_G(H)$ denote the quotient group $N_G(H)/H$. Then, the following sequence of abelian groups is exact*:

$$0 \to B(G) \xrightarrow{\varphi} C(G) \xrightarrow{\psi} \prod_{[H]\in Cl(G)} \big(\mathbb{Z}/|W_G(H)|\mathbb{Z}\big) \to 0$$

where φ is the injective ring homomorphism defined above, and the $[H]$ component of ψ is defined by

$$\psi(f)_H = \sum_{gH\in W_G(H)} f(\langle g\rangle H) \pmod{|W_G(H)|}.$$

Let $\mathbb{Q}B(G)$ and $\mathbb{Q}C(G)$ denote $\mathbb{Q} \otimes_{\mathbb{Z}} B(G)$ and $\mathbb{Q} \otimes_{\mathbb{Z}} C(G)$, respectively. By tensoring the exact sequence in the above lemma with $\mathbb{Q}$, one gets a ring isomorphism

$\mathbb{Q}\varphi : \mathbb{Q}B(G) \to \mathbb{Q}C(G)$. For each $[H] \in Cl(G)$, let $e_H^G \in \mathbb{Q}B(G)$ be the element defined by the condition that $s_K(e_H^G)$ is equal to unity when $[H] = [K]$ and zero otherwise. It is easy to see that $\mathbb{Q}\varphi$ maps $\{e_H^G \mid H \in Cl(G)\}$ to primitive idempotents of $\mathbb{Q}C(G) := \mathbb{Q}^{Cl(G)}$, hence they are primitive idempotents of $\mathbb{Q}B(G)$. Observe that each element $x \in \mathbb{Q}B(G)$ has a coordinate decomposition

$$x = \sum_{[H] \in Cl(G)} s_H(x) e_H^G.$$

The *ghost ring* of G is defined by

$$\beta(G) = (\mathbb{Q}\varphi)^{-1} C(G) = \bigoplus_{[H] \in Cl(G)} \mathbb{Z} e_H^G.$$

We often will identify $\beta(G)$ with $C(G)$ and use the notation $u(H)$ for $u \in \beta(G)$ and write

$$u = \sum_{[H] \in Cl(G)} u(H) e_H^G.$$

The Burnside ring $B(G)$ is a subring of $\beta(G)$. Therefore the group of units of $B(G)$ is a subgroup of the group of units

$$\beta(G)^{\times} = \bigoplus_{[H] \in Cl(G)} \{-1, 1\} e_H^G$$

which is an elementary abelian 2-group of rank $|Cl(G)|$. Thus $B(G)^{\times}$ is an elementary abelian 2-group of rank at most $|Cl(G)|$.

Notice that Theorem 2.1 can be used to characterize the subring $B(G)^{\times}$ in $\beta(G)^{\times}$. An element $x \in \beta(G)^{\times}$ is in $B(G)^{\times}$ if and only if

$$\sum_{gH \in W_G(H)} x(\langle g \rangle H) = 0 \pmod{|W_G(H)|}$$

for all $[H] \in Cl(G)$. But, this characterization is quite inconvenient for calculations. We often think $\beta(G)^{\times}$ as a vector space over $\mathbb{F}_2$ and $B(G)^{\times}$ as a subspace, so the characterizations given in terms of linear equations over $\mathbb{F}_2$ are usually more convenient. Such a characterization is given by Yoshida [12]:

Proposition 2.2 (Yoshida [12, Proposition 6.5]). *Let $u \in \beta(G)^{\times}$. Then, u is contained in $B(G)^{\times}$ if and only if for each subgroup H of G, the map*

$$gH \to \frac{u(\langle g \rangle H)}{u(H)}, \quad gH \in W_G(H),$$

is a linear character of $W_G(H)$.

Notice that we can rephrase Yoshida's characterization as follows:

Corollary 2.3. *Let $u \in \beta(G)^\times$. Then, u is contained in $B(G)^\times$ if and only if for each subquotient H/K of G, and for every $xK, yK \in H/K$,*

$$u(K) \cdot u(\langle x\rangle K) \cdot u(\langle y\rangle K) \cdot u(\langle xy\rangle K) = 1.$$

In the rest of the paper, we will consider $B(G)^\times$ as the subspace of $\beta(G)^\times$ satisfying the linear equations given in Corollary 2.3.

3. Maps between unit groups of Burnside rings

In this section, we briefly explain the maps between unit groups of Burnside rings and give some of the formulas involving these maps. A full account of this material can be found in [12].

Let G be a finite group, H be a subgroup and N be a normal subgroup of G, and $f : G' \to G$ be an isomorphism. Let X be a G-set, Y be an H-set, and Z be a G/N-set. Then, we have

$$\begin{aligned}
\mathrm{iso}^G_{G'} : X \to X &\quad \text{as an } G'\text{-set through } f : G' \to G \text{ (isomorphism map)},\\
\mathrm{inf}^G_{G/N} : Z \to Z &\quad \text{as a } G\text{-set through } G \to G/N \text{ (inflation map)},\\
\mathrm{inv}^G_{G/N} : X \to X^N &\quad \text{(invariant map)},\\
\mathrm{res}^G_H : X \to X &\quad \text{as an } H\text{-set (restriction map)},\\
\mathrm{jnd}^G_H : Y \to \mathrm{Map}_H(G, Y) &\quad \text{(multiplicative induction map)},
\end{aligned} \tag{1}$$

where $\mathrm{Map}_H(G, Y)$ is the set of maps $\alpha : G \to X$ such that $\alpha(h \cdot g) = h \cdot \alpha(g)$ for all $h \in H$, $g \in G$, with the action of G defined by $(k \cdot \alpha)(g) = \alpha(gk)$ for $k \in G$.

Notice that isomorphism, inflation, invariant, and restriction maps are additive and multiplicative, and hence they extend linearly to ring homomorphisms on the Burnside ring, and induce group homomorphisms on the unit group of Burnside ring. However, the multiplicative induction map is not linear, so it has to be considered separately.

Let $\mathbb{Z}^+$ denote the set of non-negative integers, and

$$B(G)^+ = \sum_{[H] \in Cl(G)} \mathbb{Z}^+[G/H]$$

be the free monoid of G-sets. The assignment $\mathrm{jnd}^G_H : Y \to \mathrm{Map}_H(G, Y)$ defines a multiplicative map from $B(H)^+$ to $B(G)$ which is not additive. In [5], Dress considers this map, and observes that the multiplicative induction is an algebraic map, and describes how one can extend it to map $\mathrm{jnd}^G_H : B(H) \to B(G)$. Unfortunately, Dress does not give many details for his arguments in [5]. A more detailed description of multiplicative induction can be found in Yoshida [12]. There is also a recent paper by Barker [1] where the multiplicative induction is defined more generally for monomial Burnside rings. Barker's paper also includes some further details on algebraic functions.

Another way to define the multiplicative induction map is to use tom Dieck's definition of the Burnside ring. In Chapter IV of [8], the Burnside ring $B(G)$ is defined as the ring of equivalence classes of finite (left) G-complexes under the equivalence relation defined as follows: $X \sim Y$ if and only if for every $H \leqslant G$, the spaces X^H and Y^H have the same Euler characteristic. Notice that now $-[X]$ can be expressed as $[Y \times X]$ where Y is a G-complex with trivial action and with Euler characteristic -1.

Given an H-complex X, one can define $\mathrm{jnd}_H^G X = \mathrm{Map}_H(G, Y)$ as the set of maps $\alpha : G \to X$ such that $\alpha(hg) = h\alpha(g)$ for all $h \in H$ and $g \in G$, with the action of G defined by $k\alpha : g \to \alpha(gk)$ for $k \in G$. To show that the assignment $X \to \mathrm{jnd}_H^G X$ from the set of H-complexes to the set of G-complexes induces a well defined map on the Burnside ring, one just needs to check that if X and Y are H-complexes such that $X \sim Y$, then $\mathrm{jnd}_H^G X \sim \mathrm{jnd}_H^G Y$. For this consider the following calculation (see [8, p. 244]):

$$\begin{aligned}
s_K\big(\mathrm{jnd}_H^G X\big) &= s_K\big(\mathrm{Map}_H(G, X)\big) = \chi\big[\big(\mathrm{Map}_H(G, X)\big)^K\big] \\
&= \chi\big[\mathrm{Map}_G\big(G/K, \mathrm{Map}_H(G, X)\big)\big] = \chi\big[\mathrm{Map}_H\big(\mathrm{res}_H^G(G/K), X\big)\big] \\
&= \chi\left[\mathrm{Map}_H\left(\coprod_{HgK \in H\backslash G/K} H/(H \cap {}^gK),\ X\right)\right] \\
&= \prod_{HgK \in H\backslash G/K} s_{H \cap {}^gK}(X).
\end{aligned} \tag{2}$$

Here $\chi(X)$ denotes the Euler characteristic of the G-complex X, and $s_K(X)$ is defined as $\chi(X^K)$ for every $K \leqslant G$. So, the assignment $X \to \mathrm{jnd}_H^G X$ induces a well-defined map on the Burnside ring. It is clear from the definition that this map is multiplicative, hence it induces a group homomorphism on the unit group of the Burnside ring. (There is a similar construction for bisets, using posets with group actions, in [2, Section 4.1].)

Considering an element $x \in B(G)$ as a class function through $x(K) = s_K(x)$, we have the following formulas:

$$\begin{aligned}
\mathrm{iso}_{G'}^G(x)(H') &= x(H) \quad \text{where } f(H') = H, \\
\mathrm{inf}_{G/N}^G(z)(K) &= z(KN/N), \\
\mathrm{inv}_{G/N}^G(x)(K/N) &= x(K), \\
\mathrm{res}_H^G(x)(K) &= x(K), \\
\mathrm{jnd}_H^G(y)(K) &= \prod_{HgK \in H\backslash G/K} y(H \cap {}^gK).
\end{aligned} \tag{3}$$

Using the definitions of these maps on G-sets (or on G-complexes), one obtains many composition formulas, such as the Mackey formula for the composition of multiplicative induction and restriction maps. These formulas are listed in Lemmas 3.1, 3.3, and 3.4 in [12]. For example, if N is a normal subgroup of G, and H is a subgroup of G containing N, we have:

$$\begin{aligned} \operatorname{res}^G_H \operatorname{inf}^G_{G/N} &= \operatorname{inf}^H_{H/N} \operatorname{res}^{G/N}_{H/N}, \\ \operatorname{jnd}^G_H \operatorname{inf}^H_{H/N} &= \operatorname{inf}^G_{G/N} \operatorname{jnd}^{G/N}_{H/N}, \\ \operatorname{inv}^H_{H/N} \operatorname{res}^G_H &= \operatorname{res}^{G/N}_{H/N} \operatorname{inv}^G_{G/N}, \\ \operatorname{inv}^G_{G/N} \operatorname{jnd}^G_H &= \operatorname{jnd}^{G/N}_{H/N} \operatorname{inv}^H_{H/N}. \end{aligned} \tag{4}$$

Notice that using the formulas in Eq. (3) as a definition, we can extend all the maps in the list to $C(G)$ or equivalently to $\beta(G)$, and hence obtain group homomorphisms on $C(G)^\times$ or on $\beta(G)^\times$ as the extension of group homomorphisms on $B(G)^\times$. Since $B(G)$ has a finite index in $\beta(G)$, the extended maps will also have the same composition formulas.

Another way to define these maps on the unit group of $\beta(G)$ is to consider the duality pairing

$$(\,,\,) : \beta(G)^\times \otimes \mathbb{F}_2 B(G) \to \{\pm 1\}$$

defined by

$$(u, x) = \prod_{H \in Cl(G)} (\gamma_H)^{\alpha_H}$$

where $u = \sum_{[H]\in Cl(G)} \gamma_H e^G_H \in \beta(G)^\times$ and $x = \sum_{[H]\in Cl(G)} \alpha_H [G/H] \in \mathbb{F}_2 B(G)$. Here $\mathbb{F}_2 B(G)$ denotes the mod 2 reduction of the Burnside ring, i.e., $\mathbb{F}_2 B(G) = \mathbb{F}_2 \otimes_{\mathbb{Z}} B(G)$. Note that the group homomorphisms we defined above as extensions of maps on $B(G)^\times$ can also be defined as duals of maps between the Burnside rings. To illustrate this duality, we will show that

$$\operatorname{jnd}^G_H : \beta(H)^\times \to \beta(G)^\times$$

is dual to the restriction map

$$\operatorname{res}^G_H : \mathbb{F}_2 B(G) \to \mathbb{F}_2 B(H).$$

First observe that for every $u \in \beta(G)^\times$, we have $u(K) = (u, [G/K])$. So, for some $y \in \beta(G)^\times$, the last formula in Eq. (3) gives

$$\big(\operatorname{jnd}^G_H y, [G/K]\big) = \left(y, \sum_{HgK \in H\backslash G/K} \big[H/(H \cap {}^g K)\big]\right) = \big(y, \operatorname{res}^G_H [G/K]\big).$$

So, by linearity, we get $(\operatorname{jnd}^G_H y, x) = (y, \operatorname{res}^G_H x)$ for every $y \in \beta(G)^\times$ and $x \in \mathbb{F}_2 B(G)$.

4. The proof of the induction theorem

The aim of this section is to prove Theorem 1.1 stated in the introduction. In the proof, we will be using Yoshida's characterization of units in $B(G)^\times$ given in Corollary 2.3. We first state a proposition from which Theorem 1.1 follows as a corollary:

Proposition 4.1. *Let G be a nontrivial 2-group which is not isomorphic to a dihedral group of order 2^n with $n \geqslant 4$. Then, the map*

$$\prod_{H/K \neq G/1} \mathrm{jnd}_H^G \inf_{H/K}^H : \prod_{H/K \neq G/1} B(H/K)^\times \to B(G)^\times$$

is surjective, where the sum is over all proper subquotients of G.

This is a general strategy for proving induction theorems. To see that Theorem 1.1 follows from Proposition 4.1, one just needs to check that the generalized induction map $\mathrm{jnd}_H^G \inf_{H/K}^H$ is transitive. This follows from the following calculation: Let H'/K' and H/K be two subquotients of G such that $K \leqslant K' \trianglelefteq H' \leqslant H$. Then, applying the second equation in Eq. (4), we get

$$\mathrm{jnd}_H^G \inf_{H/K}^H \mathrm{jnd}_{H'/K}^{H/K} \inf_{H'/K'}^{H'/K} = \mathrm{jnd}_H^G \mathrm{jnd}_{H'}^H \inf_{H'/K}^{H'} \inf_{H'/K'}^{H'/K} = \mathrm{jnd}_{H'}^G \inf_{H'/K'}^{H'}.$$

To prove the proposition, we use a well known argument used to prove similar results (see, for example, [3,4]). The idea is to reduce the proof to the case where G has no normal subgroups isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, and then use the classification of such 2-groups.

We first consider the case where G has a central subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Lemma 4.2. *Let G be a 2-group which includes a central subgroup E isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. Let H_1, H_2, and H_3 be the distinct subgroups of E of order 2. Then,*

$$\prod_{i=1}^{3} \inf_{G/H_i}^G : \prod_{i=1}^{3} B(G/H_i)^\times \to B(G)^\times$$

is surjective.

Proof. Let c_1 and c_2 be the generators of H_1 and H_2, respectively. Take $u \in B(G)^\times$, and let $u_i = \inf_{G/H_i}^G \mathrm{inv}_{G/H_i}^G u$. Consider the element $w = uu_1u_2u_3$. For every $H \leqslant G$, we have

$$\begin{aligned} w(H) &= u(H) \cdot u_1(H) \cdot u_2(H) \cdot u_3(H) \\ &= u(H) \cdot u(H_1H) \cdot u(H_2H) \cdot u(H_3H) \\ &= u(H) \cdot u(\langle c_1 \rangle H) \cdot u(\langle c_2 \rangle H) \cdot u(\langle c_1c_2 \rangle H). \end{aligned}$$

If c_1, c_2, or c_1c_2 is in H, then it is clear that $w(H) = 1$. So, assume that H is a subgroup such that $E \cap H = \{1\}$. Then, EH/H is a subquotient of G isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, and we again get $w(H) = 1$ by Corollary 2.3. This shows that $w = 1$, and hence $u = u_1u_2u_3$. Therefore, u is in the image of $\prod_{i=1}^{3} \inf_{G/H_i}^{G}$. $\square$

If G is a 2-group which has no central $\mathbb{Z}/2 \times \mathbb{Z}/2$, then the center $Z(G)$ must be cyclic. In this case, G has a unique central element of order 2, which we usually denote by c. We have the following decomposition for $B(G)^\times$.

Lemma 4.3. *Let G be a 2-group with cyclic center and let c be the unique central element of order 2. Then, $B(G)^\times = \mathrm{im}\{\inf_{G/\langle c\rangle}^{G}\} \times B(G,c)^\times$ where $B(G,c)^\times$ is the set of all units $u \in B(G)^\times$ such that $u(H) = 1$ for every $H \leqslant G$ such that $c \in H$.*

Proof. Note that for every normal subgroup $K \trianglelefteq G$, we have

$$B(G)^\times \cong \mathrm{im}\bigl\{\inf_{G/K}^{G} : B(G/K)^\times \to B(G)^\times\bigr\} \times \ker\bigl\{\mathrm{inv}_{G/K}^{G} : B(G)^\times \to B(G/K)^\times\bigr\}.$$

This is because the composite $\mathrm{inv}_{G/K}^{G} \inf_{G/K}^{G}$ is the identity homomorphism. Applying this to $K = \langle c\rangle$, we get

$$B(G)^\times = \mathrm{im}\bigl\{\inf_{G/\langle c\rangle}^{G}\bigr\} \times \ker\bigl\{\mathrm{inv}_{G/\langle c\rangle}^{G}\bigr\}.$$

If $c \in H \leqslant G$, then we have $u(H) = s_H(u) = s_{H/\langle c\rangle}(\mathrm{inv}_{G/\langle c\rangle}^{G} u)$ for every $u \in B(G)^\times$. It follows that $u \in \ker\{\mathrm{inv}_{G/\langle c\rangle}^{G}\}$ if and only if $u(H) = 1$ for every $H \leqslant G$ such that $c \in H$. Thus, $\ker\{\mathrm{inv}_{G/\langle c\rangle}^{G}\} = B(G,c)^\times$. $\square$

Lemma 4.4. *Let G be a 2-group with cyclic center. Assume that G has a normal subgroup $E \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ generated by $a, c \in E$ where c is central. Let H be the centralizer of E. Then,*

$$B(G,c)^\times \subseteq \mathrm{im}\bigl\{\mathrm{jnd}_H^G \inf_{H/\langle a\rangle}^{H} : B(H/\langle a\rangle)^\times \to B(G)^\times\bigr\}.$$

Proof. Let $u \in B(G,c)^\times$, then $u(H) = 1$ for every $H \leqslant G$ such that $c \in H$. Define

$$w = \mathrm{jnd}_H^G \inf_{H/\langle a\rangle}^{H} \mathrm{inv}_{H/\langle a\rangle}^{H} \mathrm{res}_H^G u.$$

We will show that $u = w$. First note that $H = C_G(E)$ is a normal subgroup of G with index 2. This is because $Aut(E) = GL(2,2)$ has order $(2^2-1)(2^2-2) = 6$.

For every $K \leqslant G$, we have

$$\begin{aligned} w(K) &= \bigl[\mathrm{jnd}_H^G \inf_{H/\langle a\rangle}^{H} \mathrm{inv}_{H/\langle a\rangle}^{H} \mathrm{res}_H^G u\bigr](K) \\ &= \prod_{HgK \in H\backslash G/K} \bigl[\inf_{H/\langle a\rangle}^{H} \mathrm{inv}_{H/\langle a\rangle}^{H} \mathrm{res}_H^G u\bigr](H \cap {}^gK) \\ &= \prod_{HgK \in H\backslash G/K} u\bigl(\langle a\rangle(H \cap {}^gK)\bigr) \end{aligned}$$

$$= \prod_{gHK \in G/HK} u\big(\langle a^g\rangle(H\cap K)\big).$$

Now, we consider the following two cases:

Case 1. Assume that $K \nleqslant H$. Then $HK = G$ and $w(K) = u(\langle a\rangle(H\cap K))$. If $K\cap E = \langle a\rangle$ or $\langle ac\rangle$, then a will be central in K, contradicting the assumption $K \nleqslant H = C_G(E)$. So, we either have $c \in K$ or $K \cap E = \{1\}$.

If $c \in K$, then $c \in H\cap K$, and hence $w(K) = 1 = u(K)$. So, assume $E \cap K = \{1\}$. Consider the subgroup series $(H\cap K) \trianglelefteq EK \leqslant G$. Pick an element $k \in K - (K\cap H)$, and let $\bar{k}, \bar{a}, \bar{c}$ denote the images of k, a, c in the quotient group $EK/(H\cap K)$. We have $(\bar{k})^2 = (\bar{a})^2 = 1$ and $[\bar{a}, \bar{k}] = \bar{c}$. So, $EK/(H\cap K) \cong D_8$, the dihedral group of order 8. By Corollary 2.3, we get

$$u(H\cap K)\cdot u\big(\langle a\rangle(H\cap K)\big)\cdot u\big(\langle k\rangle(H\cap K)\big)\cdot u\big(\langle ak\rangle(H\cap K)\big) = 1. \tag{5}$$

Since $(\overline{ak})^2 = \bar{c}$, we have $c \in \langle ak\rangle(H\cap K)$, and hence $u(\langle ak\rangle(H\cap K)) = 1$. Note also that $K = \langle k\rangle(H\cap K)$, so Eq. (5) reduces to

$$u(H\cap K)\cdot w(K)\cdot u(K) = 1. \tag{6}$$

To finish the proof we need to show $u(H\cap K) = 1$. For this, we consider the subquotient $E(H\cap K)/(H\cap K)$ which is isomorphic to $\mathbb{Z}/2\times\mathbb{Z}/2$. By Corollary 2.3, we have

$$u(H\cap K)\cdot u\big(\langle a\rangle(H\cap K)\big)\cdot u\big(\langle c\rangle(H\cap K)\big)\cdot u\big(\langle ac\rangle(H\cap K)\big) = 1.$$

Since $\bar{a}$ is conjugate to $\overline{ac}$, this equation reduces to $u(H\cap K) = u(\langle c\rangle(H\cap K))$. It is clear that $c \in \langle c\rangle(H\cap K)$, so we conclude that $u(H\cap K) = 1$.

Case 2. Assume that $K \leqslant H$. Then $HK = H$ and $w(K) = u(\langle a\rangle K)\cdot u(\langle ac\rangle K)$. If $c \in K$, then both $w(K)$ and $u(K)$ are equal to 1. If $K\cap E = \langle a\rangle$ or $\langle ac\rangle$, then $w(K) = u(K)\cdot u(\langle c\rangle K) = u(K)$. Finally, if $K \cap E = \{1\}$, then we consider $K \trianglelefteq KE \leqslant G$, and apply Corollary 2.3. This gives

$$u(K)\cdot u(\langle a\rangle K)\cdot u(\langle c\rangle K)\cdot u(\langle ac\rangle K) = 1$$

from which we obtain

$$w(K) = u(\langle a\rangle K)\cdot u(\langle ac\rangle K) = u(K).$$

This completes the proof of the lemma. □

For the proof of Proposition 4.1, it remains to consider the case where G is a 2-group which has no normal subgroups isomorphic to $\mathbb{Z}/2\times\mathbb{Z}/2$. In this case, G is said to have normal 2-rank one. Note that a 2-group G has normal 2-rank one if and only if every abelian normal subgroup of G is cyclic.

The classification of 2-groups with no non-cyclic abelian subgroups is given in Chapter 5 of Gorenstein [6] as Theorem 4.10. We quote this result here:

Theorem 4.5. *Let G be a 2-group with normal 2-rank equal to one. Then, G is isomorphic to one of the following groups*:

(a) *cyclic group C_{2^n} ($n \geqslant 0$)*;
(b) *generalized quaternion group Q_{2^n} ($n \geqslant 3$)*;
(c) *dihedral group D_{2^n} ($n \geqslant 4$)*;
(d) *semi-dihedral group SD_{2^n} ($n \geqslant 4$)*.

We have the following lemma:

Lemma 4.6. *Let G be a 2-group isomorphic to one of the following groups*:

(a) *cyclic group C_{2^n} ($n \geqslant 2$)*;
(b) *generalized quaternion group Q_{2^n} ($n \geqslant 3$)*;
(c) *semi-dihedral group SD_{2^n} ($n \geqslant 4$)*.

Then, $B(G, c)^\times = \{1\}$.

Proof. Let G be a cyclic group or a generalized quaternion group. Then, G has no subgroups isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, so the unique central element c is the only element of order 2 in G. This implies, in particular, that c is included in every non-trivial subgroup of G. So, if u is a unit in $B(G, c)^\times$, then $u(H) = 1$ for every non-trivial subgroup $H \leqslant G$. We claim that if $|G| > 2$, then $u(\{1\})$ is also unity. Observe that if $|G| > 2$, then G must include an element g of order 4, such that $g^2 = c$. Now, consider the subgroup series $\{1\} \trianglelefteq \langle g\rangle \leqslant G$. Applying Corollary 2.3 for $K = \{1\}$ and $x = y = g$, we get $u(\{1\}) = u(\langle g^2\rangle) = 1$, hence $u = 1$.

Now assume that $G \cong SD_{2^n}$ ($n \geqslant 4$). A presentation for G can be given as

$$G = \left\langle b, z \mid z^{2^{n-1}} = b^2 = 1,\ bzb = z^{-1+2^{n-2}}\right\rangle.$$

Note that $c = z^{2^{n-2}}$ is the unique central element of order 2. Take $u \in B(G, c)^\times$. If H is a subgroup of G such that $H \cap \langle z\rangle \neq \{1\}$, then $c \in H$, and hence $u(H) = 1$. So, assume $H \cap \langle z\rangle = \{1\}$. Since $\langle z\rangle$ has index 2 in G, the order of H is 2. Let $H = \langle h\rangle$. Then, $h = bz^m$ for some m. Since

$$(bz^m)^2 = bz^m bz^m = z^{(-1+2^{n-2})m} z^m = z^{2^{n-2}m} = c^m,$$

m must be an even integer. Note that $(hz)^2 = (bz^{m+1})^2 = c^{m+1} = c$, so $c \in \langle hz\rangle$.

Applying Corollary 2.3 to the subquotient $G/\{1\}$ we get

$$u(\{1\}) \cdot u(\langle h\rangle) \cdot u(\langle z\rangle) \cdot u(\langle hz\rangle) = 1$$

which reduces to $u(\langle h\rangle) = u(\{1\})$. Similarly, Corollary 2.3 applied to the subquotient $\langle c, b\rangle/\{1\}$ gives $u(\{1\}) = u(\langle c\rangle) = 1$. Combining these, we get $u(\langle h\rangle) = 1$. Thus, $u(H) = 1$ for all $H \leqslant G$, giving $u = 1$ as desired. $\square$

Lemma 4.6, together Theorem 4.5, completes the proof of Proposition 4.1 for all cases except the case $G \cong C_2$. Note that in this case

$$B(G)^\times = \beta(G)^\times = \left\{\alpha_1 e_1^G + \alpha_2 e_G^G \mid \alpha_1, \alpha_2 = \pm 1\right\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$$

and

$$B(G, c)^\times = \left\{\alpha e_1^G + e_G^G \mid \alpha = \pm 1\right\} \cong \mathbb{Z}/2.$$

It is easy to see that

$$\mathrm{jnd}_{\{1\}}^G(-1) \cdot \mathrm{inf}_{G/G}^G(-1) = -e_1^G + e_G^G.$$

So, the map

$$\left(\mathrm{jnd}_{\{1\}}^G, \mathrm{inf}_{G/G}^G\right) : B(\{1\})^\times \times B(G/G)^\times \to B(G)^\times$$

is surjective. This completes the proof of Proposition 4.1, and hence the proof of Theorem 1.1. We end this section with two refinements of Theorem 1.1 which we use later for applications.

Corollary 4.7. *Theorem* 1.1 *still holds if we replace each $B(H/K)^\times$ with $B(H/K, c_{H/K})^\times$ for every subquotient $H/K \in \mathcal{H}$ with $|H/K| > 1$, where $c_{H/K}$ denotes the unique central element of order* 2 *in H/K.*

Proof. By Lemma 4.3, for each subquotient for every $H/K \in \mathcal{H}$ with $|H/K| > 1$, there is a decomposition

$$B(H/K)^\times = \mathrm{im}\left\{\mathrm{inf}_{(H/K)/\langle c_{H/K}\rangle}^{H/K}\right\} \times B(H/K, c_{H/K})^\times$$

where $c_{H/K}$ is the unique central element of order 2 in H/K. Let $I(H/K)^\times$ denote the image of inflations in the above decomposition. By the transitivity of generalized induction map $\mathrm{jnd}_H^G \mathrm{inf}_{H/K}^H$, it is easy to see that for every $H/K \in \mathcal{H}$ with $|H/K| > 1$, the subgroup $\mathrm{jnd}_H^G \mathrm{inf}_{H/K}^H(I(H/K)^\times)$ is included in the image of the map

$$\prod_{H/K \in \mathcal{H}'} \mathrm{jnd}_{H'}^G \mathrm{inf}_{H'/K'}^{H'} : \prod_{H'/K' \in \mathcal{H}'} B(H'/K')^\times \to B(G)^\times$$

where $\mathcal{H}' = \{H'/K' \in \mathcal{H} \mid H'/K' < H/K\}$. So, starting from the subquotients with bigger order we can replace $B(H/K)^\times$ with $B(H/K, c_{H/K})^\times$ whenever $|H/K| > 1$. $\square$

Corollary 4.8. *Theorem* 1.1 *still holds if we replace the collection $\mathcal{H}$ with a collection of representatives of conjugacy classes of subquotients in $\mathcal{H}$.*

Proof. We say two subquotients H/K and H'/K' are conjugate if there is an elements $g \in G$ such that $H' = H^g$ and $K' = K^g$. Note that in this case the images of $\mathrm{jnd}_H^G \inf_{H/K}^H$ and $\mathrm{jnd}_{H'}^G \inf_{H'/K'}^{H'}$ are equal, so it is enough to take one representative from each conjugacy class.

5. The surjectivity of tom Dieck's homomorphism

The main purpose of this section is to prove Corollary 1.2 stated in the introduction. First we recall the definition of tom Dieck's homomorphism.

Let G be a finite group, and let $R(G,\mathbb{R})$ denote the Grothendieck ring of isomorphism classes of (left) $\mathbb{R}G$-modules where addition and multiplication are defined by direct sums and tensor products. Given an $\mathbb{R}G$-module V, consider the following element in $\beta(G)^\times$ defined as

$$\Theta(V) = \sum_{[H]\in Cl(G)} \mathrm{sgn}\big(\dim_{\mathbb{R}} V^H\big)\, e_H^G$$

where $\mathrm{sgn}(n) = (-1)^n$. Using a geometric argument, tom Dieck [8] proved that $\Theta(V)$ actually lies in $B(G)^\times$. Later, Yoshida [12] gave an algebraic proof (for a more general statement which holds for real valued characters) which uses the characterization given in Proposition 2.2. It is clear that $\Theta(V \oplus W) = \Theta(V)\Theta(W)$, so Θ defines a group homomorphism

$$\Theta : R(G,\mathbb{R}) \to B(G)^\times$$

from the underlying additive group of $R(G,\mathbb{R})$ to the multiplicative group $B(G)^\times$ which is usually referred as tom Dieck's homomorphism.

Similar to the maps defined on unit group of the Burnside ring, there are restriction, induction, isomorphism, inflation, and invariant maps defined on group rings. Given a map $f : H \to K$, an $\mathbb{R}K$-module V can be considered as an $\mathbb{R}H$-module through the map $f : H \to K$. This gives a ring homomorphism $\Phi_f : R(K,\mathbb{R}) \to R(H,\mathbb{R})$. If $f : H \to G$ is an inclusion map of a subgroup $H \leqslant G$, then this ring homomorphism is called *restriction map* and is denoted by res_H^G. When $f : G \to G/N$ is a quotient map for a normal subgroup $N \trianglelefteq G$, then the ring homomorphism we obtain is called *inflation map* and it is denoted by $\inf_{G/N}^G$. Finally, if $f : G' \to G$ is an isomorphism, we get the *isomorphism map* which is denoted by $\mathrm{iso}_{G'}^G$.

Aside from these maps, we have two more maps, induction and invariant maps, which are not ring homomorphisms, but group homomorphisms of the underlying additive group. The *induction map* $\mathrm{ind}_H^G : R(H,\mathbb{R}) \to R(G,\mathbb{R})$ is the linear extension of the assignment $V \to \mathbb{R}G \otimes_{\mathbb{R}H} V$ defined for every $\mathbb{R}H$-module V where $H \leqslant G$. The *invariant map* $\mathrm{inv}_{G/N}^G : R(G,\mathbb{R}) \to R(G/N,\mathbb{R})$ is defined as the linear extension of the assignment

$W \to W^N$ where W is an $\mathbb{R}G$-module and N is a normal subgroup of G. We will need the following result from Yoshida [12].

Lemma 5.1 (Yoshida [12, Lemma 3.5]). *The tom Dieck homomorphism commutes with induction, restriction, isomorphism, inflation, and invariant maps.*

Now, we are ready to prove Corollary 1.2.

Proof of Corollary 1.2. Consider the following diagram:

$$\begin{tikzcd}
\bigoplus_{H/K\in\mathcal{H}} R(H/K,\mathbb{R}) \arrow[r, "\bigoplus \mathrm{ind}_H^G \mathrm{inf}_{H/K}^H"] \arrow[d, "\prod \Theta_{H/K}"] & R(G,\mathbb{R}) \arrow[d, "\Theta_G"] \\
\prod_{H/K\in\mathcal{H}} B(H/K)^\times \arrow[r, "\prod \mathrm{jnd}_H^G \mathrm{inf}_{H/K}^H"] & B(G)^\times.
\end{tikzcd}$$

By Lemma 5.1, this diagram commutes. By Corollary 4.7, the horizontal map on the bottom is surjective even when each $B(H/K)^\times$ is replaced with $B(H/K, c_{H/K})^\times$ for subquotients $H/K \in \mathcal{H}$ with $|H/K| > 1$. When $H = K$, we have $B(H/H)^\times = \{\pm 1\}$, which is the image of trivial $\mathbb{R}H/H$-module $\mathbb{R}$ under $\Theta_{H/H}$. So, to prove that Θ_G is surjective, it is enough to show that $B(G, c_{H/K})$ is in the image of $\Theta_{H/K}$ for all $H/K \in \mathcal{H}$ isomorphic to a dihedral group of order 2^n with $n \geqslant 4$. Hence, the proof follows from the following lemma. □

Lemma 5.2. *Let G be a 2-group isomorphic to a dihedral group of order 2^n with $n \geqslant 4$. Then, $B(G,c)^\times \cong \mathbb{Z}/2$, and the generator of $B(G,c)^\times$ is an element of the form $\Theta(V)$ for some $V \in R(G,\mathbb{R})$.*

Proof. Let $G \cong D_{2^n}$ with $n \geqslant 4$. Consider the following presentations

$$G = \langle b, z \mid z^{2^{n-1}} = b^2 = 1,\ bzb = z^{-1} \rangle = \langle a, b \mid a^2 = b^2 = (ab)^{2^{n-1}} = 1 \rangle$$

where $z = ab$. Note that $c = z^{2^{n-2}}$ is a central element. If g is an element G which is not in $\langle z \rangle$, then $g = bz^i$ for some i, and

$$(bz^i)^{z^j} = z^{-j} b z^{i+j} = (bz^j b) b z^{i+j} = bz^{i+2j}.$$

Hence every element $g \in G$ is either conjugate to b or $a = bz^{-1}$. Let H be a non-trivial subgroup of G such that $c \notin H$. Then, $H \cap \langle z \rangle = \{1\}$, and hence H is a cyclic subgroup of order 2. If h is a generator of H, then h is conjugate to a or b, and therefore H is conjugate to $\langle a \rangle$ or $\langle b \rangle$.

Let V be 2-dimensional real representation of G where z action is a rotation by $\pi/2^{n-2}$ and b action is a reflection around the x-axis. Then c acts by multiplication with -1, so

$\dim V^H = 0$ if $c \in H$. If c is not in H, then H is conjugate to $\langle a\rangle$ or $\langle b\rangle$. It is obvious that $\dim_{\mathbb{R}} V^a = \dim_{\mathbb{R}} V^b = 1$. So, $\Theta(V) = 1 - 2(e^G_{\langle a\rangle} + e^G_{\langle b\rangle})$.

We claim that $\Theta(V)$ is the only non-trivial unit in $B(G,c)^\times$. Let $u \in B(G,c)^\times$. Then, $u(H) = 0$ for every $c \in H$. If c is not in H, then H is conjugate to $\langle a\rangle$ or $\langle b\rangle$. So,

$$u = 1 - 2\bigl(\alpha_{\langle a\rangle} e^G_{\langle a\rangle} + \alpha_{\langle b\rangle} e^G_{\langle b\rangle}\bigr)$$

for some $\alpha_{\langle a\rangle}, \alpha_{\langle b\rangle} \in \{0,1\}$. We will show that $\alpha_{\langle a\rangle} = \alpha_{\langle b\rangle}$. For this, we apply Corollary 2.3 to subquotients $G/\{1\}$ and $\langle a,c\rangle/\{1\}$, and get

$$u(\{1\}) \cdot u(\langle a\rangle) \cdot u(\langle b\rangle) \cdot u(\langle ab\rangle) = 1 \quad \text{and} \quad u(\{1\}) = u(\langle c\rangle) = 1.$$

These give $u(\langle a\rangle) = u(\langle b\rangle)$, and hence $\alpha_{\langle a\rangle} = \alpha_{\langle b\rangle}$. Thus, the proof is complete. $\square$

6. The unit group as a $B(G)$-module

In this section we define an action of $B(G)$ on $B(G)^\times$. The material is well-known, and can be found in Yoshida [12] and Dress [5]. We include it here for convenience, and to introduce the notation.

Let G be a finite group. For left G-sets X and Y, let $[Y] \uparrow [X] := [\mathrm{Map}(X,Y)]$ denote the equivalence class of the G-set consisting of all maps from X to Y with G action defined by

$$(g \cdot \alpha)(x) = g\alpha\bigl(g^{-1}x\bigr)$$

for $\alpha : X \to Y$, $g \in G$, and $x \in X$. As before let $B(G)^+$ be the monoid generated by G-sets. The assignment $([Y],[X]) \to [Y] \uparrow [X]$ gives a map

$$(\ ) \uparrow (\ ) : B(G)^+ \times B(G)^+ \to B(G)^+$$

satisfying

$$\begin{aligned}
\bigl([Y_1] \cdot [Y_2]\bigr) \uparrow [X] &= \bigl([Y_1] \uparrow [X]\bigr)\bigl([Y_2] \uparrow [X]\bigr),\\
[Y_1] \uparrow \bigl([X_1] + [X_2]\bigr) &= \bigl([Y] \uparrow [X_1]\bigr)\bigl([Y] \uparrow [X_2]\bigr),\\
[Y] \uparrow \bigl([X_1] \cdot [X_2]\bigr) &= \bigl([Y] \uparrow [X_1]\bigr) \uparrow [X_2].
\end{aligned} \tag{7}$$

When X is a transitive G-set, say $[X] = [G/H]$, we have

$$[Y] \uparrow [X] = \bigl[\mathrm{Map}(G/H, Y)\bigr] = \bigl[\mathrm{Map}_H\bigl(G, \mathrm{res}^G_H Y\bigr)\bigr] = \mathrm{jnd}^G_H \mathrm{res}^G_H [Y],$$

so the assignment $[Y] \to [Y] \uparrow [G/H]$ can be extended to a map

$$(\ ) \uparrow [G/H] : B(G) \to B(G)$$

defined by $y \uparrow [G/H] = \mathrm{jnd}_H^G \operatorname{res}_H^G y$. Hence, we obtain a map

$$(\ ) \uparrow (\ ) : B(G) \times B(G)^+ \to B(G)$$

such that

$$y \uparrow x = \prod_{H \in Cl(G)} \left(\mathrm{jnd}_H^G \operatorname{res}_H^G y\right)^{\alpha_H} \quad \text{for } x = \sum_{H \in Cl(G)} \alpha_H [G/H] \in B(G)^+. \tag{8}$$

Note that this equation makes sense only when α_H is non-negative for all $H \leqslant G$, so the action of $B(G)^+$ on $B(G)$ cannot be extended to a $B(G)$-action.

On the other hand, when y is a unit, then the formula for $y \uparrow x$ given in Eq. (8) makes sense even when α_H is a negative integer for some $H \leqslant G$. So, we have a map

$$(\ ) \uparrow (\ ) : B(G)^\times \times B(G) \to B(G)^\times$$

which defines a $B(G)$-module structure for $B(G)^\times$. Note that $B(G)^\times \uparrow 2B(G) = \{1\}$, so $B(G)^\times$ can also be considered as a module over $\mathbb{F}_2 B(G) := \mathbb{F}_2 \otimes_{\mathbb{Z}} B(G)$.

Proposition 6.1. *There is a $B(G)$-action on $B(G)^\times$ derived from the pairing $Y \uparrow X :=$ $\mathrm{Map}(X, Y)$ on G-sets satisfying the following formula*:

$$s_K(u \uparrow x) = \prod_{[H] \in Cl(G)} \left\{ \prod_{KgH \in K \backslash G/H} \left[u(K^g \cap H)\right]^{x_H} \right\} \tag{9}$$

where $u \in B(G)^\times$ and $x = \sum_{[H] \in Cl(G)} x_H [G/H] \in B(G)$.

We can extend the $B(G)$-action on $B(G)^\times$ to an action on $\beta(G)^\times$ (or equivalently on $C(G)^\times$). For this, we first extend the map $(\ ) \uparrow (\ ) : B(G) \times B(G)^+ \to B(G)$ to a map $(\ ) \uparrow (\ ) : \beta(G) \times B(G)^+ \to \beta(G)$. Since $B(G)$ has a finite index in $\beta(G)$, the extension also satisfies the identities in Eq. (7). Repeating the arguments used above, we obtain a $B(G)$ action on $\beta(G)^\times$. Note that $B(G)$ action on $\beta(G)^\times$ also satisfies the formula given in Eq. (9).

In Section 2, we introduced a duality pairing $\langle \cdot, \cdot \rangle : \beta(G)^\times \otimes \mathbb{F}_2 B(G) \to \{\pm 1\}$ where

$$\langle u, x \rangle = \prod_{[H] \in Cl(G)} (\gamma_H)^{\alpha_H}$$

for $u = \sum_{[H] \in Cl(G)} \gamma_H e_H^G \in \beta(G)^\times$ and $x = \sum_{[H] \in Cl(G)} \alpha_H [G/H] \in \mathbb{F}_2 B(G)$. Note that this is the bilinear map of elementary abelian 2-groups (written multiplicatively on the first entry and additively on the second) which satisfies

$$\left\langle e_K^G, [G/H] \right\rangle = \begin{cases} 1 & \text{if } [H] = [K], \\ 0 & \text{if } [H] \neq [K]. \end{cases}$$

This means that for every u in $\beta(G)^\times$, we have $\langle u, [G/H]\rangle = s_H(u)$. On the other hand, by Proposition 6.1, we have $s_G(u \uparrow [G/H]) = s_H(u)$. So, we conclude the following:

Lemma 6.2. *The pairing $\langle\cdot\,,\cdot\rangle : \beta(G)^\times \otimes \mathbb{F}_2 B(G) \to \{\pm 1\}$ can expressed by the formula*

$$\langle u, x\rangle = s_G(u \uparrow x)$$

for every $u \in \beta(G)^\times$ and $x \in \mathbb{F}_2 B(G)$.

As a consequence of this we obtain the following:

Proposition 6.3. *As a $\mathbb{F}_2 B(G)$-module $\beta(G)^\times$ is isomorphic to* $\mathrm{Hom}(B(G), \mathbb{F}_2)$. *So, as a $B(G)$-module, $B(G)^\times$ is a submodule of* $\mathrm{Hom}(B(G), \mathbb{F}_2)$.

Proof. This follows from the identity

$$\langle (u \uparrow x), y\rangle = s_G\big((u \uparrow x) \uparrow y\big) = s_G\big(u \uparrow (xy)\big) = \langle u, xy\rangle. \quad \square$$

7. The surjectivity of the exponential map

In this section, we define the exponential map, and study some basic properties of this map. The main objective of this section is to prove Corollary 1.3 stated in the introduction. We start with the definition of exponential map.

Definition 7.1. The map $\exp : B(G) \to B(G)^\times$ defined by $\exp(x) = (-1) \uparrow x$ is called the *exponential map*.

Notice that for a G-set $X = \sum_{H \leqslant_G G} x_H [G/H]$, we have

$$s_K\big[\exp(X)\big] = \prod_{H \leqslant_G G} \left\{ \prod_{KgH \in K\backslash G/H} (-1)^{x_H} \right\} = (-1)^{|X/K|}.$$

One can consider the exponential map as a map $\exp : \mathbb{F}_2 B(G) \to \beta(G)^\times$, where the image is in $B(G)^\times$. Then, it is possible to describe this map as a linear transformation, where the matrix of the transformation with suitable choice of basis is the mod-2 reduction of the matrix of double cosets. So, the rank of the image of the exponential map is equal to the rank of mod-2 reduction of matrix of double cosets.

Recall that, for every $x, y \in B(G)$, we have

$$\exp(x) \uparrow y = \big((-1) \uparrow x\big) \uparrow y = (-1) \uparrow (xy) = \exp(xy),$$

so the exponential is a $B(G)$-module map. In particular, the image of the exponential map is the submodule of $B(G)^\times$ generated by (-1). The image of the exponential map is usually denoted by $(-1) \uparrow B(G)$.

The exponential map is related to the tom Dieck's homomorphism in the following way:

Lemma 7.2. *Let G be a 2-group, and let $\pi_{\mathbb{R}} : B(G) \to R(G, \mathbb{R})$ be the linearization map. Then*

$$\exp = \Theta \circ \pi_{\mathbb{R}}$$

where $\Theta : R(G, \mathbb{R}) \to B(G)^{\times}$ is tom Dieck's homomorphism.

Proof. For every G-set X and $[K] \in Cl(G)$, we have

$$s_K\bigl[\exp(X)\bigr] = (-1)^{|X/K|} = \operatorname{sgn}\bigl[\dim_{\mathbb{R}}\bigl(\pi_{\mathbb{R}}(X)\bigr)^K\bigr].$$

So, the result follows. □

Let $R(G, \mathbb{Q})$ denote the ring of rational representations of G. We can consider $R(G, \mathbb{Q})$ as a subring of $R(G, \mathbb{R})$ through linear extension of the map $V \to \mathbb{R} \otimes_{\mathbb{Q}} V$. In particular tom Dieck's homomorphism restricts to map

$$\Theta_{\mathbb{Q}} : R(G, \mathbb{Q}) \to B(G)^{\times} \quad \text{where } s_K\bigl[\Theta_{\mathbb{Q}}(V)\bigr] = \operatorname{sgn}\bigl[\dim_{\mathbb{Q}} V^K\bigr].$$

We have the following:

Lemma 7.3. *Let G be a 2-group. Then,*

$$(-1) \uparrow B(G) = \operatorname{im}(\Theta_{\mathbb{Q}}).$$

Proof. This follows from the Ritter–Segal theorem which states that the linearization map $\pi_{\mathbb{Q}} : B(G) \to R(G, \mathbb{Q})$ is surjective when G is a p-group (see [3] for a new proof). □

Finally, we have

Lemma 7.4. *The exponential map commutes with induction, restriction, conjugation, inflation, and invariant maps.*

Proof. This follows from Lemmas 7.2 and 5.1. □

Note that $B(G)$ is an abelian group generated by $\{[G/H] \mid [H] \in Cl(G)\}$, so the image of the exponential map, $(-1) \uparrow B(G)$, will be generated by $(-1) \uparrow [G/H]$. Note that for each $[H] \in Cl(G)$, we can express $[G/H]$ as $\operatorname{ind}_H^G[H/H]$, and by Lemma 7.4, we have

$$(-1) \uparrow \operatorname{ind}_H^G[H/H] = \operatorname{jnd}_H^G\bigl((-1) \uparrow [H/H]\bigr) = \operatorname{jnd}_H^G(-1).$$

Thus, $(-1) \uparrow B(G)$ is generated by the set $\{\operatorname{jnd}_H^G(-1) \mid [H] \in Cl(G)\}$. So, we proved the following:

Lemma 7.5. *Let G be a 2-group. Then,*

$$(-1) \uparrow B(G) = \mathrm{im}\left\{ \prod_{[H]\in Cl(G)} \mathrm{jnd}_H^G \inf_{H/H}^H : \prod_{[H]\in Cl(G)} B(H/H)^\times \to B(G)^\times \right\}.$$

Now, we are ready to prove Corollary 1.3 stated in the introduction. In fact we will state a slightly more general version of Corollary 1.3 which will be easier to prove.

Theorem 7.6. *If G is a 2-group which has no subquotients isomorphic to D_{2^n} of order 2^n with $n \geqslant 4$, then*

(i) *the exponential map $\exp : B(G) \to B(G)^\times$ is surjective,*
(ii) *$B(G)^\times$ is generated by (-1) as a module over $B(G)$,*
(iii) *$\Theta_{\mathbb{Q}} : R(G, \mathbb{Q}) \to B(G)^\times$ is surjective,*
(iv) *$\prod \mathrm{jnd}_H^G \inf_{H/H}^H : \prod B(H/H)^\times \to B(G)^\times$ is surjective where the product is over all $[H] \in Cl(G)$.*

Proof. First observe that (i) and (ii) are equivalent because of the way we defined the exponential map. By Lemma 7.3, (iii) is equivalent to (i) and (ii). Similarly, (iv) is equivalent to first three statements by Lemma 7.5. Now, by Theorem 1.1 the last statement holds whenever G does not have a subquotient isomorphic to D_{2^n} of order 2^n with $n \geqslant 4$. So, the proof is complete. □

Remark 7.7. Note that we could give a direct proof for the surjectivity of the exponential map using the same argument used for the surjectivity of tom Dieck's homomorphism. For this consider the following diagram

$$\begin{tikzcd}
\bigoplus_{H/K\in\mathcal{H}} B(H/K) \arrow[r, "\bigoplus \mathrm{ind}_H^G \inf_{H/K}^H"] \arrow[d, "\prod \exp_{H/K}"] & B(G) \arrow[d, "\exp_G"] \\
\prod_{H/K\in\mathcal{H}} B(H/K)^\times \arrow[r, "\prod \mathrm{jnd}_H^G \inf_{H/K}^H"] & B(G)^\times.
\end{tikzcd}$$

Since G has no subquotients isomorphic to D_{2^n} of order 2^n with $n \geqslant 4$, we can take $\mathcal{H}$ as the collection of subquotients of G which are isomorphic to the trivial group. By Lemma 7.4, this diagram commutes, and by Theorem 1.1, the horizontal map on the bottom is surjective. So, to show that $\exp_G$ is surjective, it is enough to show that the exponential map is surjective for the trivial group which is obvious.

Theorem 7.6 applies, in particular, to a group with exponent less than or equal to 4. It is well known that the exponential map is not surjective in general, even for 2-groups. For example, Matsuda in [10] shows that when G is a dihedral group of order $2n$, the exponential map is surjective if and only if $n = 2, 4, p^r$ or $2p^r$, where p is an odd prime such that $p = 3 \bmod 4$. In particular, when $G \cong D_{2^n}$ with $n \geqslant 4$, the exponential map is not surjective. For convenience of the reader, we include a short argument for this last statement.

Proposition 7.8. *If G is a 2-group such that $G \cong D_{2^n}$ with $n \geqslant 4$, then the exponential map* $\exp: B(G) \to B(G)^\times$ *is not surjective.*

Proof. Let $G \cong D_{2^n}$ for some $n \geqslant 4$, and let c be the unique central element in G of order 2. We will be using the presentation given in the proof of Lemma 5.2, and carry over the calculations already done there.

Let $I(G)^\times$ denote the image of the inflation map $\mathrm{inf}_{G/\langle c\rangle}^G : B(G/\langle c\rangle)^\times \to B(G)^\times$, and $B(G,c)^\times$ denote the group of units $u \in B(G)$ such that $u(H) = 1$ for every H which includes the unique central element $c \in G$. By Lemma 4.3, we have a decomposition $B(G)^\times \cong I(G)^\times \times B(G,c)^\times$. In Lemma 5.2, we have shown that $B(G,c)$ is a cyclic group of order 2, generated by the unit $u = 1 - 2(e_{\langle a\rangle}^G + e_{\langle b\rangle}^G)$. We will show that u is not in the image of the exponential map, by showing that $\exp([G/H]) \in I(G)^\times$ for every $[H] \in Cl(G)$.

It is clear that if $c \in H$, then $\exp(G/H)$ lies in $I(G)^\times$. So, assume $c \notin H$. Then H is conjugate to $\langle a\rangle$ or $\langle b\rangle$. We complete the proof by showing that $\exp([G/\langle a\rangle] + [G/\langle a,c\rangle]) = 1$. The argument for $G/\langle b\rangle$ is similar.

Recall that for a transitive G-set G/L, we have

$$s_K\big[\exp(G/L)\big] = \mathrm{sgn}\big|(G/L)/K\big| = \mathrm{sgn}|L\backslash G/K|$$

where $|L\backslash G/K|$ denotes the number of double cosets of L and K in G. So, we just need to show that $n_K := |\langle a\rangle\backslash G/K| - |\langle a,c\rangle\backslash G/K|)$ is even for every $[K] \in Cl(G)$. Applying the formula

$$|H\backslash G/K| = \frac{1}{|H|}\sum_{h\in H}\big|(G/K)^h\big|$$

and $H = \langle a\rangle$ and $\langle a,c\rangle$, we get

$$n_K = \frac{1}{4}\big(|G/K| - \big|(G/K)^c\big|\big).$$

If $c \in K$, then $n_K = |G/K| - |(G/K)^c| = 0$. If $c \notin K$, then $|K| = 2$ and $|(G/K)^c| = 0$. So, $n_K = \frac{1}{4}(|G/K| - |(G/K)^c|) = |G|/8$ which is even since $G \cong D_{2^n}$ with $n \geqslant 4$. This completes the proof of the proposition. □

We have shown that the exponential map is not surjective when G is a dihedral 2-group of order at least 16. However, there exist 2-groups where the exponential map is surjective even though they have a dihedral section of order 16. The smallest 2-group with these properties is of order 32, and below we give an example of such a 2-group.

Lemma 7.9. *There exists a 2-group G such that G has a subquotient isomorphic to D_{16}, and the exponential map* $\exp: B(G) \to B(G)^\times$ *is surjective.*

Proof. Let G be the 2-group of order 32 generated by g_1, g_2, g_3 subject to following relations: $[g_1, g_3] = g_4$, $[g_1, g_2] = [g_3, g_4] = [g_1, g_4] = g_5$, $g_2^2 = g_3^2 = g_4^2 = g_5$, $g_1^2 = g_5^2 = 1$, $[g_2, g_3] = [g_2, g_4] = [g_i, g_5] = 1$ for every $1 \leqslant i \leqslant 4$. It is easy to see that G has a unique central element of order 2 which is g_5, so the only quotient group of order 16 is $G/\langle g_5\rangle \cong D_8 \times C_2$. The Frattini subgroup of G is the cyclic group generated by g_4 which is of order 4. We have $G/\langle g_4\rangle \cong (\mathbb{Z}/2)^3$, so the group has 7 maximal subgroups. Out of these 7, only two of them are isomorphic to D_{16}, namely $H_1 = \langle g_1, g_1g_2g_3\rangle$, and $H_2 = \langle g_1g_2, g_1g_3\rangle$. So, G has two subquotients isomorphic to D_{16}.

Now, we will show that the exponential map is surjective. For this, we will use Lemmas 4.3 and 4.4. Recall that by these lemmas, there is a surjective map

$$\inf_{G/\langle c\rangle}^{G} \times \operatorname{jnd}_H^G \inf_{H/\langle a\rangle}^{H} : B(G/\langle c\rangle)^\times \times B(H/\langle a\rangle)^\times \to B(G)^\times$$

where $\langle c\rangle$ is a central element, $E = \langle c, a\rangle$ is a non-central normal subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, and H is the centralizer of E in G. Take $c = g_5$ and $a = g_2g_4$. Then, $H = \langle g_1, g_2, g_4\rangle \cong C_2 \times D_8$, and $H/\langle a\rangle \cong D_8$. We have already observed above that $G/\langle c\rangle \cong C_2 \times D_8$. Since the exponential map is surjective for D_8 and $C_2 \times D_8$, it is also surjective for G. $\square$

We have seen that Corollary 1.3 provides a sufficient condition for the surjectivity of exponential map, but it is not a necessary condition. To find a necessary and sufficient condition, one needs to understand the contribution of each subquotient in Theorem 1.1. This can be done by considering $B(G)^\times$ as a module over the ring of $(\mathbb{Q}G, \mathbb{Q}G)$-bimodules and using an idempotent decomposition for this ring. We leave this to another paper since it requires some background on bisets and their actions on the unit group.

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