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# Clifford theory for Mackey algebras

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**Abstract**

We develop a Clifford theory for Mackey algebras. For simple Mackey functors, using their classification we prove Mackey algebra versions of Clifford's theorem and the Clifford correspondence. Let $\mu_R(G)$ be the Mackey algebra of a finite group $G$ over a commutative unital ring $R$, and let $1_N$ be the unity of $\mu_R(N)$ where $N$ is a normal subgroup of $G$. Observing that $1_N\mu_R(G)1_N$ is a crossed product of $G/N$ over $\mu_R(N)$, a number of results concerning group graded algebras are extended to the context of Mackey algebras, including Fong's theorem, Green's indecomposibility theorem and some reduction and extension techniques for indecomposable Mackey functors.


*Keywords:* Mackey functor; Mackey algebra; Clifford theory; Green's indecomposibility criterion; Graded algebra

## 1. Introduction

The notion of a Mackey functor, introduced by J.A. Green [11] and A. Dress [7], plays an important role in representation theory of finite groups, and it unifies several notions like representation rings, $G$-algebras and cohomology. During the last two decades, the theory of Mackey functors has received much attention. In [27,28], J. Thévenaz and P. Webb constructed the simple Mackey functors explicitly. Also, they introduced the Mackey algebra $\mu_R(G)$ for a finite group $G$ over a commutative unital ring $R$. The left $\mu_R(G)$-modules are identical to the Mackey functors for $G$ over $R$.

Let $N$ be a normal subgroup of $G$. A classical topic in the representation theory of finite groups is Clifford theory initiated by A.H. Clifford [2]. It consists of the repeated applications of

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three basic operations on modules of group algebras, namely restriction to $RN$, induction from $RN$ and extension from $RN$. Later, E.C. Dade [3–5] lifted much of the theory to a more general abstract system called now group graded algebras.

The goal of this paper is to develop a Clifford theory for Mackey functors. The paper can be roughly divided into three parts. The first part, the Sections 3 and 4, analyzes restriction and induction of simple Mackey functors, and the second part, the Sections 5 and 6, is concerned with the structure of Mackey algebras and Clifford type results for indecomposable Mackey functors, and the third part, the last section, deals with extension of $G$-invariant Mackey functors.

One of the main differences between the Mackey algebra $\mu_R(G)$ and the group algebra $RG$ is that in the former $\mu_R(N)$ is a nonunital subalgebra of $\mu_R(G)$ and if we want to get a unitary $\mu_R(N)$-module after restricting a $\mu_R(G)$-module $M$ to $\mu_R(N)$, we must define the restriction of $M$ as $1_N M$ where $1_N$ denotes the unity of $\mu_R(N)$. For this reason the restriction of a Mackey functor may be 0.

We attack the problem in two ways. Our first approach uses the classification of simple Mackey functors and Clifford theory for group algebras which leads to elementary proofs if simple Mackey functors are concerned. We show in Section 5 that $1_N\mu_R(G)1_N$ is a crossed product of $G/N$ over $\mu_R(N)$ where $N$ is a normal subgroup of $G$ and $1_N$ is the unity of $\mu_R(N)$, and this result allows us to attack the problem by using Clifford theory for group graded algebras. But this approach relates modules of $\mu_R(N)$ and $1_N\mu_R(G)1_N$, and for this reason Section 5 contains some results relating modules of $1_N\mu_R(G)1_N$ and $\mu_R(G)$.

A number of results pertaining to Clifford theory for group algebras are extended to the context of Mackey algebras. The results 3.10, 4.4, 5.2, 5.4, 6.1 and 6.3 are among the most important results obtained here. They include Mackey functor versions of Clifford's theorem, the Clifford correspondence, Fong's theorem and Green's indecomposibility theorem.

Character ring and Burnside ring functors are Mackey functors satisfying a special property which is not shared with some other Mackey functors, namely each coordinate module of them is a free abelian semigroup such that restriction of basis elements are nonzero. In [19], motivated by these functors, a notion of a based Mackey functor for $G$ is defined which is a Mackey functor $M$ for $G$ such that each coordinate module $M(H)$, $H \leqslant G$, is a free abelian semigroup with a basis $B(H)$ satisfying some conditions. In [19], Clifford's theorem and the Clifford correspondence for based Mackey functors are studied. It is shown that Clifford's theorem holds between $G$ and its normal subgroup $N$ for a based Mackey functor $M$ for $G$ and for a $\alpha \in B(G)$ if either $r^G_N(\alpha) = n\beta$ for some $\beta \in B(N)$ and natural number $n$ or $\alpha$ appears in $t^G_K(\delta)$ for some subgroup $K$ with $N \leqslant K < G$ and $\delta \in B(K)$. One may consider the Grothendieck rings $\mathfrak{M}(H)$ of Mackey functors for $H$, $H \leqslant G$. Then $\mathfrak{M}$ is a based Mackey functor for $G$. Given a simple Mackey functor $M$ for $G$ and a normal subgroup $N$ of $G$ our result 3.10 holds if $\mathfrak{M}$ satisfies the above property given in [19], however checking this property is not easier than proving the result itself. In particular, 3.10 and 4.4 show that the property given in [19] holds in $\mathfrak{M}$ for a simple Mackey functor $M$ for $G$ and a normal subgroup $N$ of $G$ such that $1_N M$ is nonzero. Finally, it must be remarked that the results 6.1(i) and some parts of 6.2 follow from [19, 1.5 and 2.6] because $N$-projectivity implies the property.

Throughout the paper, $G$ denotes a finite group, $R$ denotes a commutative unital ring and $\mathbb{K}$ denotes a field. We write $H \leqslant G$ (respectively $H < G$) to indicate that $H$ is a subgroup of $G$ (respectively a proper subgroup of $G$), and we write $H \trianglelefteq G$ if it is a normal subgroup. Let $H \leqslant G \geqslant K$. The notation $H =_G K$ means that $K$ is $G$-conjugate to $H$ and $H \leqslant_G K$ means that $H$ is $G$-conjugate to a subgroup of $K$. By the notation $gH \subseteq G$ we mean that $g$ ranges over a complete set of representatives of left cosets of $H$ in $G$, and by $HgK \subseteq G$ we mean that $g$ ranges over a complete set of representatives of double cosets of $(H, K)$ in $G$. Also we put $\bar{N}_G(H) = N_G(H)/H$, ${}^gH = gHg^{-1}$ and $H^g = g^{-1}Hg$ for $g \in G$. Lastly for any natural numbers $a$ and $b$, $(a, b)$ denotes their greatest common divisor.

## 2. Preliminaries

In this section, we briefly summarize some crucial material on Mackey functors. For the proofs, see Thévenaz–Webb [27,28]. Let $\chi$ be a family of subgroups of $G$, closed under subgroups and conjugation. Recall that a Mackey functor for $\chi$ over $R$ is such that, for each $H \in \chi$, there is an $R$-module $M(H)$; for each pair $H, K \in \chi$ with $H \leqslant K$, there are $R$-module homomorphisms $r_H^K : M(K) \to M(H)$ called the restriction map and $t_H^K : M(H) \to M(K)$ called the transfer map or the trace map; for each $g \in G$, there is an $R$-module homomorphism $c_H^g : M(H) \to M({}^gH)$ called the conjugation map. The following axioms must be satisfied for any $g, h \in G$ and $H, K, L \in \chi$ [1,11,27,28]:

($\mathrm{M}_1$) if $H \leqslant K \leqslant L$, $r_H^L = r_H^K r_K^L$ and $t_H^L = t_K^L t_H^K$; moreover $r_H^H = t_H^H = \mathrm{id}_{M(H)}$;
($\mathrm{M}_2$) $c_K^{gh} = c_{{}^hK}^g c_K^h$;
($\mathrm{M}_3$) if $h \in H$, $c_H^h : M(H) \to M(H)$ is the identity;
($\mathrm{M}_4$) if $H \leqslant K$, $c_H^g r_H^K = r_{{}^gH}^{{}^gK} c_K^g$ and $c_K^g t_H^K = t_{{}^gH}^{{}^gK} c_H^g$;
($\mathrm{M}_5$) (Mackey axiom) if $H \leqslant L \geqslant K$, $r_H^L t_K^L = \sum_{HgK \subseteq L} t_{H\cap {}^gK}^H r_{H\cap {}^gK}^{{}^gK} c_K^g$.

When $\chi$ is the family of all subgroups of $G$, we say that $M$ is a Mackey functor for $G$ over $R$. A homomorphism $f : M \to T$ of Mackey functors for $\chi$ is a family of $R$-module homomorphisms $f_H : M(H) \to T(H)$, where $H$ runs over $\chi$, which commutes with restriction, trace and conjugation. In particular, each $M(H)$ is an $R\bar{N}_G(H)$-module via $\bar{g}.x = c_H^g(x)$ for $\bar{g} \in \bar{N}_G(H)$ and $x \in M(H)$. Also, each $f_H$ is an $R\bar{N}_G(H)$-module homomorphism. By a subfunctor $N$ of a Mackey functor $M$ for $\chi$ we mean a family of $R$-submodules $N(H) \subseteq M(H)$, which is stable under restriction, trace, and conjugation. A Mackey functor $M$ is called simple if it has no proper subfunctor.

Another possible definition of Mackey functors for $G$ over $R$ uses the Mackey algebra $\mu_R(G)$ [1,28]: $\mu_{\mathbb{Z}}(G)$ is the algebra generated by the elements $r_H^K, t_H^K$, and $c_H^g$, where $H$ and $K$ are subgroups of $G$ such that $H \leqslant K$, and $g \in G$, with the following relations:

($\mathrm{M}_1'$) if $H \leqslant K \leqslant L$, $r_H^L = r_H^K r_K^L$ and $t_H^L = t_K^L t_H^K$;
($\mathrm{M}_2'$) if $g, h \in G$, $c_K^{gh} = c_{{}^hK}^g c_K^h$;
($\mathrm{M}_3'$) if $h \in H$, $t_H^H = r_H^H = c_H^h$;
($\mathrm{M}_4'$) if $H \leqslant K$ and $g \in G$, $c_H^g r_H^K = r_{{}^gH}^{{}^gK} c_K^g$ and $c_K^g t_H^K = t_{{}^gH}^{{}^gK} c_H^g$;
($\mathrm{M}_5'$) if $H \leqslant L \geqslant K$, $r_H^L t_K^L = \sum_{HgK \subseteq L} t_{H\cap {}^gK}^H r_{H\cap {}^gK}^{{}^gK} c_K^g$;
($\mathrm{M}_6'$) $\sum_{H \leqslant G} t_H^H = \sum_{H \leqslant G} r_H^H = 1_{\mu_{\mathbb{Z}}(G)}$;
($\mathrm{M}_7'$) any other product of $r_H^K$, $t_H^K$ and $c_H^g$ is zero.

A Mackey functor $M$ for $G$, defined in the first sense, gives a left module $\tilde{M}$ of the associative $R$-algebra $\mu_R(G) = R \otimes_{\mathbb{Z}} \mu_{\mathbb{Z}}(G)$ defined by $\tilde{M} = \bigoplus_{H \leqslant G} M(H)$. Conversely, if $\tilde{M}$ is a $\mu_R(G)$-module then $\tilde{M}$ corresponds to a Mackey functor $M$ in the first sense, defined by $M(H) = t_H^H \tilde{M}$, the maps $t_H^K$, $r_H^K$, and $c_H^g$ being defined as the corresponding elements of the $\mu_R(G)$. Moreover, homomorphisms and subfunctors of Mackey functors for $G$ are $\mu_R(G)$-module homomorphisms and $\mu_R(G)$-submodules, and conversely.

**Theorem 2.1.** [28] *Letting $H$ and $K$ run over all subgroups of $G$, letting $g$ run over representatives of the double cosets $HgK \subseteq G$, and letting $J$ runs over representatives of the conjugacy classes of subgroups of $H^g \cap K$, then $t_{{}^gJ}^H c_J^g r_J^K$ comprise, without repetition, a free $R$-basis of $\mu_R(G)$.*

For a Mackey functor $M$ for $\chi$ over $R$ and a subset $E$ of $M$, a collection of subsets $E(H) \subseteq M(H)$ for each $H \in \chi$, we denote by $\langle E \rangle$ the subfunctor of $M$ generated by $E$.

**Proposition 2.2.** [27] *Let $M$ be a Mackey functor for $G$, and let $T$ be a subfunctor of $\downarrow_\chi M$, the restriction of $M$ to $\chi$ which is the family $M(H)$, $H \in \chi$, viewed as a Mackey functor for $\chi$. Then $\langle T \rangle(K) = \sum_{X \in \chi:\ X \leqslant K} t_X^K(M(X))$ for any $K \leqslant G$. Moreover $\downarrow_\chi \langle T \rangle = T$.*

Let $M$ be a Mackey functor for $G$. Then by [27] we have the following important subfunctors of $M$, namely $\operatorname{Im} t_\chi^M$ and $\operatorname{Ker} r_\chi^M$ defined by

$$\big(\operatorname{Im} t_\chi^M\big)(K) = \sum_{X \in \chi:\ X \leqslant K} t_X^K\big(M(X)\big) \quad \text{and}$$

$$\big(\operatorname{Ker} r_\chi^M\big)(K) = \bigcap_{X \in \chi:\ X \leqslant K} \operatorname{Ker}\big(r_X^K : M(K) \to M(X)\big).$$

For a nonzero Mackey functor $M$ for $G$ over $R$, a minimal subgroup $H$ such that $M(H) \neq 0$ is called a minimal subgroup of $M$. If $H \leqslant G$ we put $\chi_H = \{K \leqslant G:\ K \leqslant_G H\}$.

The following results will be of great use later.

**Proposition 2.3.** [27] *Let $S$ be a simple Mackey functor for $G$ with a minimal subgroup $H$:*

(i) *$S$ is generated by $S(H)$, that is $S = \langle S(H) \rangle$.*
(ii) *$S(K) \neq 0$ implies that $H \leqslant_G K$, and so minimal subgroups of $S$ form a unique conjugacy class.*
(iii) *$S(H)$ is a simple $R\bar{N}_G(H)$-module.*

**Proposition 2.4.** [27] *Let $M$ be a Mackey functor for $G$ over $R$, and let $H$ be a minimal subgroup of $M$. Then, $M$ is simple if and only if $\operatorname{Im} t_{\chi_H}^M = M$, $\operatorname{Ker} r_{\chi_H}^M = 0$, and $S(H)$ is a simple $R\bar{N}_G(H)$-module.*

**Theorem 2.5.** [27] *Given a subgroup $H \leqslant G$ and a simple $R\bar{N}_G(H)$-module $V$, then there exists a simple Mackey functor $S_{H,V}^G$ for $G$, unique up to isomorphism, such that $H$ is a minimal subgroup of $S_{H,V}^G$ and $S_{H,V}^G(H) \cong V$. Moreover, up to isomorphism, every simple Mackey functor for $G$ has the form $S_{H,V}^G$ for some $H \leqslant G$ and simple $R\bar{N}_G(H)$-module $V$. Two simple Mackey functors $S_{H,V}^G$ and $S_{H',V'}^G$ are isomorphic if and only if, for some element $g \in G$, we have $H' = {}^gH$ and $V' \cong c_H^g(V)$.*

Finally, we recall the definitions of restriction, induction and conjugation for Mackey functors [1,25,27]. For any $H \leqslant G$, there is an obvious nonunital $R$-algebra homomorphism $\mu_R(H) \to \mu_R(G)$, $t^A_{gI} c^g_I r^B_I \mapsto t^A_{gI} c^g_I r^B_I$ for any basis element $t^A_{gI} c^g_I r^B_I$ of $\mu_R(H)$. Moreover this map is injective [1]. Viewing, Mackey functors as modules of Mackey algebras, we have obvious notions of restriction and induction: let $M$ and $T$ be Mackey functors for $G$ and $H$, respectively, where $H \leqslant G$, then the restricted Mackey functor $\downarrow^G_H M$ is the $\mu_R(H)$-module $1_{\mu_R(H)} M$ and the induced Mackey functor $\uparrow^G_H T$ is the $\mu_R(G)$-module $\mu_R(G) 1_{\mu_R(H)} \otimes_{\mu_R(H)} T$, where $1_{\mu_R(H)}$ denotes the unity of $\mu_R(H)$. There is a unital $R$-algebra monomorphism $\gamma : RG \to \mu_R(G)$, $g \mapsto \gamma_g = \sum_{H \leqslant G} c^g_H$, making $\mu_R(G)$ an interior $G$-algebra. For $H \leqslant G$, $g \in G$, and a Mackey functor $M$ for $H$, viewing $M$ as a $\mu_R(H)$-module, the conjugate Mackey functor $|^g_H M = {}^g M$ is the $\mu_R({}^g H)$-module $M$ with the module structure given for any $x \in \mu_R({}^g H)$ and $m \in M$ by $x.m = (\gamma_{g^{-1}} x \gamma_g) m$. Obviously, one has $|^g_L S^L_{H,V} \cong S^{{}^g L}_{{}^g H, c^g_H(V)}$.

The following equivalent definition of induction is useful [25,27]. Let $H \leqslant G$ and let $M$ be a Mackey functor for $H$. Then for any $K \leqslant G$ the induced Mackey functor $\uparrow^G_H M$ for $G$ is given by

$$\left(\uparrow^G_H M\right)(K) = \bigoplus_{KgH \subseteq G} M(H \cap K^g),$$

where, if we write $m_g$ for the component in $M(H \cap K^g)$ of $m \in (\uparrow^G_H M)(K)$, the maps are given as follows:

$$r^K_L(m)_g = r^{H \cap K^g}_{H \cap L^g}(m_g), \qquad t^K_L(n)_g = \sum_{Lu(K \cap {}^g H) \subseteq K} t^{H \cap K^{ug}}_{H \cap L^{ug}}(n_{ug}) \quad \text{and} \quad c^y_K(m)_g = m_{y^{-1}g}$$

for $L \leqslant K$, $n \in (\uparrow^G_H M)(L)$ and $y \in G$.

Let $L \leqslant G$ and $M$ be a Mackey functor for $L$ with maps $t$, $r$, $c$. Let $\tilde{t}$, $\tilde{r}$, $\tilde{c}$ be the maps of $\uparrow^G_L M$, then we have

$$\operatorname{Ker} \tilde{r}^{K_2}_{K_1} = \bigoplus_{K_2 g L \subseteq G} \operatorname{Ker} r^{L \cap K_2^g}_{L \cap K_1^g} \quad \text{and} \quad \operatorname{Im} \tilde{t}^{K_2}_{K_1} = \bigoplus_{K_2 g L \subseteq G} \left( \sum_{K_1 u (K_2 \cap {}^g L) \subseteq K_2} \operatorname{Im} t^{L \cap K_2^{ug}}_{L \cap K_1^{ug}} \right).$$

As a last result in this section, we record the Mackey decomposition formula for Mackey functors, which can be found (for example) in [28].

**Theorem 2.6.** *Given $H \leqslant L \geqslant K$ and a Mackey functor $M$ for $K$ over $R$, we have*

$$\downarrow^L_H \uparrow^L_K M \cong \bigoplus_{HgK \subseteq L} \uparrow^H_{H \cap {}^g K} \downarrow^{{}^g K}_{H \cap {}^g K} |^g_K M.$$

## 3. Clifford's theorem

In this section using the classification of simple Mackey functors we prove that restriction of a simple functor to a normal subgroup is semisimple and simple summands of it are conjugate.

For the next two results we let $M = S^G_{H,V}$ be a simple Mackey functor for $G$ over $\mathbb{K}$.

The following remark shows that any minimal subgroup of a nonzero $L$-subfunctor of $\downarrow_L^G M$ is conjugate to $H$, where $H \leqslant L \leqslant G$.

**Remark 3.1.** Let $H \leqslant L \leqslant G$. If $S$ is a nonzero $L$-subfunctor of $\downarrow_L^G M$ then $S({}^g H) \neq 0$ for some $g \in G$ with ${}^g H \leqslant L$.

**Proof.** There is a $K \leqslant L$ such that $S(K) \neq 0$. If for all $g \in G$ with ${}^g H \leqslant K$ $S({}^g H) = 0$, then $r^K_{{}^g H}(S(K)) \subseteq S({}^g H) = 0$, implying that $S(K) \subseteq (\operatorname{Ker} r^M_{\chi H})(K)$. But by 2.4 $(\operatorname{Ker} r^M_{\chi H})(K) = 0$ and so $S(K) = 0$, a contradiction. $\square$

Let $H \leqslant L$. For any $\mathbb{K}\bar{N}_L(H)$-submodule $U$ of $M(H) = V$ and any $g \in N_G(L)$, we denote by $T^L_{{}^g H, c^g_H(U)}$ the $L$-subfunctor of $\downarrow_L^G M$ generated by $c^g_H(U)$. Therefore, for any $K \leqslant L$, we have, by 2.2,

$$T^L_{{}^g H, c^g_H(U)}(K) = \sum_{x \in L:\ {}^x({}^g H) \leqslant K} t^K_{{}^{xg} H} c^x_{{}^g H} c^g_H(U) \quad \text{and} \quad T^L_{{}^g H, c^g_H(U)}({}^g H) = c^g_H(U).$$

We draw some elementary properties of these subfunctors which will be useful in our subsequent investigations, in particular in the proof of 3.10.

**Lemma 3.2.**

(i) *For any* $x \in L$

$$T^L_{{}^g H, c^g_H(U)} = T^L_{{}^{xg} H, c^{xg}_H(U)}.$$

(ii) $T^L_{{}^g H, c^g_H(U)}$ *is simple if and only if* $U$ *is simple* $\mathbb{K}\bar{N}_L(H)$*-module.*

(iii) $T^L_{{}^{g_1} H, M({}^{g_1} H)} = T^L_{{}^{g_2} H, M({}^{g_2} H)}$ *if and only if* $Lg_1 N_G(H) = Lg_2 N_G(H)$.

(iv) *If* $L \trianglelefteq G$ *then*

$$\downarrow_L^G M = \sum_{LgN_G(H) \subseteq G} T^L_{{}^g H, M({}^g H)},$$

*and each summand is distinct.*

(v) *If* $U_1$ *and* $U_2$ *are* $\mathbb{K}\bar{N}_L(H)$*-submodules of* $M(H)$*, and if* $g \in G$ *with* ${}^g H \leqslant L$*, then*

$$T^L_{{}^g H, c^g_H(U_1) + c^g_H(U_2)} = T^L_{{}^g H, c^g_H(U_1)} + T^L_{{}^g H, c^g_H(U_2)}.$$

**Proof.** (i) For any $x \in L$, it is obvious that the subsets $c^g_H(U)$ and $c^x_{{}^g H} c^g_H(U) = c^{xg}_H(U)$ of $\downarrow_L^G M$ generate the same $L$-subfunctor of $\downarrow_L^G M$.

(ii) If $T^L_{{}^g H, c^g_H(U)}$ is simple, then 2.3 implies that $U$ is simple $\mathbb{K}\bar{N}_L(H)$-module. Suppose now $U$ is simple. If $S$ is a nonzero $L$-subfunctor of $T^L_{{}^g H, c^g_H(U)}$ then $S$ is a nonzero $L$-subfunctor of $\downarrow_L^G M$, and hence, by 3.1, $S({}^y H) \neq 0$ for some $y \in G$ with ${}^y H \leqslant L$. Then, $S({}^y H)$ is a nonzero submodule of $T^L_{{}^g H, c^g_H(U)}({}^y H)$, implying that the index set $\{x \in L\colon\ {}^x({}^g H) \leqslant {}^y H\}$ of the sum expressing $T^L_{{}^gH,c^g_H(U)}({}^yH)$ is nonempty, and so $xg = yu$ for some $x \in L$ and $u \in N_G(H)$. Then, by (i), we have

$$T^L_{{}^gH,c^g_H(U)} = T^L_{{}^{xg}H,c^{xg}_H(U)} = T^L_{{}^{yu}H,c^{yu}_H(U)} = T^L_{{}^yH,c^y_H(U)}.$$

Thus, $S$ is a nonzero subfunctor of $T^L_{{}^yH,c^y_H(U)}$, and so $S({}^yH)$ is a nonzero submodule of $c^y_H(U)$. Then simplicity of $U$ implies that $S({}^yH) = c^y_H(U)$. Now,

$$T^L_{{}^yH,c^y_H(U)} = \langle c^y_H(U)\rangle = \langle S({}^yH)\rangle$$

implies that

$$T^L_{{}^gH,c^g_H(U)} = T^L_{{}^yH,c^y_H(U)} = S.$$

Hence, $T^L_{{}^gH,c^g_H(U)}$ is simple.

(iii) Suppose that $T^L_{{}^{g_1}H,M({}^{g_1}H)} = T^L_{{}^{g_2}H,M({}^{g_2}H)}$. Then $0 \neq M({}^{g_1}H) = T^L_{{}^{g_2}H,M({}^{g_2}H)}({}^{g_1}H)$, implying that the index set $\{x \in L\colon {}^x({}^{g_2}H) \leqslant {}^{g_1}H\}$ of the sum expressing $T^L_{{}^{g_2}H,M({}^{g_2}H)}({}^{g_1}H)$ is nonempty, and so ${}^x({}^{g_2}H) = {}^{g_1}H$ for some $x \in L$. Hence $Lg_1N_G(H) = Lg_2N_G(H)$. Conversely, if $Lg_1N_G(H) = Lg_2N_G(H)$ then $g_2 = xg_1u$ for some $x \in L$ and $u \in N_G(H)$. Thus, by (i),

$$T^L_{{}^{g_1}H,M({}^{g_1}H)} = T^L_{{}^{g_2}H,M({}^{g_2}H)}.$$

(iv) For $K \leqslant L$, it is clear that

$$\sum_{g\in G} T^L_{{}^gH,M({}^gH)}(K) = \sum_{g\in G}\ \sum_{x\in L\colon {}^x({}^gH)\leqslant K} t^K_{{}^x({}^gH)} c^x_{{}^gH} M({}^gH) = \sum_{g\in G\colon {}^gH\leqslant K} t^K_{{}^gH} c^g_H M(H) = M(K),$$

where the last equality follows by 2.4. The result now follows by (iii).

(v) It is clear because trace maps are additive. □

**Corollary 3.3.** *Let $H \leqslant L \trianglelefteq G$, and let a simple Mackey functor $S^G_{H,V}$ for $G$ be given. Then, $\downarrow^G_L S^G_{H,V}$ is semisimple if and only if $\downarrow^{\bar N_G(H)}_{\bar N_L(H)} V$ is semisimple.*

**Proof.** By 3.2

$$\downarrow^G_L S^G_{H,V} = \sum_{LgN_G(H)\subseteq G} T^L_{{}^gH,c^g_H(V)}.$$

Suppose $\downarrow^{\bar N_G(H)}_{\bar N_L(H)} V = \bigoplus_i W_i$ where each $W_i$ is a simple $\bar N_L(H)$-module. For any $g \in G$,

$$\downarrow^{\bar N_G({}^gH)}_{\bar N_L({}^gH)} c^g_H(V) = c^g_H\big(\downarrow^{\bar N_G(H)}_{\bar N_L(H)} V\big) = \bigoplus_i c^g_H(W_i),$$

implying by 3.2 that

$$\downarrow_L^G S_{H,V}^G = \sum_{LgN_G(H)\subseteq G} \sum_i T^L_{{}^gH, c_H^g(W_i)}$$

where each summand $T^L_{{}^gH, c_H^g(W_i)}$ is simple. Thus, $\downarrow_L^G S_{H,V}^G$ is semisimple.

Conversely, suppose $\downarrow_L^G S_{H,V}^G = \bigoplus_i S_i$ where each $S_i$ is a simple Mackey functor for $L$. Then, by 3.1, each $S_i$ has a minimal subgroup $G$-conjugate to $H$, and so $S_i(H)$, if nonzero, is a simple $\bar{N}_L(H)$-module. Therefore,

$$V = \downarrow_L^G S_{H,V}^G(H) = \bigoplus_i S_i(H)$$

is a direct sum of simple $\bar{N}_L(H)$-modules, proving that $\downarrow_{\bar{N}_L(H)}^{\bar{N}_G(H)} V$ is semisimple. $\square$

If $N$ is a normal subgroup of $G$, 3.3 implies that $\downarrow_N^G S$ is semisimple for any simple Mackey functor $S$ for $G$ whose minimal subgroup is contained in $N$.

The next two results will play a crucial role in the proofs of some of the later results.

**Lemma 3.4.** *Let $H \leqslant L \leqslant G$ be such that ${}^gH \leqslant L$ for every $g \in G$, and let a simple Mackey functor $S_{H,U}^L$ for L be given. Then, letting $\uparrow_L^G S_{H,U}^L = \tilde{S}$:*

(i) *$H$ is a minimal subgroup of $\tilde{S}$.*
(ii) $\tilde{S} = \operatorname{Im} t_{\chi H}^{\tilde{S}}$.
(iii) $\operatorname{Ker} r_{\chi H}^{\tilde{S}} = 0$.
(iv) $\tilde{S}(H) \cong \uparrow_{\bar{N}_L(H)}^{\bar{N}_G(H)} U$.

**Proof.** We write $\tilde{t}, \tilde{r}, \tilde{c}$ for the maps on $\tilde{S}$:

(i) First note that, if the module

$$\tilde{S}(K) = \bigoplus_{KgL\subseteq G} S_{H,U}^L(L \cap K^g)$$

is nonzero, then $S_{H,U}^L(L \cap K^g) \neq 0$ for some $g \in G$, hence $H \leqslant_G K$. Plainly, $\tilde{S}(H) \neq 0$. So the minimal subgroups for $\tilde{S}$ are precisely the $G$-conjugates of $H$.

(ii) Let $K \leqslant G$. We must show that

$$\tilde{S}(K) \subseteq \operatorname{Im} t_{\chi H}^{\tilde{S}}(K) = \sum_{g\in G:\ {}^gH\leqslant K} \operatorname{Im} \tilde{t}_{{}^gH}^K.$$

For an $x \in G$,

$$\bigl(\tilde{S}(K)\bigr)_x = S_{H,U}^L(L\cap K^x) = \sum_{y\in L:\ {}^yH\leqslant L\cap K^x} \operatorname{Im} t_{{}^yH}^{L\cap K^x} \quad \text{and}$$

$$\left(\operatorname{Im} t_{\chi_H}^{\tilde{S}}(K)\right)_x = \sum_{g\in G:\ {}^gH\leqslant K}\ \sum_{({}^gH)u(K\cap {}^xL)\subseteq K} \operatorname{Im} t_{L\cap({}^gH)^{ux}}^{L\cap K^{ux}}.$$

Now, by the assumption on $L$, we see that $L\cap({}^gH)^{ux} = {}^{x^{-1}u^{-1}g}H$. And if $y\in L$ with ${}^yH\leqslant L\cap K^x$ then, putting $g = xy$ and $u = 1$, we see that ${}^gH\leqslant K$ and $x^{-1}u^{-1}g = y$. Therefore, every summand in $(\tilde{S}(K))_x$ appears in $(\operatorname{Im} t_{\chi_H}^{\tilde{S}}(K))_x$.

(iii) Let $K\leqslant G$. If

$$m\in \operatorname{Ker} r_{\chi_H}^{\tilde{S}}(K) = \bigcap_{g\in G:\ {}^gH\leqslant K} \operatorname{Ker} \tilde{r}_{{}^gH}^{K}$$

then, for any $x\in G$,

$$m_x \in \bigcap_{g\in G:\ {}^gH\leqslant K} \operatorname{Ker} r_{L\cap({}^gH)^x}^{L\cap K^x},$$

and by the assumption on $L$, $L\cap({}^gH)^x = {}^{x^{-1}g}H$. Consequently,

$$m_x \in \bigcap_{g\in G:\ {}^gH\leqslant K} \operatorname{Ker} r_{{}^{xg^{-1}}H}^{L\cap K^x}.$$

Simplicity of $S_{H,U}^L$ implies that

$$\bigcap_{y\in L:\ {}^yH\leqslant L\cap K^x} \operatorname{Ker} r_{{}^yH}^{L\cap K^x} = 0.$$

If $y\in L$ with ${}^yH\leqslant L\cap K^x$, putting $g = xy$, we have ${}^gH\leqslant K$ and $x^{-1}g = y$. Hence, any set appearing in the intersection

$$\bigcap_{y\in L:\ {}^yH\leqslant L\cap K^x} \operatorname{Ker} r_{{}^yH}^{L\cap K^x}\ (=0)$$

appears also in the intersection

$$\bigcap_{g\in G:\ {}^gH\leqslant K} \operatorname{Ker} r_{{}^{xg^{-1}}H}^{L\cap K^x}.$$

Therefore, $m_x = 0$.

(iv) Firstly, for any $g\in G$, if $S_{H,U}^L({}^gH)\neq 0$ then $g\in N_G(H)L$. Also $L\cap H^g = H^g$, and if $x\in N_G(H)L$ then $HxL = xL$. Thus,

$$\tilde{S}(H) = \bigoplus_{HgL\subseteq G} S_{H,U}^L(L\cap H^g) = \bigoplus_{HgL\subseteq N_G(H)L} S_{H,U}^L(H^g) = \bigoplus_{gL\subseteq N_G(H)L} S_{H,U}^L(H^g).$$

As $S^L_{H,U}(H^g) = c_H^{g^{-1}}(U)$,

$$\tilde{S}(H) = \bigoplus_{gL\subseteq N_G(H)L} c_H^{g^{-1}}(U), \quad \text{a direct sum of } \mathbb{K}\text{-modules.}$$

Moreover, since $k \in \bar{N}_G(H)$ acts on an element

$$x = \bigoplus_{gL\subseteq N_G(H)L} x_g \quad \text{of } \tilde{S}(H) \quad \text{as}$$

$$k.x = \tilde{c}_H^k(x) = \bigoplus_{gL\subseteq N_G(H)L} \tilde{c}_H^k(x)_g \quad \text{where } \tilde{c}_H^k(x)_g = x_{k^{-1}g},$$

we see that $\bar{N}_G(H)$ permutes the summands $c_H^{g^{-1}}(U)$ of $\tilde{S}(H)$ transitively and that the stabilizer of the summand $c_H^1(U) = U$ is $\bar{N}_L(H)$. Hence we proved that if $L \neq N_G(H)L$ then $\tilde{S}(H)$ is an imprimitive $\bar{N}_G(H)$-module with a system of imprimitivity

$$\left\{c_H^{g^{-1}}(U):\ gL \subseteq N_G(H)L\right\}$$

on which $\bar{N}_G(H)$ acts transitively, implying that

$$\tilde{S}(H) \cong \uparrow_{\bar{N}_L(H)}^{\bar{N}_G(H)} U \quad \text{as } \mathbb{K}\bar{N}_G(H)\text{-modules.}$$

On the other hand, if $L = N_G(H)L$ then $\bar{N}_L(H) = \bar{N}_G(H)$ and $\tilde{S}(H) = U$. So the result is trivial in this case. □

**Proposition 3.5.** *Let $H \leqslant L \leqslant G$ be such that ${}^gH \leqslant L$ for every $g \in G$, and let a simple Mackey functor $S^L_{H,U}$ for $L$ be given. Put $V = \uparrow_{\bar{N}_L(H)}^{\bar{N}_G(H)} U$. Then $\uparrow_L^G S^L_{H,U}$ is simple if and only if $V$ is simple, and if this is the case then $\uparrow_L^G S^L_{H,U} \cong S^G_{H,V}$.*

**Proof.** If $\uparrow_L^G S^L_{H,U}$ is simple then 3.4(iv) implies that $V$ is simple. Conversely, suppose $V \cong (\uparrow_L^G S^L_{H,U})(H)$ is simple. Then 3.4 and 2.4 imply that $\uparrow_L^G S^L_{H,U}$ is simple. Finally the last assertion follows by 2.5 and 3.4. □

We have now accumulated all the information necessary to prove one of our main results, Clifford's theorem for Mackey functors. But we first state some consequences of 3.4 and 3.5.

**Remark 3.6.** Let $S$ be Mackey functor for $G$, and $T$ be a $G$-subfunctor of $S$, and let $\chi$ be a family of subgroups of $G$ closed under taking subgroups and conjugation. Then we have

$$\operatorname{Ker} r_\chi^T = T \cap \operatorname{Ker} r_\chi^S, \qquad \operatorname{Im} t_\chi^T \leqslant T \cap \operatorname{Im} t_\chi^S, \quad \text{and} \quad \operatorname{Im} t_\chi^{\operatorname{Im} t_\chi^T} = \operatorname{Im} t_\chi^T.$$

**Proof.** Since $T$ is a subfunctor it must be stable under restriction and trace, implying that

$$\operatorname{Ker}\bigl(r_X^K : T(K) \to T(X)\bigr) = T(K) \cap \operatorname{Ker}\bigl(r_X^K : S(K) \to S(X)\bigr),$$
$$t_X^K\bigl(T(X)\bigr) \subseteq T(K) \cap t_X^K\bigl(S(X)\bigr)$$

for any $K \leqslant G$ and $X \in \chi$ with $X \leqslant K$. Then the result follows easily. $\square$

**Corollary 3.7.** *Let $H \leqslant L \leqslant G$ be such that ${}^gH \leqslant L$ for every $g \in G$, and let a simple Mackey functor $S_{H,U}^L$ for $L$ be given. Then, $\uparrow_L^G S_{H,U}^L$ is semisimple if and only if $\uparrow_{\bar{N}_L(H)}^{\bar{N}_G(H)} U$ is semisimple.*

**Proof.** Let $\tilde{S} = \uparrow_L^G S_{H,U}^L$. Suppose $\tilde{S} = \bigoplus_{i \in I} S_i$ is a decomposition into simple $G$-subfunctors. If for a $K \leqslant G$ and $i \in I$ $S_i(K)$ is nonzero then

$$\tilde{S}(K) = \bigoplus_{KgL \subseteq G} S_{H,U}^L(L \cap K^g)$$

is nonzero, and so $S_{H,U}^L(L \cap K^g) \neq 0$ for some $g \in G$, and by 2.3, $H \leqslant_G K$. Then by evaluating at $H$ we get $\tilde{S}(H) = \bigoplus_{i \in J} S_i(H)$ where $J$ is the subset of $I$ containing those $i \in I$ for which $S_i(H) \neq 0$. And $H$ is a minimal subgroup of $S_i$ for each $i \in J$, so $S_i(H)$ is a simple $\bar{N}_G(H)$-module for any $i \in J$. Therefore, $\tilde{S}(H)$ is semisimple, and so is $\uparrow_{\bar{N}_L(H)}^{\bar{N}_G(H)} U$ by 3.4.

Conversely, suppose now $\uparrow_{\bar{N}_L(H)}^{\bar{N}_G(H)} U = \bigoplus_i V_i$ where each $V_i$ is a simple $\mathbb{K}\bar{N}_G(H)$-module. We let $S_i$ be the $G$-subfunctor of $\tilde{S}$ generated by $V_i$. In particular $S_i(H) = V_i$, $H$ is a minimal subgroup of $S_i$ and $\operatorname{Im} t_{\chi H}^{S_i} = S_i$ for each $i$. Also by 3.4 $\operatorname{Ker} r_{\chi H}^{\tilde{S}} = 0$. Then 3.6 implies that $\operatorname{Ker} r_{\chi H}^{S_i} = 0$ for each $i$. Hence each $S_i$ is a simple Mackey functor for $G$. More to the point,

$$\Bigl(\sum_i S_i\Bigr)(H) = \sum_i V_i = \uparrow_{\bar{N}_L(H)}^{\bar{N}_G(H)} U \cong \tilde{S}(H)$$

by 3.4, and this implies that $\tilde{S} = \sum_i S_i$ because we know by 3.4 that $\tilde{S}$ is generated by $\tilde{S}(H)$. Consequently $\uparrow_L^G S_{H,U}^L$ is semisimple. $\square$

**Corollary 3.8.** *Let $\mathbb{K}$ be of characteristic $p > 0$, and let $N$ be a normal subgroup of $G$ such that $(|G : N|, p) = 1$, and let $N \leqslant L \leqslant G$. Then, if $S_{H,U}^L$ is a simple Mackey functor for $L$ over $\mathbb{K}$ with $H \leqslant N$ then $\uparrow_L^G S_{H,U}^L$ is semisimple.*

**Proof.** We know that $U$ is simple $\mathbb{K}\bar{N}_L(H)$-module. Note that $\bar{N}_N(H) \trianglelefteq \bar{N}_G(H)$, $\bar{N}_N(H) \leqslant \bar{N}_L(H) \leqslant \bar{N}_G(H)$, and $(|\bar{N}_G(H) : \bar{N}_N(H)|, p) = 1$. Therefore, by [20, Theorem 11.2], $\uparrow_{\bar{N}_L(H)}^{\bar{N}_G(H)} U$ is semisimple. The result now follows by 3.7. $\square$

Over algebraically closed fields, simple modules of nilpotent groups are monomial. The following is a Mackey functor version of this result.

**Corollary 3.9.** *Let $G$ be a nilpotent group, and $\mathbb{K}$ be algebraically closed. Then, for any simple Mackey functor $S^G_{H,V}$ for $G$ over $\mathbb{K}$, there is a simple Mackey functor $S^L_{H,W}$ for some subgroup $L$ with $H \trianglelefteqslant L \leqslant G$ such that* $\dim_{\mathbb{K}} W = 1$ *and* $\uparrow^G_L S^L_{H,W} \cong S^G_{H,V}$.

**Proof.** As $\mathbb{K}$ is algebraically closed, $\bar{N}_G(H)$ is nilpotent, and $V$ is simple $\mathbb{K}\bar{N}_G(H)$-module, $V$ must be monomial, see [21, Theorem 3.7, p. 205]. Therefore, there is a subgroup $\bar{L}$ of $\bar{N}_G(H)$ and a one-dimensional $\mathbb{K}\bar{L}$-module $W$ such that $\uparrow^{\bar{N}_G(H)}_{\bar{L}} W \cong V$. Now, $H \trianglelefteqslant L \leqslant N_G(H)$ implies that $\bar{N}_L(H) = \bar{L}$, and so we may consider the simple Mackey functor $S^L_{H,W}$ for $L$. Since $G$ is nilpotent, we can find a subnormal series: $L = L_0 \lhd L_1 \lhd \cdots \lhd L_n = G$ for some natural number $n$. For $j = 1, \ldots, n-1$ we let

$$W_j = \uparrow^{\bar{N}_{L_j}(H)}_{\bar{N}_{L_{j-1}}(H)} \cdots \uparrow^{\bar{N}_{L_1}(H)}_{\bar{N}_L(H)} W.$$

Since

$$V \cong \uparrow^{\bar{N}_G(H)}_{\bar{N}_{L_{n-1}}(H)} \uparrow^{\bar{N}_{L_{n-1}}(H)}_{\bar{N}_{L_{n-2}}(H)} \cdots \uparrow^{\bar{N}_{L_1}(H)}_{\bar{N}_L(H)} W$$

is simple, it follows that $W_{n-1}, \ldots, W_1$ are all simple. Then, by a repeated application of 3.5

$$\begin{aligned}\uparrow^G_L S^L_{H,W} &\cong \uparrow^G_{L_{n-1}} \cdots \uparrow^{L_2}_{L_1} \uparrow^{L_1}_L S^L_{H,W} \cong \uparrow^G_{L_{n-1}} \cdots \uparrow^{L_2}_{L_1} S^{L_1}_{H,W_1} \cong \cdots \\ &\cong \uparrow^G_{L_{n-1}} S^{L_{n-1}}_{H,W_{n-1}} \cong S^G_{H,V}. \qquad \square\end{aligned}$$

We now state Clifford's theorem for Mackey functors. We state it over a filed, but it is true over any commutative base ring. Of course, restriction of a simple Mackey functor may be 0. Indeed, $\downarrow^G_K S^G_{H,V} \neq 0$ implies that $H \leqslant_G K$. And note that if $H \leqslant N \trianglelefteqslant G$ then $\bar{N}_N(H) \trianglelefteqslant \bar{N}_G(H)$.

**Theorem 3.10.** *Let $N \trianglelefteqslant G$, and let $S^G_{H,V}$ be a simple Mackey functor for $G$ over $\mathbb{K}$ such that $H \leqslant N$. Then*:

(i) *There is a simple $N$-subfunctor $S^N_{H,W}$ of $\downarrow^G_N S^G_{H,V}$.*
(ii) *Let $L = \{g \in G\colon S^N_{{}^gH, c^g_H(W)} \cong S^N_{H,W}\}$ be the inertia group of $S^N_{H,W}$. Then, there is a positive integer $e = e(S^G_{H,V})$, called the ramification index of $S^G_{H,V}$ relative to $N$, such that*

$$\downarrow^G_N S^G_{H,V} \cong e \bigoplus_{gL \subseteq G} \big|^g_N S^N_{H,W} \cong e \bigoplus_{gL \subseteq G} S^N_{{}^gH, c^g_H(W)}.$$

*Moreover, if $\bar{T} = \{\bar{g} \in \bar{N}_G(H)\colon c^g_H(W) \cong W\}$ is the inertia group of the $\bar{N}_N(H)$-module $W$ in $\bar{N}_G(H)$, then $L = NT$ and*

$$\downarrow^{\bar{N}_G(H)}_{\bar{N}_N(H)} V \cong e \bigoplus_{gT \subseteq N_G(H)} c^g_H(W).$$

*Furthermore $S^N_{{}^gH, c^g_H(W)}$, for $gL \subseteq G$, form, without repetition, a complete set of nonisomorphic $G$-conjugates of $S^N_{H,W}$. And $c^g_H(W)$, for $gT \subseteq N_G(H)$, form, without repetition, a complete set of nonisomorphic $\bar{N}_G(H)$-conjugates of $W$.*

(iii) *$N_L(H) = T$ and there is a simple Mackey functor $S$ for $L$ such that $S \cong S^L_{H,U}$ where $U$ is the sum of all $\mathbb{K}\bar{N}_N(H)$-submodules of $\downarrow^{\bar{N}_G(H)}_{\bar{N}_N(H)} V$ isomorphic to $W$. Moreover, $S$ is a simple $L$-subfunctor of $\downarrow^G_L S^G_{H,V}$ such that*

$$\downarrow^L_N S \cong eS^N_{H,W} \quad \textit{and} \quad \uparrow^G_L S \cong S^G_{H,V}.$$

*Furthermore $U$ is a simple $\mathbb{K}\bar{N}_L(H)$-submodule of $\downarrow^{\bar{N}_G(H)}_{\bar{N}_L(H)} V$ satisfying*

$$\downarrow^{\bar{N}_L(H)}_{\bar{N}_N(H)} U \cong eW \quad \textit{and} \quad \uparrow^{\bar{N}_G(H)}_{\bar{N}_L(H)} U \cong V.$$

**Proof.** As $V$ is a simple $\mathbb{K}\bar{N}_G(H)$-module and $\bar{N}_N(H) \trianglelefteq \bar{N}_G(H)$, by Clifford's theorem for group algebras [21], there is a positive integer $e$, and a simple $\mathbb{K}\bar{N}_N(H)$-submodule $W$ of $V$ such that

$$\downarrow^{\bar{N}_G(H)}_{\bar{N}_N(H)} V \cong e \bigoplus_{\bar{g}\bar{T} \subseteq \bar{N}_G(H)} c^g_H(W) = e \bigoplus_{gT \subseteq N_G(H)} c^g_H(W),$$

where $\bar{T} = \{\bar{g} \in \bar{N}_G(H)\colon c^g_H(W) \cong W\}$ is the inertia group of the $\bar{N}_N(H)$-module $W$ in $\bar{N}_G(H)$. Moreover $c^g_H(W)$, $gT \subseteq N_G(H)$, form, without repetition, a complete set of nonisomorphic $\bar{N}_G(H)$-conjugates of $W$. Also, if $U$ is the sum of all $\mathbb{K}\bar{N}_N(H)$-submodules of $\downarrow^{\bar{N}_G(H)}_{\bar{N}_N(H)} V$ isomorphic to $W$ then $U$ is a simple $\mathbb{K}\bar{T}$-module such that

$$\downarrow^{\bar{T}}_{\bar{N}_N(H)} U \cong eW \quad \text{and} \quad \uparrow^{\bar{N}_G(H)}_{\bar{T}} U \cong V.$$

For any $x \in G$, it is clear that

$$\downarrow^{\bar{N}_G({}^xH)}_{\bar{N}_N({}^xH)} c^x_H(V) = c^x_H\big(\downarrow^{\bar{N}_G(H)}_{\bar{N}_N(H)} V\big) \cong e \bigoplus_{gT \subseteq N_G(H)} c^{xg}_H(W).$$

We now use 3.2 with $L = N$ and $M = S^G_{H,V}$. The parts (iv) and (v) of 3.2 imply

$$\downarrow^G_N S^G_{H,V} = \sum_{NxN_G(H) \subseteq G} T^N_{{}^xH, c^x_H(V)} \quad \text{and}$$

$$T^N_{{}^xH, c^x_H(V)} \cong e \sum_{gT \subseteq N_G(H)} T^N_{{}^xH, c^{xg}_H(W)} = e \sum_{gT \subseteq N_G(H)} T^N_{{}^{xg}H, c^{xg}_H(W)},$$

where we use $g \in N_G(H)$ for the last equality. Therefore, we have the decomposition

$$\downarrow^G_N S^G_{H,V} \cong e \sum_{NxN_G(H) \subseteq G} \sum_{gT \subseteq N_G(H)} T^N_{{}^{xg}H, c^{xg}_H(W)}.$$

Letting $G = \biguplus_i Nx_iN_G(H)$ and $N_G(H) = \biguplus_j g_jT$ we see that $G = \biguplus_i \biguplus_j Nx_ig_jT$. Thus,

$$\downarrow_N^G S_{H,V}^G \cong e \sum_i \sum_j T^N_{{}^{x_ig_j}H, c_H^{x_ig_j}(W)} = e \sum_{NgT \subseteq G} T^N_{{}^gH, c_H^g(W)}.$$

Moreover, by 3.2 and 2.5, we know that $T^N_{{}^gH,c_H^g(W)}$, $NgT \subseteq G$, are all simple and distinct. Hence, $T^N_{{}^gH,c_H^g(W)} \cong S^N_{{}^gH,c_H^g(W)}$ and we have the direct sum

$$\downarrow_N^G M \cong e \bigoplus_{gNT \subseteq G} S^N_{{}^gH,c_H^g(W)},$$

where we use $NgT = gNT$. Furthermore, by 2.5, $S^N_{{}^gH,c_H^g(W)} \cong S^N_{H,W}$ if and only if, for some $n \in N$, ${}^{ng}H = H$ and $c_H^{ng}(W) \cong W$, equivalently $g \in NT = L$. Hence, $S^N_{{}^gH,c_H^g(W)}$, $gL \subseteq G$, form, without repetition, a complete set of nonisomorphic $G$-conjugates of $S^N_{H,W}$ and $L = NT$.

Now $U$ is a simple $\mathbb{K}\bar{T}$-submodule of $M(H) = V$. If we apply the modular law to the tower $T \leqslant N_G(H) \leqslant G \geqslant N$ we see that

$$N_L(H) = N_G(H) \cap L = N_G(H) \cap TN = T\big(N_G(H) \cap N\big) = TN_N(H) = T.$$

As a result, $U$ is a simple $\mathbb{K}\bar{N}_L(H)$-submodule of $V$. We put $S = T^L_{H,U}$. It is a simple $L$-subfunctor of $\downarrow_L^G M$, by 3.2, and so $S \cong S^L_{H,U}$.

As $\uparrow_{\bar{N}_L(H)}^{\bar{N}_G(H)} U \cong V$ is simple, 3.5 implies that $\uparrow_L^G S^L_{H,U} \cong S^G_{H,V}$.

Finally, since $U$ is a $\mathbb{K}\bar{T}$-module we have $c_H^x(U) = c_H^n(U)$ for any $x = nt \in L = NT$, $n \in N$, $t \in T$. If $K \leqslant N$,

$$\big(\downarrow_N^L S^L_{H,U}\big)(K) = \sum_{x \in L:\ {}^xH \leqslant K} t^K_{{}^xH} c_H^x(U) = \sum_{n \in N:\ {}^nH \leqslant K} t^K_{{}^nH} c_H^n(U) = T^N_{H,U}(K),$$

thus $\downarrow_N^L S^L_{H,U} = T^N_{H,U}$. Because $\downarrow_{\bar{N}_N(H)}^{\bar{N}_L(H)} U \cong eW$, 3.2 implies that $T^N_{H,U} \cong eS^N_{H,W}$. Hence, $\downarrow_N^L S^L_{H,U} \cong eS^N_{H,W}$. $\square$

## 4. The Clifford correspondence

Our aim in this section is to prove a Mackey functor version of the Clifford correspondence. Namely, if $N$ is a normal subgroup of $G$ and $S$ is a simple Mackey functor for $N$ whose inertia group is $L$ then we show that there is a bijection between certain simple Mackey functors for $L$ and for $G$.

The following result shows that given any simple Mackey functor $S$ for $N$, $N \trianglelefteq G$, we can find a simple Mackey functor $M$ for $G$ such that $S$ is a direct summand of $\downarrow_N^G M$.

**Lemma 4.1.**

(i) *Let $K \leqslant G$, and let $S$ be a simple Mackey functor for $K$ over $\mathbb{K}$. Then there exists a simple Mackey functor $M$ for $G$ such that $S$ is a $K$-subfunctor of $\downarrow_K^G M$.*
(ii) *Let $N \trianglelefteq G$, and let $S$ be a simple Mackey functor for $N$ over $\mathbb{K}$. Then there exists a simple Mackey functor $M$ for $G$ such that $S$ is a direct summand of $\downarrow_N^G M$ as $\mu_{\mathbb{K}}(N)$-modules.*

**Proof.** (i) Let $S = S_{H,W}^K$. So $W$ is a simple $\mathbb{K}\bar{N}_K(H)$-module and $\bar{N}_K(H) \leqslant \bar{N}_G(H)$. Then, by [24, Lemma 1.2, p. 224], there is a simple $\bar{N}_G(H)$-module $V$ such that $W$ is a $\bar{N}_K(H)$-submodule of $\downarrow_{\bar{N}_K(H)}^{\bar{N}_G(H)} V$. We let $M = S_{H,V}^G$. Now since $W$ is a submodule of $(\downarrow_K^G M)(H) = V$, we see that $S$ is a $K$-subfunctor of $\downarrow_K^G M$ because $S$ is generated by $S(H) = W$.

(ii) This follows from (i) and 3.10. □

**Remark 4.2.** Given a Mackey functor $M$ for $K$ where $K \leqslant G$, then $M$ is a direct summand of $\downarrow_K^G \uparrow_K^G M$.

**Proof.** By the Mackey decomposition formula, 2.6. □

For a ring $A$ and a subring $B$, we denote by $\mathrm{Irr}(A)$ a complete set of representatives for the isomorphism classes of simple $A$-modules, for $S \in \mathrm{Irr}(B)$ we denote by $\mathrm{Irr}(A|S)$ the set $\{M \in \mathrm{Irr}(A)\colon\ S|\downarrow_B^A M\}$ where the notation $S|\downarrow_B^A M$ means that $S$ is a direct summand of $1_B M$ as $B$-module where $1_B$ is the unity of $B$.

Given any simple Mackey functor $S_{H,W}^N$ for $N$ over $\mathbb{K}$ where $N \trianglelefteq G$, 4.1 implies that the sets $\mathrm{Irr}(\mu_{\mathbb{K}}(X)|S_{H,W}^N)$ are nonempty for any $X$ with $N \leqslant X \leqslant G$.

**Lemma 4.3.** *Let $N \trianglelefteq G$ and $N \leqslant X \leqslant G$. Then*:

(i) *If $M \in \mathrm{Irr}(\mu_{\mathbb{K}}(X)|S_{H,W}^N)$ then $M \cong S_{H,V}^X$ for some $V \in \mathrm{Irr}(\mathbb{K}\bar{N}_X(H)|W)$.*
(ii) *If $V \in \mathrm{Irr}(\mathbb{K}\bar{N}_X(H)|W)$ then $S_{H,V}^X \in \mathrm{Irr}(\mu_{\mathbb{K}}(X)|S_{H,W}^N)$.*
(iii) *$S_{H,V}^X \cong S_{H,V'}^X$ as Mackey functors for $X$ if and only if $V \cong V'$ as $\mathbb{K}\bar{N}_X(H)$-modules.*
(iv) *The map $\mathrm{Irr}(\mu_{\mathbb{K}}(X)|S_{H,W}^N) \to \mathrm{Irr}(\mathbb{K}\bar{N}_X(H)|W)$, given by $S_{H,V}^X \leftrightarrow V$, is a bijection preserving ramification indexes.*

**Proof.** (i) If $M = S_{K,V}^X \in \mathrm{Irr}(\mu_{\mathbb{K}}(X)|S_{H,W}^N)$ then 3.10 implies that $H$ and $K$ are $X$-conjugate which gives the desired result.

(ii) It is an immediate consequence of 3.10.

(iii) Follows by 2.5.

(iv) Follows by (i), (ii) and 2.5. □

The following result is a Mackey functor version of the Clifford correspondence for group algebras, see [21, Theorem 3.2, p. 203].

**Theorem 4.4.** *Let $N \trianglelefteq G$, and a simple Mackey functor $S_{H,W}^N$ for $N$ over $\mathbb{K}$ be given, and let $L$ be the inertia group of $S_{H,W}^N$ in $G$. Then*:

(i) *If* $S \in \operatorname{Irr}(\mu_{\mathbb{K}}(L)|S^N_{H,W})$ *then* $\uparrow^G_L S \in \operatorname{Irr}(\mu_{\mathbb{K}}(G)|S^N_{H,W})$.

(ii) *The map* $\operatorname{Irr}(\mu_{\mathbb{K}}(L)|S^N_{H,W}) \to \operatorname{Irr}(\mu_{\mathbb{K}}(G)|S^N_{H,W})$, *given by* $S \mapsto \uparrow^G_L S$, *is a bijection preserving ramification indexes.*

**Proof.** (i) Let $S \in \operatorname{Irr}(\mu_{\mathbb{K}}(L)|S^N_{H,W})$. Then, by 4.3, $S = S^L_{H,U}$ for some $U \in \operatorname{Irr}(\mathbb{K}\bar{N}_L(H)|W)$. Also 3.10 implies that $L = NT$ and $\bar{N}_L(H) = \bar{T}$ where $\bar{T}$ is the inertia group of the simple $\mathbb{K}\bar{N}_N(H)$-module $W$ in $\bar{N}_G(H)$. The Clifford correspondence for group algebras implies that $V = \uparrow^{\bar{N}_G(H)}_{\bar{N}_L(H)} U \in \operatorname{Irr}(\mathbb{K}\bar{N}_G(H)|W)$. Then, by 4.3, $S^G_{H,V} \in \operatorname{Irr}(\mu_{\mathbb{K}}(G)|S^N_{H,W})$. Finally, because of 3.5, $\uparrow^G_L S^L_{H,U} \cong S^G_{H,V}$.

(ii) By 4.3, the Clifford correspondence for group algebras, and again 4.3, respectively, the following composition of maps

$$\operatorname{Irr}\big(\mu_{\mathbb{K}}(L)|S^N_{H,W}\big) \to \operatorname{Irr}\big(\mathbb{K}\bar{N}_L(H)|W\big) \to \operatorname{Irr}\big(\mathbb{K}\bar{N}_G(H)|W\big) \to \operatorname{Irr}\big(\mu_{\mathbb{K}}(G)|S^N_{H,W}\big),$$
$$S = S^L_{H,U} \mapsto U \mapsto V = \uparrow^{\bar{N}_G(H)}_{\bar{N}_L(H)} U \mapsto S^G_{H,V} \cong \uparrow^G_L S^L_{H,U}$$

is a bijection preserving ramification indexes where for the last isomorphism we use 3.5. □

The inverse of the bijection in 4.4 will be described in the next section.

**Corollary 4.5.** *Let* $N \trianglelefteq G$, *and let* $S$ *be a simple Mackey functor for* $N$ *over* $\mathbb{K}$. *If the inertia group of* $S$ *in* $G$ *is* $N$ *then* $\uparrow^G_N S$ *is a simple Mackey functor for* $G$.

**Proof.** A simple consequence of 4.4. □

**Remark 4.6.** Let $N \trianglelefteq G$, and let $S_1$ and $S_2$ be simple Mackey functors for $N$ over $\mathbb{K}$. Then, $\uparrow^G_N S_1 \cong \uparrow^G_N S_2$ if and only if $S_1 \cong |^g_N S_2$ for some $g \in G$.

**Proof.** It is an easy consequence of 2.6. □

**Corollary 4.7.** *Let* $N \trianglelefteq G$, *and a simple Mackey functor* $S^N_{H,W}$ *for* $N$ *over* $\mathbb{K}$ *be given, and let* $L$ *be the inertia group of* $S^N_{H,W}$ *in* $G$. *Then, for any* $X$ *with* $L \leqslant X \leqslant G$, *the map* $\operatorname{Irr}(\mu_{\mathbb{K}}(X)|S^N_{H,W}) \to \operatorname{Irr}(\mu_{\mathbb{K}}(G)|S^N_{H,W})$, *given by* $S \mapsto \uparrow^G_X S$, *is a bijection preserving ramification indexes.*

**Proof.** This follows easily from 4.4. □

## 5. Group grading method

In this section, we first show that a certain subalgebra of $\mu_R(G)$ is a group graded algebra over $\mu_R(N)$ where $N$ is a normal subgroup of $G$. After obtaining a Mackey algebra version of Fong's theorem, we use Clifford theory results on group graded algebras to study restriction and induction of Mackey functors. We also study the subalgebras $e\mu_R(G)e$ of $\mu_R(G)$ for some special kinds of idempotents of $\mu_R(G)$.

For a ring $A$ and its subset $B$, we let $C_A(B) = \{a \in A\colon\ ab = ba,\ \text{for all } b \in B\}$, and $Z(A) = C_A(A)$, and $U(A)$ be the unit group of $A$.

An $R$-algebra $A$ is called strongly $G$-graded algebra if $A = \bigoplus_{x\in G} A_x$, direct sum of $R$-submodules of $A$, and $A_x A_y = A_{xy}$ for all $x, y \in G$; here $A_x A_y$ is the $R$-submodule of $A$ consisting of all finite sums $\sum_i a_i b_i$ with $a_i \in A_x$ and $b_i \in A_y$. The trivial component $A_1$ is a unital subring of $A$. If $u \in U(A)$ lies in $A_x$ for some $x \in G$ then $u$ is called graded unit and $x$ is called the degree of $u$, written $\deg(u) = x$. Letting $\operatorname{Gr} U(A)$ be the set of all graded units of $A$ we see that $\operatorname{Gr} U(A)$ is a subgroup of $U(A)$ and $\deg : \operatorname{Gr} U(A) \to G$, $u \mapsto \deg(u)$, is a group homomorphism with kernel $U(A_1)$. If $U(A) \cap A_x$ is nonempty for all $x \in G$ then $A$ is called a crossed product of $G$ over $A_1$. Let $A$ be a crossed product of $G$ over $A_1$, choosing $u_x \in U(A) \cap A_x$ for any $x \in G$, we see that $A_x = A_1 u_x = u_x A_1$ [3,17,22,23].

From now on, for $K \leqslant G$ we let $\chi_K$ denote the set $\{H \leqslant G\colon\ H \leqslant_G K\}$, and we let $1_K$ denote the unity of $\mu_R(K)$ which is a nonunital subring of $\mu_R(G)$, if $K \neq G$, and a unital subring of $1_K \mu_R(G) 1_K$. Finally, for $g \in G$ we let $\gamma_g = \sum_{L\leqslant G} c_L^g$, and we let $\beta_g = \sum_{L\leqslant N} c_L^g \in 1_N \mu_R(G) 1_N$ whenever $N$ is a normal subgroup of $G$.

**Lemma 5.1.** *Let $N$ be a normal subgroup of $G$. Then*:

(i) $\beta_x \mu_R(N) = \beta_y \mu_R(N)$ *if and only if* $xN = yN$.
(ii) $\beta_x \mu_R(N) = \mu_R(N) \beta_x$.
(iii) $1_N \mu_R(G) 1_N = \bigoplus_{gN\in G/N} \beta_{\bar g} \mu_R(N)$.

**Proof.** (i) Noting that $\beta_x 1_N = \beta_x = 1_N \beta_x$ for any $x \in G$, we see that $\beta_x \mu_R(N) = \beta_y \mu_R(N)$ if and only if $\beta_{y^{-1}x} \mu_R(N) = \mu_R(N)$, and so $\beta_{y^{-1}x} = \beta_{y^{-1}x} 1_N \in \mu_R(N)$, implying that $y^{-1}x \in N$. Conversely, $y^{-1}x \in N$ implies that $\beta_{y^{-1}x}$ is a unit of $\mu_R(N)$. Thus $\beta_{y^{-1}x} \mu_R(N) = \mu_R(N)$.

(ii) By 2.1, an $R$-basis element of $\mu_R(N)$ is of the form $t_{nJ}^H c_J^n r_J^K$ where $H \leqslant N \geqslant K$, $n \in N$, and $J \leqslant H^n \cap K$. For any $x \in G$ we have

$$\beta_x t_{nJ}^H c_J^n r_J^K = c_H^x t_{nJ}^H c_J^n r_J^K = t_{xnx^{-1}({}^xJ)}^{{}^xH} c_{({}^xJ)}^{xnx^{-1}} r_{({}^xJ)}^{{}^xK} c_K^x = t_{xnx^{-1}({}^xJ)}^{{}^xH} c_{({}^xJ)}^{xnx^{-1}} r_{({}^xJ)}^{{}^xK} \beta_x.$$

By the normality of $N$, $t_{nJ}^H c_J^n r_J^K$ is an element of $\mu_R(N)$ if and only if $t_{xnx^{-1}({}^xJ)}^{{}^xH} c_{({}^xJ)}^{xnx^{-1}} r_{({}^xJ)}^{{}^xK}$ is an element of $\mu_R(N)$. Therefore, $\beta_x \mu_R(N) = \mu_R(N) \beta_x$.

(iii) 2.1 implies that the elements $t_{gJ}^H c_J^g r_J^K$, where $H \leqslant N \geqslant K$, $HgK \subseteq G$, and $J$ is a subgroup of $H^g \cap K$ up to conjugacy, form, without repetition, a free $R$-basis of $1_N \mu_R(G) 1_N$. Now $g \in G$ is in a unique coset $xN$, and if $g = xn$ with $n \in N$ then

$$t_{gJ}^H c_J^g r_J^K = c_{H^x}^x t_{nJ}^{H^x} c_J^n r_J^K = \beta_x t_{nJ}^{H^x} c_J^n r_J^K \in \beta_x \mu_R(N).$$

Hence,

$$1_N \mu_R(G) 1_N = \sum_{gN\in G/N} \beta_{\bar g} \mu_R(N).$$

Furthermore, since $\beta_x$ is a unit of $1_N \mu_R(G) 1_N$ we see that the elements $\beta_x t_{nJ}^H c_J^n r_J^K$, where $H \leqslant N \geqslant K$, $HnK \subseteq N$, and $J$ is a subgroup of $H^n \cap K$ up to conjugacy, form, without repetition, a free $R$-basis of $\beta_x \mu_R(N)$. If $\beta_x t_{nJ}^H c_J^n r_J^K = \beta_y t_{mI}^{H'} c_I^m r_I^{K'}$ then

$$\beta_{y^{-1}x} t_{nJ}^H c_J^n r_J^K = t_{y^{-1}xnJ}^{{}^{y^{-1}x}H} c_J^{y^{-1}xn} r_J^K = t_{mI}^{H'} c_I^m r_I^{K'}.$$

Then, by 2.1, $K' = K$, ${}^{y^{-1}x}H = H'$ and $H'mK' = H'y^{-1}xnK'$, implying that $N = y^{-1}xN$. So (i) implies that $\beta_x \mu_R(N) = \beta_y \mu_R(N)$. Hence, any basis element of $1_N \mu_R(G) 1_N$ lies in a unique summand $\beta_{\bar{x}} \mu_R(N)$. Therefore, the sum $\sum_{gN \in G/N} \beta_{\bar{g}} \mu_R(N)$ must be direct. □

Lemma 5.1 implies

**Theorem 5.2.** *If $N \trianglelefteq G$ then*

$$1_N \mu_R(G) 1_N = \bigoplus_{gN \in G/N} \beta_{\bar{g}} \mu_R(N)$$

*is a crossed product of $G/N$ over $\mu_R(N)$.*

We state the following elementary result whose proof is straightforward, see [3,14,17].

**Remark 5.3.** Let $A$ be a crossed product of $G$ over $A_1$. Then:

(i) For each $y \in C_A(A_1)$ and $g \in G$, let ${}^g y = u_g y u_g^{-1}$ where $u_g$ is any element of $U(A) \cap A_g$. Then, with this action $G$ acts as automorphism of the algebras $C_A(A_1)$ and $Z(A_1)$. Furthermore, the above action does not depend on the choice of $u_g$.
(ii) Let $e$ be a $G$-invariant block idempotent of $A_1$, that is, $u_g e u_g^{-1} = e$, for all $g \in G$. Then $e$ is a central idempotent of $A$, and $Ae = \bigoplus_{g \in G} A_g e$ is a crossed product of $G$ over $A_1 e$.

Let $N$ be a normal subgroup of $G$. Then we note that $\gamma_g a = \beta_g a$ for any $a \in \mu_R(N)$. If $e$ is a block idempotent of $\mu_R(N)$ corresponding to a $G$-invariant simple $\mu_R(N)$-module $S$ then $\beta_g e = e \beta_g$ for all $g \in G$ where, by $G$-invariant, we mean that the inertia group is $G$.

If $A = \bigoplus_{g \in G} A_g$ is a strongly $G$-graded algebra and $W$ is an $A_1$-module, the conjugate of $W$ is defined to be the $A_1$-module $A_g \otimes_{A_1} W$ with obvious $A_1$-action [3,17,23]. Let $A_1 = \mu_R(N)$ and $A = 1_N \mu_R(G) 1_N$. Then, by 5.2, $A$ is a strongly $G/N$-graded algebra, and note that the notion of conjugation of $A_1$-modules described above coincides with the conjugation of $\mu_R(N)$-modules defined in Section 2, because if $S$ is a $\mu_{\mathbb{K}}(N)$-module we defined its conjugate $|_N^g S$ in Section 2 as $|_N^g S = S$ with $\mu_{\mathbb{K}}(N)$ action given as $x.s = \gamma_{g^{-1}} x \gamma_g s$ for $x \in \mu_{\mathbb{K}}(N)$, $s \in S$. On the other hand, we defined its conjugate here as ${}^g S = \beta_{\bar{g}} \mu_{\mathbb{K}}(N) \otimes_{\mu_{\mathbb{K}}(N)} S$. Now it is clear that there is a $\mu_{\mathbb{K}}(N)$-module isomorphism $|_N^g S \to {}^g S$ given by $s \mapsto \beta_{\bar{g}} \otimes s$.

We now proceed to obtain one of our main results, a Mackey algebra version of Fong's theorem, see [21, Theorem 7.4, p. 355].

**Theorem 5.4.** *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$, and let $N$ be a normal $p'$-subgroup of $G$. If $e$ is a $G$-invariant block idempotent of $\mu_{\mathbb{K}}(N)$, then*:

(i) *$e$ is a central idempotent of $1_N \mu_{\mathbb{K}}(G) 1_N$.*
(ii) *$\mu_{\mathbb{K}}(N) e \cong \mathrm{Mat}_d(\mathbb{K})$, the algebra of $d \times d$ matrices over $\mathbb{K}$.*
(iii)

$$\big(1_N \mu_{\mathbb{K}}(G) 1_N\big) e = \bigoplus_{gN \in G/N} \beta_{\bar{g}} \mu_{\mathbb{K}}(N) e$$

*is a crossed product of $G/N$ over $\mu_{\mathbb{K}}(N) e$.*

(iv)

$$e\mu_{\mathbb{K}}(G)e \cong \mu_{\mathbb{K}}(N)e \otimes_{\mathbb{K}} C_{\mu_{\mathbb{K}}(G)}\big(\mu_{\mathbb{K}}(N)\big)e.$$

(v) *There is a central extension $\tilde{G}$ of $G/N$ by a cyclic $p'$-group $Z$ and a linear character $\lambda$ of $Z$ such that*

$$e\mu_{\mathbb{K}}(G)e \cong \mu_{\mathbb{K}}(N)e \otimes_{\mathbb{K}} \mathbb{K}\tilde{G}e_\lambda,$$

*where $e_\lambda = \frac{1}{|Z|}\sum_{z\in Z}\lambda(z^{-1})z$ is the corresponding block idempotent of $\mathbb{K}Z$, which is also a central idempotent of $\mathbb{K}\tilde{G}$. Moreover we can express the above isomorphism as*

$$e\mu_{\mathbb{K}}(G)e \cong \mu_{\mathbb{K}}(N)e \otimes_{\mathbb{K}} \big(\epsilon_\lambda \mu_{\mathbb{K}}(\tilde{G})\epsilon_\lambda\big),$$

*where $\epsilon_\lambda = \frac{1}{|Z|}\sum_{z\in Z}\lambda(z^{-1})c_1^z$, an idempotent of $\mu_{\mathbb{K}}(\tilde{G})$.*

**Proof.** (i) and (iii) They follow by 5.3.

(ii) Since $N$ is a $p'$-group, $\mu_{\mathbb{K}}(N)$ is semisimple by [27], implying the result.

(iv) As $(1_N\mu_{\mathbb{K}}(G)1_N)e$ is a crossed product of $G/N$ over a matrix algebra $\mu_{\mathbb{K}}(N)e$, [21, Theorem 7.2, p. 352] implies that

$$\big(1_N\mu_{\mathbb{K}}(G)1_N\big)e \cong \mu_{\mathbb{K}}(N)e \otimes_{\mathbb{K}} C_{(1_N\mu_{\mathbb{K}}(G)1_N)e}\big(\mu_{\mathbb{K}}(N)e\big).$$

Now it is clear that

$$\big(1_N\mu_{\mathbb{K}}(G)1_N\big)e = e\mu_{\mathbb{K}}(G)e \quad \text{and} \quad C_{(1_N\mu_{\mathbb{K}}(G)1_N)e}\big(\mu_{\mathbb{K}}(N)e\big) = C_{\mu_{\mathbb{K}}(G)}\big(\mu_{\mathbb{K}}(N)\big)e.$$

(v) The same argument in [21, pp. 352–354] with $A = (1_N\mu_{\mathbb{K}}(G)1_N)e$ and $A_1 = \mu_{\mathbb{K}}(N)e$ shows that there is a central extension $\tilde{G}$ of $G/N$ by a cyclic $p'$-group $Z$ and a linear character $\lambda$ of $Z$ such that $C_A(A_1) = \mathbb{K}\tilde{G}e_\lambda$, and we know that

$$C_A(A_1) = C_{(1_N\mu_{\mathbb{K}}(G)1_N)e}\big(\mu_{\mathbb{K}}(N)e\big) = C_{\mu_{\mathbb{K}}(G)}\big(\mu_{\mathbb{K}}(N)\big)e.$$

Moreover, the basis Theorem 2.1 shows that

$$t_1^1\mu_{\mathbb{K}}(\tilde{G})t_1^1 = \bigoplus_{g\in\tilde{G}}\mathbb{K}c_1^g \cong \mathbb{K}\tilde{G}, \quad c_1^g \leftrightarrow g, \quad \text{as } \mathbb{K}\text{-algebras}.$$

Letting $\epsilon_\lambda$ corresponds to $e_\lambda$ under this isomorphism, we see that $\epsilon_\lambda$ is a central idempotent of $t_1^1\mu_{\mathbb{K}}(\tilde{G})t_1^1$, because $e_\lambda$ is a central idempotent of $\mathbb{K}\tilde{G}$. As $t_1^1$ is the unity of $t_1^1\mu_{\mathbb{K}}(\tilde{G})t_1^1$, we have $(t_1^1\mu_{\mathbb{K}}(\tilde{G})t_1^1)\epsilon_\lambda = \epsilon_\lambda\mu_{\mathbb{K}}(\tilde{G})\epsilon_\lambda$. □

Mackey functors for $G$ over $R$ and left $\mu_R(G)$-modules are identical as described in Section 2. The same identification shows that

**Remark 5.5.** Let $N$ be a normal subgroup of $G$. Then, Mackey functors for $\chi_N$ over $R$ and left $1_N\mu_R(G)1_N$-modules are identical.

Before going further we need the following result, see [12, pp. 83–87].

**Remark 5.6.** Let $e$ be an idempotent in a ring $A$. Then:

(i) If $V$ is a simple $A$-module then $eV$ is either zero or a simple $eAe$-module.
(ii) Let $W$ is a simple $eAe$-module, and let $V = Ae \otimes_{eAe} W$. Then $eV \cong W$. Moreover, if $I$ is the sum of all $A$-submodules of $V$ killed by $e$ then $I$ is the unique maximal $A$-submodule of $V$ and $e(V/I) \cong W$.
(iii) Let $\mathrm{Irr}(A|e)$ be the set $\{V \in \mathrm{Irr}(A)\colon\ eV \neq 0\}$. Then, there is a bijection $\mathrm{Irr}(A|e) \leftrightarrow \mathrm{Irr}(eAe)$, given by $V \to eV$ and $(Ae \otimes_{eAe} W)/I \leftarrow W$, where $I$ is the unique maximal $A$-submodule of $Ae \otimes_{eAe} W$.

Clifford theory for group graded algebras in [3, Section 18] applied to the crossed product $1_N\mu_{\mathbb{K}}(G)1_N = \bigoplus_{gN\in G/N} \beta_{\bar g}\mu_{\mathbb{K}}(N)$ of $G/N$ over $\mu_{\mathbb{K}}(N)$ implies the following result.

**Proposition 5.7.** *Let $N \trianglelefteq G$, and $\mathfrak{N}$ be simple $1_N\mu_{\mathbb{K}}(G)1_N$-module, and let $\mathfrak{S}$ be a simple $\mu_{\mathbb{K}}(N)$-submodule of $\mathfrak{N}$. Assume that $\mathfrak{S}'$ is a simple $\mu_{\mathbb{K}}(N)$-module whose inertia group $\{gN \subseteq G\colon\ \beta_{\bar g}\mu_{\mathbb{K}}(N) \otimes_{\mu_{\mathbb{K}}(N)} \mathfrak{S}' \cong \mathfrak{S}'\}$ is $L'/N$. Then*:

(i) *If $L/N$ is the inertia group of $\mathfrak{S}$ there is a positive integer $d$ such that*

$$\mathfrak{N} \cong d \bigoplus_{gL\subseteq G} {}^{g}\mathfrak{S}.$$

(ii) *Let $\mathfrak{P}$ be the sum of all $\mu_{\mathbb{K}}(N)$-submodules of $\mathfrak{N}$ isomorphic to $\mathfrak{S}$. Then $\mathfrak{P}$ is a simple $1_N\mu_{\mathbb{K}}(L)1_N$-submodule of $\mathfrak{N}$ such that*

$$1_N\mu_{\mathbb{K}}(G)1_N \otimes_{(1_N\mu_{\mathbb{K}}(L)1_N)} \mathfrak{P} \cong \mathfrak{N} \quad \textit{as } 1_N\mu_{\mathbb{K}}(G)1_N\textit{-modules}, \quad \textit{and}$$
$$\mathfrak{P} \cong d\mathfrak{S} \quad \textit{as } \mu_{\mathbb{K}}(N)\textit{-modules}.$$

(iii) *The map*

$$\mathrm{Irr}\big(1_N\mu_{\mathbb{K}}(L')1_N|\mathfrak{S}'\big) \to \mathrm{Irr}\big(1_N\mu_{\mathbb{K}}(G)1_N|\mathfrak{S}'\big), \qquad \mathfrak{P}' \mapsto 1_N\mu_{\mathbb{K}}(G)1_N \otimes_{(1_N\mu_{\mathbb{K}}(L')1_N)} \mathfrak{P}',$$

*is a bijection. The inverse map sends $\mathfrak{N}'$ to the sum of all $\mu_{\mathbb{K}}(N)$-submodules of $\mathfrak{N}'$ isomorphic to $\mathfrak{S}'$.*

Let $M = S^G_{H,V}$ be a simple Mackey functor for $G$ over $\mathbb{K}$ with a minimal subgroup $H$ contained in a normal subgroup $N$ of $G$. By 5.6 $\mathfrak{N} = 1_N M$ is a simple $1_N\mu_{\mathbb{K}}(G)1_N$-module. Then 5.6 and 5.7 imply some parts of Clifford's theorem for Mackey functors, 3.10. For 3.10 we have the following result.

**Remark 5.8.** The simple $\mu_{\mathbb{K}}(L)$-module $S = S^L_{H,U}$ in 3.10 and the simple $1_N\mu_{\mathbb{K}}(L)1_N$-module $\mathfrak{P}$ in 5.7 correspond to each other under the bijection $\mathrm{Irr}(\mu_{\mathbb{K}}(L)|1_N) \to \mathrm{Irr}(1_N\mu_{\mathbb{K}}(L)1_N)$ described in 5.6, that is $\mathfrak{P} = 1_N S$.

**Proof.** $L$ is the inertia group of $S$ as described in 3.10 if and only if $L/N$ is the inertia group of $S$ as described in 5.7.

We use the notations of 3.10. So $M = S^G_{H,V}$ and $S = S^L_{H,U}$ where $U$ is the sum of all $\mathbb{K}\bar{N}_N(H)$-submodules of $\downarrow^{\bar{N}_G(H)}_{\bar{N}_N(H)} V$ isomorphic to $W$. Moreover by 5.7 $\mathfrak{P}$ is the sum of all $\mu_{\mathbb{K}}(N)$-submodules of $1_N M$ isomorphic to $S^N_{H,W}$. Let $W' \cong W$ be a summand of $U$. Then, $S^N_{H,W'}$ is a $N$-subfunctor of $S$ isomorphic to $S^N_{H,W}$, and so $1_N S^N_{H,W'} = S^N_{H,W'}$ is a summand of $\mathfrak{P}$. Hence, $1_N S \subseteq \mathfrak{P}$ as $1_N\mu_{\mathbb{K}}(L)1_N$-modules from which the equality $1_N S = \mathfrak{P}$ follows by simplicity of $\mathfrak{P}$. $\square$

The next result describes the inverse of the bijective map given in 4.4

**Proposition 5.9.** *Let $N \trianglelefteq G$, and a simple Mackey functor $S^N_{H,W}$ for $N$ over $\mathbb{K}$ be given, and let $L$ be the inertia group of $S^N_{H,W}$ in $G$. For an $M \in \operatorname{Irr}(\mu_{\mathbb{K}}(G)|S^N_{H,W})$ we let $P_M$ be the sum of all $N$-subfunctors of $\downarrow^G_N M$ isomorphic to $S^N_{H,W}$, and we let $I_M$ be the unique maximal $L$-subfunctor of $\mu_{\mathbb{K}}(L)1_N \otimes_{1_N\mu_{\mathbb{K}}(L)1_N} P_M$. Then*:

(i) *If $M \in \operatorname{Irr}(\mu_{\mathbb{K}}(G)|S^N_{H,W})$, then $(\mu_{\mathbb{K}}(L)1_N \otimes_{1_N\mu_{\mathbb{K}}(L)1_N} P_M)/I_M = \bar{P}_M \in \operatorname{Irr}(\mu_{\mathbb{K}}(L)|S^N_{H,W})$ and $\uparrow^G_L \bar{P}_M \cong M$.*
(ii) *The map $\operatorname{Irr}(\mu_{\mathbb{K}}(G)|S^N_{H,W}) \to \operatorname{Irr}(\mu_{\mathbb{K}}(L)|S^N_{H,W})$, which maps $M$ to $\bar{P}_M$, is a bijection. The inverse map is given by $S \mapsto \uparrow^G_L S$.*
(iii) *If $M \in \operatorname{Irr}(\mu_{\mathbb{K}}(G)|S^N_{H,W})$, then $\downarrow^G_L M$ has a unique simple $L$-subfunctor $S$ such that $S \in \operatorname{Irr}(\mu_{\mathbb{K}}(L)|S^N_{H,W})$ and $\uparrow^G_L S \cong M$.*

**Proof.** The first two parts are obvious consequences of 5.6, 5.7 and 5.8. The last part follows easily from the adjointness of restriction and induction functors, see [27]. $\square$

To use the results in the context of group graded algebras concerning indecomposable modules, we first need the following two lemmas to get a relationship between the indecomposable modules of $1_N\mu_R(G)1_N$ and $\mu_R(G)$, where $N$ is a normal subgroup of $G$.

**Lemma 5.10.** *Let $M$ be a Mackey functor for $L$ where $L \leqslant N \trianglelefteq G$. Put $\tilde{M} = \uparrow^G_L M$. Then $\operatorname{Ker} r^{\tilde{M}}_{\chi_N} = 0$ and $\operatorname{Im} t^{\tilde{M}}_{\chi_N} = \tilde{M}$.*

**Proof.** We write $\tilde{t}, \tilde{r}, \tilde{c}$ for the maps of $\tilde{M}$.

For any $K \leqslant G$,

$$\operatorname{Ker} r^{\tilde{M}}_{\chi_N}(K) = \bigcap_{X\leqslant N\colon X\leqslant K} \operatorname{Ker} \tilde{r}^K_X = \bigoplus_{KgL\subseteq G} \left( \bigcap_{X\leqslant N\colon X\leqslant K} \operatorname{Ker} r^{L\cap K^g}_{L\cap X^g} \right).$$

For any $g \in G$, put $X = {}^gL \cap K$. Then $X \leqslant N$ with $X \leqslant K$, and $L \cap X^g = L \cap K^g$, implying that $\operatorname{Ker} r^{L\cap K^g}_{L\cap X^g} = 0$. So $\operatorname{Ker} r^{\tilde{M}}_{\chi_N} = 0$.

For any $K \leqslant G$,

$$\operatorname{Im} t^{\tilde{M}}_{\chi_N}(K) = \sum_{X\leqslant N\colon X\leqslant K} \operatorname{Im} \tilde{t}^K_X = \bigoplus_{KgL\subseteq G} \sum_{X\leqslant N\colon X\leqslant K} \sum_{Xu(K\cap {}^gL)\subseteq K} \operatorname{Im} t^{L\cap K^{ug}}_{L\cap X^{ug}}.$$

As $u \in K$, $L \cap K^{ug} = L \cap K^g$. For any $g \in G$, putting $X = {}^gL \cap K$ and $u = 1$, we see that $X \leqslant N$ with $X \leqslant K$, and $L \cap X^{ug} = L \cap K^g$, implying that $\operatorname{Im} t_{L\cap X^{ug}}^{L\cap K^{ug}} = M(L \cap K^g)$. So $\operatorname{Im} t_{\chi_N}^{\tilde{M}} = \tilde{M}$. $\square$

**Lemma 5.11.**

(i) *Let $e$ be an idempotent in a ring $A$. If $W$ is an indecomposable $eAe$-module, and if $I$ is the sum of all $A$-submodules of $V = Ae \otimes_{eAe} W$ killed by $e$, then $V/I$ is an indecomposable $A$-module such that $e(V/I) \cong W$ as $eAe$-modules.*

(ii) *Let $M$ be a Mackey functor for $G$, and $N$ be a normal subgroup of $G$. If $M'$ is a $G$-subfunctor of $M$ killed by $1_N$ then $M' \leqslant \operatorname{Ker} r_{\chi_N}^M$.*

(iii) *Let $M$ be a Mackey functor for $G$, and $N$ be a normal subgroup of $G$. Assume that $\operatorname{Ker} r_{\chi_N}^M = 0$ and $\operatorname{Im} t_{\chi_N}^M = M$. If $M$ is indecomposable then $1_N M$ is an indecomposable $1_N \mu_R(G) 1_N$-module.*

**Proof.** (i) Suppose that $V/I = X \oplus Y$ as $A$-modules. Then

$$eX \oplus eY = e(V/I) = (eV + I)/I \cong eV/(eV \cap I) = eV/0 \cong eV \cong eAe \otimes_{eAe} W \cong W,$$

where we use $(eV \cap I) = e(eV \cap I) \subseteq eI = 0$ to see that $eV \cap I = 0$. Then, since $W$ is indecomposable, $eX = 0$ or $eY = 0$, say $eX = 0$. Now $X = \tilde{X}/I$ for some $A$-submodule $\tilde{X}$ of $V$ containing $I$. Then $eX = 0$ implies that $e\tilde{X} \subseteq I$, and so $e\tilde{X} = e^2\tilde{X} \subseteq eI = 0$. Thus $\tilde{X}$ is an $A$-submodule of $V$ killed by $e$ which means $\tilde{X} \subseteq I$ and $X = 0$.

(ii) Let $K \leqslant G$. Then for any $X \in \chi_N$ with $X \leqslant K$, since $M'$ is a subfunctor of $M$ killed by $1_N$ and $X \leqslant N$, $r_X^K(M'(K)) \subseteq M'(X) = 0$. Hence, $M' \leqslant \operatorname{Ker} r_{\chi_N}^M$.

(iii) Since $\operatorname{Ker} r_{\chi_N}^M = 0$, $\operatorname{Im} t_{\chi_N}^M = M$, and $M$ is indecomposable it follows by [27, Proposition 3.2] that $\downarrow_{\chi_N}^G M$ is an indecomposable Mackey functor for $\chi_N$. The result now follows by 5.5. $\square$

**Proposition 5.12.** *Let $N$ be a normal subgroup of $G$. Given a Mackey functor $S$ for $N$ over $R$, $\uparrow_N^G S$ is an indecomposable $\mu_R(G)$-module if and only if $1_N \uparrow_N^G S$ is an indecomposable $1_N \mu_R(G) 1_N$-module.*

**Proof.** Firstly, 5.10 implies that $\operatorname{Ker} r_{\chi_N}^{\tilde{S}} = 0$ and $\operatorname{Im} t_{\chi_N}^{\tilde{S}} = \tilde{S}$, where $\tilde{S} = \uparrow_N^G S$. Let $A = \mu_R(G)$ and $B = \mu_R(N)$. If $\tilde{S}$ is an indecomposable $A$-module then 5.11 implies that $1_N\tilde{S}$ is an indecomposable $1_N A 1_N$-module. Conversely, suppose that $1_N\tilde{S}$ is an indecomposable $1_N A 1_N$-module. Since $\operatorname{Ker} r_{\chi_N}^{\tilde{S}} = 0$, 5.11 implies that $\tilde{S}$ has no nonzero $A$-submodule killed by $1_N$. Moreover

$$\tilde{S} = \uparrow_N^G S \cong A1_N \otimes_B S \cong A1_N \otimes_{1_N A 1_N} (1_N A 1_N \otimes_B S) \cong A1_N \otimes_{1_N A 1_N} 1_N \tilde{S}.$$

Then by 5.11 $\tilde{S}$ is an indecomposable $A$-module. $\square$

For simple modules we have the following version of the previous result.

**Proposition 5.13.** *Let $N$ be a normal subgroup of $G$. Given a Mackey functor $S$ for $N$ over $R$, $\uparrow_N^G S$ is a simple (respectively semisimple) $\mu_R(G)$-module if and only if $1_N \uparrow_N^G S$ is a simple (respectively semisimple) $1_N\mu_R(G)1_N$-module.*

**Proof.** Let $\tilde{S} = \uparrow_N^G S$, $A = \mu_R(G)$ and $B = \mu_R(N)$. 5.10 implies that $\operatorname{Ker} r_{\chi_N}^{\tilde{S}} = 0$, and by 5.11 $\tilde{S}$ has no nonzero $A$-submodule killed by $1_N$. In particular $1_N\tilde{S}$ is nonzero.

Suppose $\tilde{S}$ is simple. Since $1_N\tilde{S}$ is nonzero, 5.6 implies that $1_N\tilde{S}$ is simple. Conversely, suppose $1_N\tilde{S}$ is simple. As in the proof of 5.12 we have $\tilde{S} \cong A1_N \otimes_B 1_N\tilde{S}$. Since $\tilde{S}$ has no nonzero $A$-submodule killed by $1_N$ and $1_N\tilde{S}$ is simple, it follows by 5.6 that $\tilde{S}$ is simple.

Because $\tilde{S} = \bigoplus_i S_i$ implies $1_N\tilde{S} = \bigoplus_i 1_N S_i$, and $A1_N \otimes_{1_N A 1_N} (\bigoplus_j P_j) = \bigoplus_j (A1_N \otimes_{1_N A 1_N} P_j)$ for $A$-modules $S_i$ and $1_N A 1_N$-modules $P_j$, it follows from what we have proved that $\uparrow_N^G S$ is semisimple if and only if $1_N \uparrow_N^G S$ is semisimple. $\square$

We now provide some necessary and sufficient conditions for simplicity of induced Mackey functors.

**Theorem 5.14.** *Let $R$ be commutative complete noetherian local ring whose residue field $R/\mathrm{J}(R)$ is algebraically closed and is of characteristic $p > 0$, and $N$ be a normal subgroup of $G$. Then:*

(i) *For any finitely generated nonzero Mackey functor $S$ for $N$ over $R$, $\uparrow_N^G S$ is semisimple if and only if $S$ is semisimple and, for any simple $N$-subfunctor $P$ of $S$, $p$ does not divide $|L : N|$, where $L$ is the inertia group of $P$.*

(ii) *For any nonzero Mackey functor $S$ for $N$ over $R$, $\uparrow_N^G S$ is simple if and only if $S$ is simple and ${}^gS \ncong S$ for all $g \in G - N$.*

**Proof.** We let $A = \mu_R(G)$, $B = 1_N\mu_R(G)1_N$ and $B_1 = \mu_R(N)$. So $B$ is a crossed product of $G/N$ over $B_1$ and $B_1$ is a finite-dimensional $R$-algebra:

(i) [18, Theorem 6.13, p. 525] implies that $B \otimes_{B_1} S$ is semisimple if and only if the desired conditions are satisfied. The result follows by 5.13 because $B \otimes_{B_1} S \cong 1_N \uparrow_N^G S$.

(ii) By [18, Theorem 6.14, p. 526] $B \otimes_{B_1} S$ is simple if and only if the conditions above hold. Again the result is immediate by the virtue of 5.13. $\square$

We next study the primitivity of the idempotents $t_K^K \in \mu_R(G)$ where $K \leqslant G$.

**Remark 5.15.** Let $N \trianglelefteq G$. Then $t_N^N\mu_R(G)t_N^N = \bigoplus_{gN\in G/N} A_{\bar{g}}$ is a crossed product of $G/N$ over $A_{\bar{1}} = \bigoplus_{J\leqslant_N N} Rt_J^N r_J^N$, where $A_{\bar{g}} = c_N^g A_{\bar{1}}$. Moreover, $A_{\bar{1}}$ is isomorphic to the Burnside algebra $B_R(N)$.

**Proof.** 2.1 implies that the elements $t_{{}^gJ}^N c_J^g r_J^N$ where $gN \subseteq G$, and $J$ is a subgroup of $N$ up to conjugacy, form, without repetition, a free $R$-basis of $t_N^N\mu_R(G)t_N^N$. It is obvious that $t_{{}^gJ}^N c_J^g r_J^N = c_N^g t_J^N r_J^N$. Thus we have the direct sum

$$t_N^N\mu_R(G)t_N^N = \bigoplus_{gN\subseteq G} c_N^g \biggl( \bigoplus_{J\leqslant_N N} Rt_J^N r_J^N \biggr).$$

Finally it is clear that the map

$$B_R(N) = \bigoplus_{J \leqslant_N N} R[N/J] \to \bigoplus_{J \leqslant_N N} Rt_J^N r_J^N \quad \text{given by } [N/J] \mapsto t_J^N r_J^N$$

is an $R$-algebra isomorphism, where $[N/J]$ denotes the isomorphism class of transitive $N$-sets with stabilizers $N$-conjugate to $J$. $\square$

The characterization of solvable groups given in [6] becomes

**Proposition 5.16.** *The idempotent $t_G^G \in \mu_{\mathbb{Z}}(G)$ is primitive if and only if $G$ is solvable.*

**Proof.** The idempotent $t_G^G \in \mu_{\mathbb{Z}}(G)$ is primitive if and only if $t_G^G \mu_{\mathbb{Z}}(G) t_G^G$ has no nontrivial idempotent, which is, by 5.15, equivalent to $B_{\mathbb{Z}}(G)$ has no nontrivial idempotent. Moreover, in [6] it is proven that $B_{\mathbb{Z}}(G)$ has no nontrivial idempotent if and only if $G$ is solvable, finishing the proof. $\square$

The following result contains a characterization of $p$-groups.

**Proposition 5.17.** *Let $G$ be nontrivial*:

(i) *The idempotent $t_1^1 \in \mu_{\mathbb{K}}(G)$ is primitive if and only if $\mathbb{K}$ is of characteristic $p > 0$ and $G$ is a $p$-group.*
(ii) *The idempotent $t_1^1 \in \mu_{\mathbb{Z}}(G)$ is primitive.*
(iii) *Let $H \leqslant G$. If the idempotent $t_H^H \in \mu_{\mathbb{K}}(G)$ is primitive then $\mathbb{K}$ is of characteristic $p > 0$ and $N_G(H)$ is a $p$-group.*

**Proof.** 5.15 shows that $t_1^1 \mu_R(G) t_1^1 \cong RG$ and $t_H^H \mu_R(H) t_H^H$ is a subalgebra of $t_H^H \mu_R(G) t_H^H$ isomorphic to the Burnside algebra $B_R(H)$:

(i) The idempotent $t_1^1$ is primitive if and only if $\mathbb{K}G$ has no nontrivial idempotent which is equivalent to $\mathbb{K}G$ is a local algebra. So the result is obtained.

(ii) It is clear because $\mathbb{Z}G$ has no nontrivial idempotent.

(iii) Suppose that $t_H^H$ is primitive. By (i) we may assume that $H \neq 1$. Then the Burnside algebra $B_{\mathbb{K}}(H)$ has no nontrivial idempotents. It follows by [6] that $\mathbb{K}$ is of characteristic $p > 0$ and $H$ is a $p$-group. Let $X$ be any subgroup of $N_G(H)$ containing $H$ and put $e = \sum_{gH \subseteq X} c_H^g$. Then $e$ is an element of $t_H^H \mu_{\mathbb{K}}(G) t_H^H$ such that $e^2 = |X : H| e$. Hence $N_G(H)/H$ must be $p$-group. $\square$

Let $\mathbb{K}$ be of characteristic $p > 0$. It is proved in [28, Theorem (19.2)] that $\mu_{\mathbb{K}}(G)$ is selfinjective if and only if $p^2$ does not divide $|G|$. Concerning the similar topics we have the following result.

**Remark 5.18.** Let $\mathbb{K}$ be of characteristic $p > 0$, $N$ be a normal subgroup of $G$. Then:

(i) Let $\mathbb{K}$ be algebraically closed, and $N$ be a $p'$-group. If $e$ is a $G$-invariant block idempotent of $\mu_{\mathbb{K}}(N)$, then $e\mu_{\mathbb{K}}(G)e$ is symmetric.
(ii) If $p^2$ does not divide $|N|$, then $t_N^N \mu_{\mathbb{K}}(G) t_N^N$ is Frobenius.
(iii) If $G$ is abelian and $p^2$ does not divide $|N|$, then $t_N^N \mu_{\mathbb{K}}(G) t_N^N$ is symmetric.

(iv) If $\mu_{\mathbb{K}}(N)$ is Frobenius then $1_N\mu_{\mathbb{K}}(G)1_N$ is Frobenius.
(v) If $N$ is a $p'$-group, then $t_N^N\mu_{\mathbb{K}}(G)t_N^N$ and $1_N\mu_{\mathbb{K}}(G)1_N$ are symmetric.
(vi) If $\mu_{\mathbb{K}}(G)$ is symmetric then $p^2$ does not divide $|G|$.

**Proof.** We begin by recalling some basic results. Semisimple algebras and group algebras are symmetric, tensor product of two symmetric algebras is again symmetric. If $A$ is symmetric and $e \in A$ is idempotent then $eAe$ is symmetric. Moreover, by [13], the Burnside algebra $B_{\mathbb{K}}(N)$ is symmetric if and only if $p^2$ does not divide $|N|$:

(i) By 5.4, $e\mu_{\mathbb{K}}(G)e = \mu_{\mathbb{K}}(N)e \otimes_{\mathbb{K}} \mathbb{K}\tilde{G}e_\lambda$ where $e_\lambda$ is a central idempotent of $\mathbb{K}\tilde{G}$. Since $\mu_{\mathbb{K}}(N)$ is semisimple, the basic results above imply that $e\mu_{\mathbb{K}}(G)e$ is symmetric.

(ii), (iii) and (iv) We know that $t_N^N\mu_{\mathbb{K}}(G)t_N^N$ and $1_N\mu_{\mathbb{K}}(G)1_N$ are crossed products of $G/N$ over $B_{\mathbb{K}}(N)$ and $\mu_{\mathbb{K}}(N)$, respectively. Moreover in (ii) and (iii) $B_{\mathbb{K}}(N)$ is symmetric. Then the results follows by [14, Lemma 1].

(v) If $N$ is a $p'$-group then $B_{\mathbb{K}}(N)$ and $\mu_{\mathbb{K}}(N)$ are both semisimple, and by [14, Proposition 2] the crossed products $t_N^N\mu_{\mathbb{K}}(G)t_N^N$ and $1_N\mu_{\mathbb{K}}(G)1_N$ must be symmetric.

(vi) If $\mu_{\mathbb{K}}(G)$ is symmetric then $t_G^G\mu_{\mathbb{K}}(G)t_G^G \cong B_{\mathbb{K}}(G)$ is symmetric, and the result follows by the result of [13] mentioned above. □

## 6. Indecomposable Mackey functors

In this section we prove some Clifford type results for indecomposable Mackey functors and a Mackey functor version of Green's indecomposibility criterion.

A Mackey functor $M$ for $G$ over $\mathbb{K}$ is $H$-projective for some $H \leqslant G$ if and only if $M$ is a direct summand of $\uparrow_H^G\downarrow_H^G M$, equivalently $M$ is a direct summand of $\uparrow_H^G P$ for some Mackey functor $P$ for $H$, see [25, Lemma 2.3].

If $M$ is $H$-projective then it follows that $\downarrow_H^G M \neq 0$, and so $1_H M \neq 0$.

We show that Clifford's theorem holds for $N$-projective indecomposable Mackey functors. To prove the last part of the following result, we use the corresponding result in crossed products which was first obtained in [26, Theorem 2].

**Theorem 6.1.** *Let $N$ be a normal subgroup of $G$, let $M$ be an $N$-projective finitely generated indecomposable Mackey functor for $G$ over $\mathbb{K}$ and let $S$ be an indecomposable direct summand of $\downarrow_N^G M$. Then*:

(i) *There is a positive integer $d$ such that*

$$\downarrow_N^G M \cong d \bigoplus_{gL\subseteq G} {}^gS,$$

*where $L$ is the inertia group of $S$.*

(ii) *$1_N M$ is an indecomposable $1_N\mu_{\mathbb{K}}(G)1_N$-module.*

(iii) *Let $\mathfrak{P}$ be the sum of all $N$-subfunctors of $\downarrow_N^G M$ isomorphic to $S$. Then $\mathfrak{P}$ is an indecomposable $1_N\mu_{\mathbb{K}}(L)1_N$-module such that*

$$1_N\mu_{\mathbb{K}}(G)1_N \otimes_{1_N\mu_{\mathbb{K}}(L)1_N} \mathfrak{P} \cong 1_N M \quad \text{and} \quad \mathfrak{P} \cong dS$$

*as $1_N\mu_{\mathbb{K}}(G)1_N$-modules and as $\mu_{\mathbb{K}}(N)$-modules, respectively.*

**Proof.** Let $A = 1_N \mu_{\mathbb{K}}(G) 1_N$ and $A_1 = \mu_{\mathbb{K}}(N)$, and $B = 1_N \mu_{\mathbb{K}}(L) 1_N$.

(i) $\downarrow_N^G M$ is nonzero because $M$ is a direct summand of $\uparrow_N^G \downarrow_N^G M$. Let $\downarrow_N^G M = S_1 \oplus \cdots \oplus S_n$ for some indecomposable Mackey functors $S_1, \ldots, S_n$ for $N$. As induction respects direct summands, $\uparrow_N^G \downarrow_N^G M = \uparrow_N^G S_1 \oplus \cdots \oplus \uparrow_N^G S_n$. Since $M$ is a direct summand of $\uparrow_N^G \downarrow_N^G M$, the functor $M$ is a direct summand of $\uparrow_N^G S_i$ for some $i \in \{1, \ldots, n\}$, say $i = 1$. Then, as restriction respects direct summands, $\downarrow_N^G M$ is a direct summand of $\downarrow_N^G \uparrow_N^G S_1$ which is, by the Mackey decomposition formula, 2.6, equal to $\bigoplus_{gN \subseteq G} {}^g S_1$. Therefore, each $S_i$ is conjugate to $S_1$.

As $\gamma_g$ is a unit of $\mu_{\mathbb{K}}(G)$, we have $\gamma_g M = M$. Moreover, $\gamma_g S_1$ is a $\mu_{\mathbb{K}}(N)$-module for any $g \in G$, because if $x \in \mu_{\mathbb{K}}(N)$ then $x \gamma_g S_1 = \gamma_g (\gamma_{g^{-1}} x \gamma_g) S_1$, and the normality of $N$ implies $\gamma_{g^{-1}} x \gamma_g \in \mu_{\mathbb{K}}(N)$. Also $\gamma_g S_1$ and ${}^g S_1$ are isomorphic $\mu_{\mathbb{K}}(N)$-modules via the isomorphism given by $\gamma_g s \leftrightarrow s$ for $s \in S_1$.

Now, $\downarrow_N^G M = \gamma_g \downarrow_N^G M = \gamma_g S_1 \oplus \cdots \oplus \gamma_g S_n$ for any $g \in G$. Hence every conjugate of $S_1$ is isomorphic to some $S_i$.

Finally, for a fixed $i \in \{1, \ldots, n\}$, let $S_i \cong {}^g S_1 \cong \gamma_g S_1$. Since $\downarrow_N^G M = S_1 \oplus \cdots \oplus S_n = \gamma_g S_1 \oplus \cdots \oplus \gamma_g S_n$, all the $S_i$ in $\downarrow_N^G M$ occurs with the same multiplicity $d$.

(ii) We know that $1_N M$ is nonzero and $\downarrow_N^G M = 1_N M \cong d \oplus_{gL \subseteq G} {}^g S$. If $1_N M = X \oplus Y$ is a direct sum decomposition as $A$-modules then $S$ is a direct summand of $X$ or $Y$, say $X$, considered as $A_1$-modules. $\gamma_h 1_N$ is a unit in $A$ for any $h \in G$, and so $\gamma_h X = \gamma_h 1_N X = X$ implying that $\gamma_h S \cong {}^h S$ is a direct summand of $X$. Hence $X = 1_N M$ and $1_N M$ is an indecomposable $A$-module.

(iii) We saw above that $M$ is a direct summand of $\uparrow_N^G S = \mu_{\mathbb{K}}(G) 1_N \otimes_{A_1} S$ implying that $1_N M$ is a direct summand of $A \otimes_{A_1} S$. Thus $1_N M$ is $A_1$-projective.

Now, by 5.2 $A$ is a crossed product of $G/N$ over $A_1$. Then Clifford theory for indecomposable modules in the context of crossed products, see [26, Theorem 2], implies that if $\mathfrak{P}$ is the sum of all $A_1$-submodules of $\downarrow_N^G M$ isomorphic to $S$ then $\mathfrak{P}$ is an indecomposable $B$-module such that $\mathfrak{P} \cong dS$ and $A \otimes_{A_1} \mathfrak{P} \cong 1_N M$. $\square$

Given a normal subgroup $N$ of $G$ and a finitely generated indecomposable Mackey functor $S$ for $N$ over $\mathbb{K}$, in the next result we relate direct decompositions of the induced functors $\uparrow_N^G S$ and $\uparrow_N^L S$ where $L$ is the inertia group of $S$. The following result is a Mackey functor version of [15, Theorem 9.6, p. 126].

**Theorem 6.2.** *Let $N \trianglelefteq G$, and $S$ be a finitely generated indecomposable Mackey functor for $N$ over $\mathbb{K}$ whose inertia group is $L$. Let $\uparrow_N^L S = P_1 \oplus \cdots \oplus P_k$ where the $P_i$ are indecomposable Mackey functors for $L$. Then*:

(i) *$1_N P_i$ is an indecomposable $1_N \mu_{\mathbb{K}}(L) 1_N$-module for any $i \in \{1, \ldots, k\}$.*
(ii) *For any $i \in \{1, \ldots, k\}$, there is a positive integer $n_i$ such that $\downarrow_N^L P_i \cong n_i S$. Moreover, $\sum_i^k n_i = |L : N|$.*
(iii) *$\uparrow_N^G S \cong \uparrow_L^G P_1 \oplus \cdots \oplus \uparrow_L^G P_k$, and the $\uparrow_L^G P_i$ are indecomposable. Furthermore, each $1_N \uparrow_L^G P_i$ is indecomposable $1_N \mu_{\mathbb{K}}(G) 1_N$-module.*
(iv) *$\uparrow_L^G P_i \cong \uparrow_L^G P_j$ if and only if $P_i \cong P_j$.*
(v) *If $P_i$ is simple then $\uparrow_L^G P_i$ and $S$ are simple.*

**Proof.** (i) Each $P_i$ is an indecomposable and $N$-projective $\mu_{\mathbb{K}}(L)$-module. Then 6.1 implies the desired result.

(ii) Since restriction respects direct summands, by the Mackey decomposition formula, 2.6,

$$\downarrow_N^L P_1 \oplus \cdots \oplus \downarrow_N^L P_k = \downarrow_N^L \uparrow_N^L S \cong |L:N| S.$$

Also 6.1 implies that $\downarrow_N^L P_i \cong n_i S$ for some positive integer $n_i$. Hence $\sum_i^k n_i = |L:N|$.

(iii) Firstly by using the Mackey decomposition formula we get

$$\downarrow_N^G \uparrow_L^G P_i \cong n_i \bigoplus_{gL \subseteq G} {}^g S.$$

Let $\uparrow_L^G P_i = X_1 \oplus \cdots \oplus X_m$ as indecomposable $\mu_{\mathbb{K}}(G)$-modules. For any $j \in \{1, \dots, m\}$, since $P_i$ is a direct summand of $\uparrow_N^L S$ we see that $X_j$ is a direct summand of $\uparrow_N^G S$ implying that $X_j$ is $N$-projective. Then, by 6.1, $1_N X_j$ is an indecomposable $1_N \mu_{\mathbb{K}}(G) 1_N$-module and

$$\downarrow_N^G X_j \cong m_{i,j} \bigoplus_{gL \subseteq G} {}^g S$$

for some positive integer $m_{i,j}$ satisfying $m_{i,j} \leqslant n_i$.

By 4.2 $P_i$ is a direct summand of $\downarrow_L^G \uparrow_L^G P_i$ implying that $\downarrow_N^L P_i$ is a direct summand of $\downarrow_N^G X_n$ for some $n \in \{1, \dots, m\}$. As

$$\downarrow_N^L P_i \cong n_i S \quad \text{and} \quad \downarrow_N^G X_n \cong m_{i,n} \bigoplus_{gL \subseteq G} {}^g S$$

we must have $n_i \leqslant m_{i,n}$. Hence, $m_{i,n} = n_i$ which means that $1_N \uparrow_L^G P_i = 1_N X_n$. Moreover,

$$1_N \uparrow_L^G P_i = 1_N X_1 \oplus \cdots \oplus 1_N X_m$$

is a direct sum into indecomposable $1_N \mu_{\mathbb{K}}(G) 1_N$-modules and $1_N X_j \neq 0$ for all $j \in \{1, \dots, m\}$. Consequently, $m = n = 1$ and $\uparrow_L^G P_i = X_n$ is indecomposable $\mu_{\mathbb{K}}(G)$-module, and $1_N \uparrow_L^G P_i$ is indecomposable $1_N \mu_{\mathbb{K}}(G) 1_N$-module.

(iv) Suppose $\uparrow_L^G P_i \cong \uparrow_L^G P_j$. We observed in 4.2 that for any Mackey functor $M$ for $H$, $H \leqslant G$, $M$ is a direct summand of $\downarrow_H^G \uparrow_H^G M$. Now, if $P_i$ and $P_j$ are not isomorphic then we can write

$$\downarrow_L^G \uparrow_L^G P_i \cong P_i \oplus P_j \oplus Y \quad \text{for some } \mu_{\mathbb{K}}(L)\text{-module } Y.$$

Then considering this isomorphism as $\mu_{\mathbb{K}}(N)$-modules we get

$$n_i \bigoplus_{gL \subseteq G} {}^g S \cong n_i S \oplus n_j S \oplus \downarrow_N^L Y$$

from which it follows that $n_i \geqslant n_i + n_j$, which is a contradiction.

(v) If $P_i$ is simple then by 3.10

$$\downarrow_N^L P_i \cong d \bigoplus_{gL' \subseteq L} {}^g S' \cong n_i S$$

for some simple $\mu_{\mathbb{K}}(N)$-module $S'$ and positive integer $d$. In particular, both of the decompositions are into indecomposable $\mu_{\mathbb{K}}(N)$-modules, and so the unique decomposition property implies $S \cong S'$ is simple. Finally, the Clifford correspondence 4.4 implies that $\uparrow_L^G P_i$ is simple. $\square$

The following result contains a Mackey functor version of Green's indecomposibility theorem.

**Theorem 6.3.** *Let $R$ be a commutative complete noetherian local ring whose residue field $R/\mathrm{J}(R)$ is algebraically closed and is of characteristic $p > 0$, and $N$ be a normal subgroup of $G$. Let $S$ be a finitely generated indecomposable Mackey functor for $N$ over $R$, and let $L$ be the inertia group of $S$. Then, $\uparrow_N^G S$ is an indecomposable Mackey functor for $G$ over $R$ if and only if $L/N$ is a $p$-group.*

**Proof.** Let $A = 1_N \mu_R(G) 1_N$ and $A_1 = \mu_R(N)$. Then we know that $A$ is a crossed product of $G/N$ over $A_1$, and in the context of crossed products $S$ is an indecomposable $A_1$-module whose inertia group is $L/N$. Then [18, Corollary 6.12, p. 524] implies that $A \otimes_{A_1} S$ is indecomposable if and only if $(L/N)/(N/N)$ is a $p$-group. Moreover, $A \otimes_{A_1} S = 1_N \uparrow_N^G S$. Now, by 5.12, $1_N \uparrow_N^G S$ is indecomposable if and only if $\uparrow_N^G S$ is indecomposable. Hence the result is proved. $\square$

If we do not assume $R/\mathrm{J}(R)$ is algebraically closed in the previous result, we get

**Theorem 6.4.** *Let $R$ be a commutative complete noetherian semilocal ring such that $R/\mathrm{J}(R)$ is of prime characteristic $p > 0$, and $N$ be a normal subgroup of $G$. Let $S$ be a finitely generated indecomposable Mackey functor for $N$ over $R$, and let $L$ be the inertia group of $S$ and $L/N$ be a $p$-group. Then, there is an indecomposable Mackey functor $M$ for $G$ over $R$ such that $\uparrow_N^G S \cong dM$ for some positive integer $d$.*

**Proof.** The corresponding result in crossed products, see [18, Theorem 6.38, p. 546], shows that $1_N \uparrow_N^G S \cong dM'$ for some indecomposable $1_N \mu_R(G) 1_N = A$-module $M'$ and positive integer $d$. Then $d(\mu_R(G)1_N \otimes_A M') \cong \mu_R(G)1_N \otimes_A 1_N \uparrow_N^G S \cong \uparrow_N^G S$. We put $M = \mu_R(G)1_N \otimes_A M'$. By 5.10 and 5.11 $\uparrow_N^G S$ has no nonzero submodule killed by $1_N$, and so the same is true for $M$ because $dM \cong \uparrow_N^G S$. Then, since $M'$ is indecomposable, by 5.11 $M$ must be indecomposable. $\square$

## 7. Extension of Mackey functors

In this final section we provide some results on extending Mackey functors. That is, given a $G$-invariant Mackey functor $S$ for $N$ where $N$ is a normal subgroup of $G$ we give some conditions on $S$ and $G$ to guarantee the existence of Mackey functor $M$ for $G$ satisfying $\downarrow_N^G M \cong S$.

The following result collects some consequences of extension theorems of [4,5].

**Theorem 7.1.** *Let $N \trianglelefteq G$, and $S$ be a nonzero $G$-invariant Mackey functor for $N$ over $\mathbb{K}$. Then, each of conditions* (i)–(iii) *below implies that the $\mu_{\mathbb{K}}(N)$-module $S$ extends to a $1_N \mu_{\mathbb{K}}(G) 1_N$-module, and there is a Mackey functor $M$ for $G$ over $\mathbb{K}$ such that $\downarrow_N^G M \cong S$. Moreover, both of conditions* (ii) *and* (iii) *imply that $M$ can be taken to be simple*:

(i) *The automorphism group* $\mathrm{Aut}_{\mu_{\mathbb{K}}(N)}(S)$ *is an abelian group uniquely divisible by* $|G/N|$, *that is, the map* $f \mapsto f^{|G/N|}$ *is a group automorphism of* $\mathrm{Aut}_{\mu_{\mathbb{K}}(N)}(S)$.
(ii) $\mathbb{K}$ *is a perfect field of characteristic* $p > 0$, $G/N$ *is a* $p$*-group, and* $S$ *is absolutely simple.*
(iii) $\mathbb{K}$ *is algebraically closed,* $S$ *is simple, and at least one of the following three hold*:
  (1) *all Sylow subgroups of* $G/N$ *are cyclic,*
  (2) $G/N$ *is generalized quaternion or semidihedral,*
  (3) $\mathbb{K}$ *is of characteristic* $p > 0$ *and* $G/N$ *is a cyclic extension of a* $p$*-group.*

**Proof.** $1_N\mu_{\mathbb{K}}(G)1_N$ is a strongly $G/N$-graded algebra. Then, the results in [18, pp. 610–612] applied to the above strongly $G/N$-graded algebra gives in each case a $1_N\mu_{\mathbb{K}}(G)1_N$-module $\tilde{S}$ which is an extension of the $\mu_{\mathbb{K}}(N)$-module $S$. So $\tilde{S} = S$ as sets and as $\mu_{\mathbb{K}}(N)$-modules. We put $M = \mu_{\mathbb{K}}(G)1_N \otimes_{1_N\mu_{\mathbb{K}}(G)1_N} \tilde{S}$. Then we obviously have $\downarrow^G_{\chi_N} M = 1_N M \cong \tilde{S}$. Thus, $\downarrow^G_N M = \downarrow^{\chi_N}_N \downarrow^G_{\chi_N} M \cong \downarrow^{\chi_N}_N \tilde{S} = S$, as desired. Finally, the last assertion follows by 5.6. □

The next result is a consequence of the extension results concerning indecomposable modules for group graded algebras.

**Theorem 7.2.** *Let* $R$ *be a commutative complete noetherian local ring whose residue field* $R/\mathrm{J}(R)$ *is algebraically closed and is of characteristic coprime to* $|G : N|$, *where* $N$ *is a normal subgroup of* $G$. *Assume that* $S$ *is finitely generated* $G$*-invariant indecomposable Mackey functor for* $N$ *over* $R$. *Then, both of conditions* (i) *and* (ii) *below imply that there is an indecomposable Mackey functor* $M$ *for* $G$ *over* $R$ *such that* $\downarrow^G_N M \cong S$:

(i) *All Sylow subgroups of* $G/N$ *are cyclic.*
(ii) $G/N$ *is generalized quaternion or semidihedral.*

**Proof.** Let $A = 1_N\mu_R(G)1_N$ and $A_1 = \mu_R(N)$. Then $A$ is a crossed product of $G/N$ over $A_1$, and so in each case [18, Theorem 2.10, p. 612] implies the existence of an $A$-module $M'$ such that $S = M'$ as sets and as $A_1$-modules. Since $A_1$ is a unital subring of $A$ and our modules are unitary, it is obvious that $M'$ is an indecomposable $A$-module. We let $M'' = \mu_R(G)1_N \otimes_A M'$ and $M = M''/I$ where $I$ is the sum of all $\mu_R(G)$-submodules of $M''$ killed by $1_N$. Then the result follows by 5.11. □

Using the classification of simple Mackey functors we get the following immediate consequence of the extension results of modules of group algebras.

**Theorem 7.3.** *Let* $N \trianglelefteq G$, *and* $S = S^N_{H,W}$ *be a* $G$*-invariant simple Mackey functor for* $N$ *over* $\mathbb{K}$. *Then, each of conditions* (i)–(iv) *below implies that there is a simple Mackey functor* $M$ *for* $G$ *over* $\mathbb{K}$ *such that* $\downarrow^G_N M \cong S$:

(i) $\mathbb{K}$ *is algebraically closed and* $G/N$ *is cyclic.*
(ii) $\mathbb{K}$ *is algebraically closed and is of characteristic* 0, *and* $(|G : N|, |N|/\dim_{\mathbb{K}} W) = 1$.
(iii) $\mathbb{K}$ *is of characteristic* $p > 0$, $N$ *is a Hall subgroup of* $G$, *and either* $N$ *is* $p$*-solvable or* $(|G : N|, \dim_{\mathbb{K}} W) = 1$.
(iv) $\mathbb{K}$ *is of characteristic* 0 *and* $N$ *is a nilpotent Hall subgroup of* $G$.

**Proof.** For any simple Mackey functor $S$ for $N$, by 3.10, it follows that $L = NT$ and $N_L(H) = T$ where $H$ is a minimal subgroup of $S$, $L$ is the inertia group of $S$ in $G$, and $\bar{T}$ is the inertia group of the simple $\mathbb{K}\bar{N}_N(H)$-module $S(H)$ in $\bar{N}_G(H)$. So, in our case $S = S^N_{H,W}$ and $L = G$, implying that the inertia group of the simple $\mathbb{K}\bar{N}_N(H)$-module $W$ in $\bar{N}_G(H)$ is $\bar{N}_G(H)$. Moreover, we have the group isomorphism $G/N = (NN_G(H))/N \cong N_G(H)/N_N(H)$.

Suppose (i) holds. Then $\bar{N}_G(H)/\bar{N}_N(H)$ is cyclic and [9, Theorem 2.14, p. 102] implies that we can find a simple $\mathbb{K}\bar{N}_G(H)$-module $V$ such that $\downarrow^{\bar{N}_G(H)}_{\bar{N}_N(H)} V = W$.

Suppose (ii) holds. Since $N_N(H) \leqslant N$ and $G/N \cong N_G(H)/N_N(H)$ it follows that $(|\bar{N}_G(H) : \bar{N}_N(H)|, |\bar{N}_N(H)|/\dim_{\mathbb{K}} W) = 1$. Then by [10] there is a simple $\mathbb{K}\bar{N}_G(H)$-module $V$ such that $\downarrow^{\bar{N}_G(H)}_{\bar{N}_N(H)} V = W$.

Suppose (iii) holds. As $N_N(H) \leqslant N$ and $G/N \cong N_G(H)/N_N(H)$, $\bar{N}_N(H)$ is a normal Hall subgroup of $\bar{N}_G(H)$, and if $(|G : N|, \dim_{\mathbb{K}} W) = 1$ then $(|\bar{N}_G(H) : \bar{N}_N(H)|, \dim_{\mathbb{K}} W) = 1$. Moreover, if $N$ is $p$-solvable then $\bar{N}_N(H)$ is $p$-solvable. Consequently [8] implies that there is a simple $\mathbb{K}\bar{N}_G(H)$-module $V$ such that $\downarrow^{\bar{N}_G(H)}_{\bar{N}_N(H)} V = W$.

Finally suppose that (iv) holds. Then $\bar{N}_N(H)$ is a normal nilpotent Hall subgroup of $\bar{N}_G(H)$ and [16] implies the existence of a simple $\mathbb{K}\bar{N}_G(H)$-module $V$ such that $\downarrow^{\bar{N}_G(H)}_{\bar{N}_N(H)} V = W$.

We saw that each condition implies the existence of a simple $\mathbb{K}\bar{N}_G(H)$-module $V$ satisfying $\downarrow^{\bar{N}_G(H)}_{\bar{N}_N(H)} V = W$. We put $M = S^G_{H,V}$. Then $M$ is a simple Mackey functor for $G$.

Moreover, 2.4 implies that for any $K \leqslant N$

$$\big(\downarrow^G_N M\big)(K) = \sum_{g\in G:\ {}^gH\leqslant K} t^K_{{}^gH} c^g_H(V).$$

We now, using $G = NN_G(H)$, see that if $g = nu \in NN_G(H)$ with $n \in N, u \in N_G(H)$ then ${}^gH \leqslant K$ is equivalent to ${}^nH \leqslant K$. Also $c^g_H(V) = c^n_H c^u_H(V) = c^n_H(V)$ implying

$$\big(\downarrow^G_N M\big)(K) = \sum_{g\in G:\ {}^gH\leqslant K} t^K_{{}^gH} c^g_H(V) = \sum_{n\in N:\ {}^nH\leqslant K} t^K_{{}^nH} c^n_H(V) = \sum_{n\in N:\ {}^nH\leqslant K} t^K_{{}^nH} c^n_H(W),$$

where we use $\downarrow^{\bar{N}_G(H)}_{\bar{N}_N(H)} V = W$ for the last equality. On the other hand, by 2.4,

$$S^N_{H,W}(K) = \sum_{n\in N:\ {}^nH\leqslant K} t^K_{{}^nH} c^n_H(W)$$

for any $K \leqslant N$. Hence we proved that each condition implies $\downarrow^G_N M = S^N_{H,W}$. □

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