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Generalized Burnside rings and group cohomology

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Abstract

We define the cohomological Burnside ring $B^n(G,M)$ of a finite group G with coefficients in a $\mathbb{Z}G$-module M as the Grothendieck ring of the isomorphism classes of pairs $[X,u]$ where X is a G-set and u is a cohomology class in a cohomology group $H^n_X(G,M)$. The cohomology groups $H^*_X(G,M)$ are defined in such a way that $H^*_X(G,M)\cong\bigoplus_i H^*(H_i,M)$ when X is the disjoint union of transitive G-sets G/H_i. If A is an abelian group with trivial action, then $B^1(G,A)$ is the same as the monomial Burnside ring over A, and when M is taken as a G-monoid, then $B^0(G,M)$ is equal to the crossed Burnside ring $B^c(G,M)$. We discuss the generalizations of the ghost ring and the mark homomorphism and prove the fundamental theorem for cohomological Burnside rings. We also give an interpretation of $B^2(G,M)$ in terms of twisted group rings when $M=k^\times$ is the unit group of a commutative ring.

Keywords: Cohomology of groups; Monomial G-sets; Generalized Burnside rings

1. Introduction

Let G be a finite group and X be a finite G-set. Given a $\mathbb{Z}G$-module M, we define $H^*_X(G,M)$, the cohomology of G associated to X with coefficients in M, as the cohomology of a cochain complex, where the n-cochains are the maps $f:G^n\times X\to M$ and derivations are given by

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$$\begin{aligned}(\delta f)(g_0,\ldots,g_n;x) = {} & g_0\cdot f(g_1,\ldots,g_n;x)\\ & - f(g_0g_1,\ldots,g_n;x)\\ & \ldots\\ & + (-1)^n f(g_0,\ldots,g_{n-1}g_n;x)\\ & + (-1)^{n+1} f(g_0,\ldots,g_{n-1};g_nx).\end{aligned}$$

The cohomology group $H^n_X(G,M)$ can be described in terms of the usual group cohomology of subgroups of G. In particular, when X is the disjoint union of transitive G-sets G/H_i for $i=1,\ldots,k$, then we have

$$H^n_X(G,M)\cong\bigoplus_{i=1}^{k}H^n(H_i,M).$$

Given a G-set map $f:X\to Y$, we define $f^*:H^n_Y(G,M)\to H^n_X(G,M)$ and $f_*:H^n_X(G,M)\to H^n_Y(G,M)$ on the chain level in such a way that the assignment $X\to H^n_X(G,M)$ together with $(\ )^*$ and $(\ )_*$ defines a Mackey functor in the sense described in [10]. In Section 3, we show that this Mackey functor is naturally equivalent to the cohomology of groups Mackey functor $H^n(?,M)$.

The motivation for this definition comes from a classification problem for monomial G-sets, where the cocycles defined above appear in a natural way. This example also motivates the definition of cohomological Burnside rings. Recall that the Burnside ring $B(G)$ of a finite group G is defined as the Grothendieck ring of the isomorphism classes of G-sets, with addition given by disjoint unions and multiplication by cartesian products. We generalize this definition as follows:

A *positioned G-set* is a pair of the form (X,u), where X is a G-set and u is a class in $H^n_X(G,M)$. A map $f:(X,u)\to(Y,v)$ is called a positioned G-set map if $f:X\to Y$ is a G-set map such that $f^*(v)=u$. We say that two positioned G-sets (X,u) and (Y,v) are isomorphic if there is a positioned G-set map $f:(X,u)\to(Y,v)$ such that $f:X\to Y$ is an isomorphism as a G-set map. We denote the isomorphism class of a positioned G-set (X,u) simply by $[X,u]$. The set of isomorphism classes of positioned G-sets is a semi-ring with addition and multiplication defined by

$$[X,u]+[Y,v]=[X\amalg Y,u\oplus v],$$
$$[X,u]\cdot[Y,v]=[X\times Y,u\otimes v],$$

where $u\oplus v\in H^n_{X\amalg Y}(G,M)$ and $u\otimes v\in H^n_{X\times Y}(G,M)$ are defined in the following way: for $u\in H^n_X(G,M)$ and $v\in H^n_Y(G,M)$,

$$u\oplus v=(i_X)_*(u)+(i_Y)_*(v)$$

where $i_X:X\to X\amalg Y$ and $i_Y:Y\to X\amalg Y$ are the usual inclusion maps of X and Y, and

$$u\otimes v=(\pi_X)^*(u)+(\pi_Y)^*(v)$$

where $\pi_X:X\times Y\to X$ and $\pi_Y:X\times Y\to Y$ are the projection maps.

The *cohomological Burnside ring* $B^n(G,M)$ of degree n of the group G with coefficients in M is defined as the Grothendieck ring of this semi-ring. This is a generalization of earlier constructions of generalized Burnside rings such as the crossed Burnside ring and the monomial Burnside ring. Indeed, if we take $n=0$ and if we extend our definition of the zero-dimensional cohomological Burnside ring in a suitable way to include the case where M is a G-monoid, then $B^0(G,M)$ becomes isomorphic to the crossed Burnside ring $B^c(G,M)$ described by Oda and Yoshida in [7,8] (see also Bouc [3]). Also, in the case $n=1$, if we take M as an abelian group A with trivial G-action, then the cohomological Burnside ring $B^1(G,A)$ coincides with the monomial Burnside ring over A defined by Dress [5] (see also Boltje [2] and Barker [1]).

This is all proved in Section 5. We also give a $\mathbb{Z}$-basis for $B^n(G,M)$ and prove that

$$B^n(G,M) \cong \bigoplus_{[H]\in Cl(G)} \big(\mathbb{Z}H^n(H,M)\big)_{W_G(H)},$$

where $Cl(G)$ denotes the set of conjugacy classes of subgroups of G and $W_G(H)=N_G(H)/H$ is the Weyl group of H in G. Motivated by this description, we define the ghost ring of the cohomological Burnside ring as

$$\beta^n(G,M) \cong \bigoplus_{[H]\in Cl(G)} \big(\mathbb{Z}H^n(H,M)\big)^{W_G(H)}$$

and describe the mark homomorphism explicitly.

Section 7 is devoted to the proof of the fundamental theorem for cohomological Burnside rings. Here we use a different approach than the earlier results for monomial Burnside rings and crossed Burnside rings. Instead of choosing a basis of the ghost ring as an abelian group, we use the direct sum decomposition coming from the conjugacy classes of subgroups and express the mark homomorphism as a matrix of homomorphisms instead of a matrix of scalars. In other words, we replace the classical table of marks for Burnside rings with a table of marks where each mark is a homomorphism. We calculate the cokernel of the mark homomorphism as the direct sum over conjugacy classes of subgroups where each summand is the 0th Tate cohomology of the Weyl group $W_G(H)=N_G(H)/H$ with coefficients in the $\mathbb{Z}W_G(H)$-module $\mathbb{Z}H^n(H,M)$. We state the fundamental theorem for cohomological Burnside rings in the following form.

Theorem 1.1 *(Fundamental theorem). Let G be a finite group, let M be a $\mathbb{Z}G$-module, and let n be a non-negative integer. Then the following sequence of abelian groups is exact*

$$0 \to B^n(G,M) \xrightarrow{\varphi} \beta^n(G,M) \xrightarrow{\psi} \mathrm{Obs}^n(G,M) \to 0$$

with

$$\mathrm{Obs}^n(G,M) = \bigoplus_{[H]\in Cl(G)} \widehat{H}^0\big(W_G(H), \mathbb{Z}H^n(H,M)\big),$$

where φ is the mark homomorphism, and the $[K]$th component of ψ is defined by

$$\psi_{[K]}(f) = \sum_{K \leqslant L} \mu(K, L)\, \mathrm{res}^L_K\, f(L)$$

modulo the image of the trace map $\mathrm{Tr}_1^{W_G(K)}$.

It is easy to calculate the Tate group $\widehat{H}^0(W_G(H), \mathbb{Z}H^n(H, M))$ for each $[H] \in Cl(G)$ using an appropriate basis for $\mathbb{Z}H^n(H, M)$. We obtain

Proposition 1.2. *We have*

$$\mathrm{Obs}^n(G, M) \cong \bigoplus_{[G/H,u]} \mathbb{Z}/\big|N_G(H, u) : H\big|\mathbb{Z}$$

where the sum is taken over the isomorphism classes of positioned G-sets with transitive G-sets.

Thus, our obstruction groups for $n = 0$ and $n = 1$ are the same as the ones given earlier for crossed Burnside rings and monomial Burnside rings, respectively. This is all done in Section 7.

Finally, in Section 8 we give an interpretation of $B^2(G, M)$ in terms of twisted group rings when $M = k^\times$ is the unit group of a commutative ring.

2. The definition of $H^*_X(G, M)$

Let G be a finite group, and let X be a finite G-set. For a given $\mathbb{Z}G$-module M, we define the cochain complex $(C^*_X(G, M), \delta)$ as follows: The n-cochains are the functions

$$f : G^n \times X \to M$$

and the coboundary $\delta : C^n_X(G, M) \to C^{n+1}_X(G, M)$ is defined by

$$\begin{aligned}(\delta f)(g_0, \dots, g_n; x) = {} & g_0 \cdot f(g_1, \dots, g_n; x) \\ & - f(g_0 g_1, \dots, g_n; x) \\ & \dots \\ & + (-1)^n f(g_0, \dots, g_{n-1} g_n; x) \\ & + (-1)^{n+1} f(g_0, \dots, g_{n-1}; g_n x).\end{aligned}$$

It is easy to check that $\delta^2 = 0$, so we define

$$H^*_X(G, M) := H^*\big(C^*_X(G, M), \delta\big),$$

and call it the *cohomology of G associated to X with coefficients in M*. Note that the cohomology of G associated to the trivial G-set G/G is just the usual cohomology of the group G.

Given a 0-cochain $f : X \to M$, we see that $(\delta f)(g; x) = gf(x) - f(gx)$. So, f is a 0-cocycle if and only if f is a G-map. Thus, $H^0_X(G, M) = \mathrm{Map}_G(X, M)$. Note that a 1-cochain $f : G \times X \to M$ is a cocycle (or a derivation) if and only if

$$f(g_0 g_1; x) = g_0 f(g_1; x) + f(g_0; g_1 x) \tag{1}$$

for every $g_0, g_1 \in G$ and $x \in X$. We say that $f : G \times X \to M$ is a *trivial derivation* (or a inner derivation) if $f = \delta t$ for some function $t : X \to M$.

The motivation for this definition comes from the classification problem for monomial G-sets. A monomial G-set with coefficients in A is an A-free $A \times G$-set Γ with Γ/A isomorphic to a given G-set X. As a set, Γ is isomorphic to $A \times X$, where the action can be described as

$$(a, g)(a', x) = \bigl(a + a' + \alpha(g, x), gx\bigr) \tag{2}$$

with $\alpha : G \times X \to M$ being a derivation in the above sense. In fact, Eq. (1) holds for α if and only if Eq. (2) defines an action. We study this problem in detail in Section 4.

In the rest of the section we will prove the following:

Theorem 2.1. *Suppose that G is a finite group, X is a G-set, and M is a $\mathbb{Z}G$-module. Then, we have*

$$H^*_X(G, M) \cong H^*\bigl(G, \mathrm{Map}(X, M)\bigr),$$

where $\mathrm{Map}(X, M)$ *is the abelian group of functions $f : X \to M$ considered as a (left) $\mathbb{Z}G$-module with the action given by $(gf)(x) = gf(g^{-1}x)$.*

We will prove Theorem 2.1 using the Hochschild cohomology of $\mathbb{Z}G$. Given a $(\mathbb{Z}G, \mathbb{Z}G)$-bimodule B, the Hochschild cohomology $HH^*(\mathbb{Z}G, B)$ is defined as the cohomology of the cochain complex

$$C^n(\mathbb{Z}G, B) = \mathrm{Hom}_{\mathbb{Z}}\bigl((\mathbb{Z}G)^{\otimes n}, B\bigr)$$

with coboundary

$$\begin{aligned}(\delta f)(a_0, \dots, a_n) = {} & a_0 \cdot f(a_1, \dots, a_n) \\ & - f(a_0 a_1, \dots, a_n) \\ & \cdots \\ & + (-1)^n f(a_0, \dots, a_{n-1} a_n) \\ & + (-1)^{n+1} f(a_0, \dots, a_{n-1}) \cdot a_n.\end{aligned}$$

Proposition 2.2. *Suppose that G is a finite group, X is a G-set, and M is a $\mathbb{Z}G$-module. Then, we have*

$$H^*_X(G, M) \cong HH^*\bigl(\mathbb{Z}G, \mathrm{Map}(X, M)\bigr)$$

where $\mathrm{Map}(X, M)$ *is the abelian group of functions $f : X \to M$ considered as a $(\mathbb{Z}G, \mathbb{Z}G)$-bimodule with the $\mathbb{Z}G$-action given by*

$$(g_1 \cdot f \cdot g_2)(x) = g_1 f(g_2 x).$$

Proof. Consider the map

$$\Phi^n : C^n_X(G, M) \to C^n\big(\mathbb{Z}G, \operatorname{Map}(X, M)\big)$$

defined for all $n \geqslant 0$ by the formula $\Phi^n : f \to \bar{f}$ where

$$\bar{f}(g_0, \dots, g_{n-1})(x) = f(g_0, \dots, g_{n-1}; x).$$

Note that

$$\begin{aligned}\big[\bar{f}(g_0, \dots, g_{n-1}) \cdot g_n\big](x) &= \bar{f}(g_0, \dots, g_{n-1})(g_n x)\\ &= f(g_0, \dots, g_{n-1}; g_n x).\end{aligned}$$

So, we have $\overline{\delta f} = \delta \bar{f}$, i.e., Φ is a cochain map. Note that the obvious inverse is also a cochain map, hence Φ induces an isomorphism on cohomology. □

We recall the following fact about Hochschild cohomology:

Lemma 2.3. *Let B be a $(\mathbb{Z}G, \mathbb{Z}G)$-bimodule. Then the Hochschild cohomology $HH^*(\mathbb{Z}G, B)$ is isomorphic to the usual group cohomology $H^*(G, L(B))$ where $L(B) = B$ is the (left) $\mathbb{Z}G$-module with the action given by $g \cdot b = gbg^{-1}$.*

Proof. See Theorem 5.5 on page 292 in Mac Lane [6]. □

Theorem 2.1 follows now from Proposition 2.2 and Lemma 2.3.

Remark 2.4. Note that the isomorphism in Lemma 2.3 is induced by the chain isomorphism $\Psi^n : C^n(\mathbb{Z}G, B) \to C^n(G, L(B))$ defined by $f \to \tilde{f}$ where

$$\tilde{f}(g_1, \dots, g_n) = f(g_1, \dots, g_n) g_n^{-1} \cdots g_1^{-1}.$$

So, the isomorphism given in Theorem 2.1 is induced by the chain map

$$\Psi^n \circ \Phi^n : C^n_X(G, M) \to C^n\big(G, \operatorname{Map}(X, M)\big)$$

defined by $f \to \bar{f}$ where

$$(\bar{f})(g_1, \dots, g_n)(x) = f\big(g_1, \dots, g_n; g_n^{-1} \cdots g_1^{-1} x\big).$$

One can also prove Theorem 2.1 directly using this chain map.

Note that as a $\mathbb{Z}G$-module

$$\operatorname{Map}(X, M) \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}X, M)$$

where $\mathbb{Z}X$ denotes the permutation module with basis given by X. When X is equal to $G/H = \{gH \mid g \in G\}$, then we have

$$\mathrm{Map}(G/H, M) \cong \mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z}[G/H], M\big)$$

as $\mathbb{Z}G$-modules.

Lemma 2.5. *Let G be a finite group, $H \leqslant G$ a subgroup, and M a $\mathbb{Z}G$-module. Then there is an isomorphism of $\mathbb{Z}G$-modules*

$$\sigma : \mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z}[G/H], M\big) \to \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$$

defined by $\sigma : f \to \bar{f}$ where $\bar{f}(g) = gf(g^{-1}H)$.

Proof. It is easy to check that $f \to \bar{f}$ is a $\mathbb{Z}G$-module homomorphism and its inverse can be defined by $f \to \tilde{f}$ where $\tilde{f}(gH) = gf(g^{-1})$. Note that the (left) G-action on $\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$ is given by $[g \cdot f](g') = f(g'g)$. □

We conclude this section with the following corollary.

Corollary 2.6. *Suppose that G is a finite group, $H \leqslant G$ a subgroup, and M a $\mathbb{Z}G$-module. Then, we have*

$$H^*_{G/H}(G, M) \cong H^*(H, M).$$

Proof. By Theorem 2.1 and Lemma 2.5, we have

$$H^n_{G/H}(G, M) \cong H^n\big(G, \mathrm{Map}(G/H, M)\big) \cong H^n\big(G, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)\big).$$

Using the Eckmann–Shapiro isomorphism

$$H^n\big(G, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)\big) \cong H^n(H, M)$$

we conclude that $H^n_{G/H}(G, M) \cong H^n(H, M)$. □

3. Functorial properties of $H^*_X(G, M)$

In this section we will be listing the basic functorial properties of the cohomology of groups associated to a G-set. We will see later that the assignment $X \mapsto H^n_X(G, M)$ is just the description of the cohomology of groups Mackey functor for G as a bifunctor from G-sets to abelian groups. But note that all our constructions are done explicitly on the chain level.

Throughout this section we assume that G is a fixed finite group and M is a fixed $\mathbb{Z}G$-module. We also fix a non-negative integer n, and consider the cohomology groups $H^n_X(G, M)$ of degree n for various G-sets.

Let X and Y be two G-sets and let $f: X \to Y$ be a G-set map. There are two ways to obtain maps between $C_X^n(G, M)$ and $C_Y^n(G, M)$ associated to f. Given $\gamma : G^n \times Y \to M$, we define $f^*(\gamma): G^n \times X \to M$ by

$$f^*(\gamma)(g_1, \dots, g_n; x) = \gamma\bigl(g_1, \dots, g_n; f(x)\bigr)$$

for $g_1, \dots, g_n \in G$ and $x \in X$. Since f is a G-map, this defines a chain map $f^*: C_Y^*(G, M) \to C_X^*(G, M)$, hence it induces a group homomorphism

$$f^*: H_Y^n(G, M) \to H_X^n(G, M).$$

For the other direction, for $\alpha : G^n \times X \to M$ we define $f_*(\alpha): G^n \times Y \to M$ by

$$f_*(\alpha)(g_1, \dots, g_n; y) = \sum_{x \in f^{-1}(y)} \alpha(g_1, \dots, g_n; x)$$

for $g_1, \dots, g_n \in G$ and $y \in Y$. It is easy to check that this defines a chain map $f_*: C_X^*(G, M) \to C_Y^*(G, M)$, hence a group homomorphism

$$f_*: H_X^n(G, M) \to H_Y^n(G, M).$$

Lemma 3.1. *For every pullback diagram of G-sets*

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "f_2"] & X_2 \arrow[d, "f_3"] \\ X_3 \arrow[r, "f_4"] & X_4 \end{tikzcd}$$

we have $(f_4)^*(f_3)_* = (f_2)_*(f_1)^*$ as chain maps $C_{X_2}^*(G, M) \to C_{X_3}^*(G, M)$ and hence as group homomorphisms $H_{X_2}^n(G, M) \to H_{X_3}^n(G, M)$ for each $n \geqslant 0$.

Proof. Let $\alpha : G^n \times X_2 \to M$ be a cocycle. For $g_1, \dots, g_n \in G$ and $x \in X_3$, we have

$$(f_4)^*(f_3)_*(\alpha)(g_1, \dots, g_n; x) = \sum_{z \in X_2,\ f_3(z) = f_4(x)} \alpha(g_1, \dots, g_n; z)$$

whereas

$$(f_2)_*(f_1)^*(\alpha)(g_1, \dots, g_n; x) = \sum_{y \in X_1,\ f_2(y) = x} \alpha\bigl(g_1, \dots, g_n; f_1(y)\bigr).$$

The equality of these sums follows from the pullback condition. □

We also note that $H_X^n(G, M)$ decomposes as an abelian group if X decomposes as a G-set. We state this fact as follows:

Lemma 3.2. *For every pair of finite G-sets X and Y and inclusion maps $X \xrightarrow{i_X} X \amalg Y$ and $Y \xrightarrow{i_Y} X \amalg Y$, the chain map*

$$\Phi : C^*_X(G, M) \oplus C^*_Y(G, M) \to C_{X\amalg Y}(G, M)$$

defined by $\Phi(u, v) = (i_X)_*(u) + (i_Y)_*(v)$ is a chain isomorphism and hence induces a group isomorphism

$$\Phi : H^n_X(G, M) \oplus H^n_Y(G, M) \to H^n_{X\amalg Y}(G, M)$$

for all $n \geqslant 0$. In particular, if X is the disjoint union of transitive G-sets G/H_i for $i = 1, \ldots, k$, then

$$H^n_X(G, M) \cong \bigoplus_{i=1}^{k} H^n(H_i, M).$$

Proof. It is easy to check that the chain map

$$C^*_{X\amalg Y}(G, M) \to C^*_X(G, M) \oplus C^*_Y(G, M)$$

defined by $w \to ((i_X)^*(w), (i_Y)^*(w))$ is the inverse of Φ. $\square$

The above two lemmas show that the assignment $X \to C^*_X(G, M)$ together with $(\,)^*$ and $(\,)_*$ defines a Mackey functor in the category of chain complexes in the sense described on page 5 of Webb [10]. As a consequence of this, or directly from the Lemmas 3.1 and 3.2, we see that the assignment $X \to H^n_X(G, M)$ together with the corresponding induced maps $(\,)^*$ and $(\,)_*$ defines a Mackey functor (of abelian groups) for each n. Let us denote this Mackey functor by $H^n_?(G, M)$. We will show later that the Mackey functor $H^n_?(G, M)$ is equivalent to the cohomology of groups Mackey functor $H^n(?, M)$ via the isomorphism

$$H^n_{G/H}(G, M) \cong H^n(H, M)$$

given in Corollary 2.6.

For a Mackey functor defined as a functor from G-sets to abelian groups, there is a standard way to obtain restriction, induction, and conjugation maps. If we apply these definitions, we obtain the following maps:

Let $K \leqslant H \leqslant G$ and $g \in G$, consider the G-maps $f_{H,K} : G/K \to G/H$ defined by $xK \to xH$ and $f_{H,g} : G/H \to G/{}^gH$ defined by $xH \to xg^{-1}\,{}^gH$. Then, the induced maps

$$\mathrm{r}_{H,K} : H^n_{G/H}(G, M) \xrightarrow{(f_{H,K})^*} H^n_{G/K}(G, M),$$
$$\mathrm{i}_{H,K} : H^n_{G/K}(G, M) \xrightarrow{(f_{H,K})_*} H^n_{G/H}(G, M),$$
$$c_{H,g} : H^n_{G/H}(G, M) \xrightarrow{(f_{H,g})_*} H^n_{G/{}^gH}(G, M)$$

are the restriction, induction, and conjugation maps for $H^n_?(G, M)$. By the equivalence of the different definitions for Mackey functors, we can also consider $H^n_?(G, M)$ as a Mackey functor $H \to H^n_{G/H}(G, M)$ with the above restriction, induction, and conjugation maps.

Theorem 3.3. *The Mackey functor $H^n_?(G,M)$ is equivalent to the cohomology of groups Mackey functor $H^n(?,M)$.*

Proof. For every $H \leqslant G$, we have an isomorphism

$$H^n_{G/H}(G,M) \cong H^n(H,M)$$

by Corollary 2.6. We just need to show that this isomorphism commutes with restriction, induction, and conjugation maps. This follows from the following lemma. $\square$

Lemma 3.4. *Let $K \leqslant H \leqslant G$, and $g \in G$. Then, the induced maps*

$$\mathrm{res}^H_K : H^n(H,M) \cong H^n_{G/H}(G,M) \xrightarrow{\mathrm{r}_{H,K}} H^n_{G/K}(G,M) \cong H^n(K,M),$$
$$\mathrm{tr}^H_K : H^n(K,M) \cong H^n_{G/K}(G,M) \xrightarrow{\mathrm{i}_{H,K}} H^n_{G/H}(G,M) \cong H^n(H,M),$$
$$c^H_g : H^n(H,M) \cong H^n_{G/H}(G,M) \xrightarrow{c_{H,g}} H^n_{G/{}^gH}(G,M) \cong H^n\bigl({}^gH,M\bigr)$$

are the usual restriction, transfer, and conjugation maps in group cohomology.

Proof. First let us consider the restriction map. We have

$$\begin{tikzcd}
H^n_{G/H}(G,M) \arrow[r, "f^*"] \arrow[d, "\cong"', "\text{Theorem 2.1}"] & H^n_{G/K}(G,M) \arrow[d, "\cong"', "\text{Theorem 2.1}"] \\
H^n\bigl(G,\mathrm{Map}(G/H,M)\bigr) \arrow[r, "f^*"] \arrow[d, "\cong"', "\sigma_*"] & H^n\bigl(G,\mathrm{Map}(G/K,M)\bigr) \arrow[d, "\cong"', "\sigma_*"] \\
H^n\bigl(G,\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G,M)\bigr) \arrow[r, "\mathrm{res}^H_K"] \arrow[d, "\cong"', "\text{Eckmann–Shapiro}"] & H^n\bigl(G,\mathrm{Hom}_{\mathbb{Z}K}(\mathbb{Z}G,M)\bigr) \arrow[d, "\cong"', "\text{Eckmann–Shapiro}"] \\
H^n(H,M) \arrow[r, "\mathrm{res}^H_K"] & H^n(K,M)
\end{tikzcd}$$

where the vertical composition is the isomorphism given in Corollary 2.6, and res^H_K is the homomorphism induced from

$$\mathrm{res}^H_K : \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G,M) \to \mathrm{Hom}_{\mathbb{Z}K}(\mathbb{Z}G,M)$$

which is defined by mapping a $\mathbb{Z}H$-homomorphism $f:\mathbb{Z}G \to M$ to itself considered as a $\mathbb{Z}K$-homomorphism. The first diagram commutes for obvious reasons, and the third diagram commutes by standard results in homological algebra. One can show that the second diagram commutes by showing that the corresponding diagram for modules

$$\begin{tikzcd} \mathrm{Map}(G/H, M) \arrow[r, "f^*"] \arrow[d, "\sigma", "\cong"'] & \mathrm{Map}(G/K, M) \arrow[d, "\sigma", "\cong"'] \\ \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) \arrow[r, "\mathrm{res}^H_K"] & \mathrm{Hom}_{\mathbb{Z}K}(\mathbb{Z}G, M) \end{tikzcd}$$

commutes. This can be done by a direct calculation as follows: Let $\varphi : G/H \to M$ be a function. Then,

$$[\sigma f^*](\varphi)(x) = xf^*(\varphi)\big(x^{-1}K\big) = x\varphi\big(x^{-1}H\big) = \big[\mathrm{res}^H_K \circ \sigma\big](\varphi)(x).$$

This completes the proof of Lemma 3.4 for the restriction map.

For transfer and conjugation, the arguments are similar. For each of these we will replace the second commuting diagram with an appropriate one and show that they commute on the module level. For the transfer map, we need to check the commutativity of the following diagram:

$$\begin{tikzcd} \mathrm{Map}(G/K, M) \arrow[r, "f_*"] \arrow[d, "\sigma", "\cong"'] & \mathrm{Map}(G/H, M) \arrow[d, "\sigma", "\cong"'] \\ \mathrm{Hom}_{\mathbb{Z}K}(\mathbb{Z}G, M) \arrow[r, "\mathrm{tr}^H_K"] & \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) \end{tikzcd}$$

where $\mathrm{tr}^H_K(\psi)(g) = \sum_{hK \in H/K} h\psi(h^{-1}g)$ for every $\mathbb{Z}H$-module $\psi : \mathbb{Z}G \to M$. For every $\varphi : G/K \to M$, we have

$$\big[\mathrm{tr}^H_K \sigma(\varphi)\big](g) = \sum_{hK \in H/K} h\sigma(\varphi)\big(h^{-1}g\big) = \sum_{hK \in H/K} g\varphi\big(g^{-1}hK\big) = \big[\sigma f_*(\varphi)\big](g),$$

so the diagram commutes.

For conjugation, we have the following diagram:

$$\begin{tikzcd} \mathrm{Map}(G/H, M) \arrow[r, "f_*"] \arrow[d, "\sigma", "\cong"'] & \mathrm{Map}\big(G/{}^gH, M\big) \arrow[d, "\sigma", "\cong"'] \\ \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) \arrow[r, "c^H_g"] & \mathrm{Hom}_{\mathbb{Z}{}^gH}(\mathbb{Z}G, M) \end{tikzcd}$$

where $c^H_g(\psi)(x) = g\psi(g^{-1}x)$ for every $\mathbb{Z}H$-module $\psi : \mathbb{Z}G \to M$. Take φ in $\mathrm{Map}(G/H, M)$. We have

$$\big[c^H_g \sigma\big](\varphi)(x) = g\sigma(\varphi)\big(g^{-1}x\big) = gg^{-1}x\varphi\big(x^{-1}gH\big) = x\varphi\big(x^{-1}gH\big)$$

and

$$\big[\sigma(f_{H,g})_*\big](\varphi)(x) = x(f_{H,g})_*(\varphi)\big(x^{-1}\,{}^{g}H\big) = x\varphi\big(x^{-1}gH\big).$$

So, the diagram commutes. This completes the proof of Lemma 3.4. □

The cohomology of groups functor $H^n(?, M)$ for a group G is known to be a cohomological functor, i.e., it satisfies $\mathrm{tr}_K^H \mathrm{res}_K^H u = |H : K|u$ for every $K \leqslant H \leqslant G$ and $u \in H^n(H, M)$. By the above equivalence, the functor $H^n_?(G, M)$ is also cohomological. One can also see this directly from the definitions of $\mathrm{i}_{H,K}$ and $\mathrm{r}_{H,K}$.

In the next section, we will consider the classification problem for A-free $A \times G$-sets where A is an abelian group. The groups $H^1_X(G, A)$ will appear naturally in this classification.

4. *A*-free *A* × *G*-sets

Let G be a finite group and A be an abelian group. Throughout this section we assume that G acts trivially on A, although most of the results in this section still hold for a non-trivial $\mathbb{Z}G$-module M when the group $A \times G$ is replaced by the semi-direct product $M \times_\varphi G$. We do not state our results in this generality since the case of trivial G-action is sufficient for all the applications we know.

Given an $A \times G$-set Γ with a free A-action, let $X = \Gamma/A$ denote the quotient of Γ by the A-action. This gives a map $\pi : \Gamma \to X$ with fibers isomorphic to A and base space X which is a G-set. Note also that π is a G-map. Any map $\pi : \Gamma \to X$ which is obtained in this way is called a *fibration with fibre group A*.

Observe that there is always a bijection $\Gamma \cong A \times X$, but in general it is not an isomorphism of $A \times G$-sets, where $A \times X$ is considered as an $A \times G$-set by the product action

$$(a, g)(a', x) = (a + a', gx).$$

In other words, there is always a set theoretical splitting $s : X \to \Gamma$, but s is not a map of $A \times G$-sets in general when X is considered as an $A \times G$-set through the projection $A \times G \to G$.

Our first result in this section classifies the fibrations $\pi : \Gamma \to X$ over a fixed G-set X up to isomorphism. We say the fibrations $\pi_1 : \Gamma_1 \to X$ and $\pi_2 : \Gamma_2 \to X$ are isomorphic if there is an $A \times G$-map $F : \Gamma_1 \to \Gamma_2$ such that $\pi_1 = \pi_2 F$.

Proposition 4.1. *Suppose G is a finite group, X is a G-set, and A is an abelian group with trivial G-action. There is a one-to-one correspondence between isomorphism classes of fibrations $\Gamma \to X$ with fibre group A and the cohomology classes in $H^1_X(G, A)$.*

Proof. We will first show that given a fibration $\pi : \Gamma \to X$, there is a unique cohomology class in $H^1_X(G, A)$ associated to it. Let $s : X \to \Gamma$ be a set theoretical section. We define $\alpha : G \times X \to A$ by

$$(0, g)s(x) = \alpha(g, x)s(gx).$$

The identity

$$\big((0, g_1)(0, g_2)\big) \cdot \gamma = (0, g_1) \cdot \big((0, g_2) \cdot \gamma\big)$$

gives the derivation condition

$$\alpha(g_1g_2;x)=\alpha(g_2;x)+\alpha(g_1;g_2x).$$

So, $\alpha: G\times X\to A$ is a 1-cocycle of the chain complex $(C^*_X(G,A),\delta)$ described earlier.

Let s_1 and s_2 be two different splittings. Since $\pi s_1=\pi s_2$, there exists a function $t: X\to A$, such that $s_2(x)=t(x)\cdot s_1(x)$ for all $x\in X$. Let α_1 and α_2 be derivations associated to the sections s_1 and s_2, respectively. Then, an easy calculation shows that

$$\alpha_2(g;x)-\alpha_1(g;x)=t(gx)-t(x)=(\delta t)(g;x).$$

So, the cohomology class $[\alpha]\in H^*_X(G,A)$ does not depend on the choice of the section.

Conversely, given a cohomology class $[\alpha]\in H^1_X(G,A)$ represented by a derivation $\alpha: G\times X\to A$, we define the $A\times G$-set Γ as the set $A\times X$ with the action given by

$$(a,g)\cdot(a',x)=\bigl(a+a'+\alpha(g,x),gx\bigr)$$

for $a,a'\in A$, $g\in G$ and $x\in X$. It is easy to verify that this defines an action by using the fact that α is a derivation. Note that if we choose another representative, $\alpha'=\delta t+\alpha$, then we obtain $\Gamma'=A\times X$ with the action given by

$$(a,g)\cdot(a',x)=\bigl(a+a'+\alpha'(g;x),gx\bigr)=\bigl(a+a'+\alpha(g;x)+t(x)-t(gx),gx\bigr).$$

Then, the map $F:\Gamma\to\Gamma'$ defined by $(a,x)\to(a+t(x),x)$ is an isomorphism of fibrations.

We also need to show that two isomorphic fibrations give the same cohomology class. For this, observe that we can fix splittings for both $\pi_1:\Gamma_1\to X$ and $\pi_2:\Gamma_2\to X$ and assume that $\Gamma_i=A\times X$ with action given by

$$(a,g)\cdot(a',x)=\bigl(a+a'+\alpha_i(g;x),gx\bigr)$$

for $i=1,2$. Let $F:\Gamma_1\to\Gamma_2$ be a isomorphism of fibrations, then there exists a function $t: X\to A$ such that $F(a,x)=(a+t(x),x)$ for all $x\in X$. Note that F is a $A\times G$-set map if and only if

$$\alpha_1(g;x)-\alpha_2(g;x)=t(x)-t(gx)=(\delta t)(g;x).$$

Thus, π_1 and π_2 are assigned to the same cohomology class when they are isomorphic as fibrations with fiber group A.

We have seen that there are well defined maps between isomorphism classes of fibrations $\pi:\Gamma\to X$ with fibre group A and cohomology classes in $H^1_X(G,A)$. From the way these maps were constructed it is easy to see that they are inverse to each other. $\square$

Next, we will classify all A-free $A\times G$-sets up to isomorphism. Let Γ be an A-free $A\times G$-set, then taking the orbit space of the A-action as before, we obtain a G-set X and a fibration $\pi:\Gamma\to X$ with fibers isomorphic to A. By fixing a set theoretical splitting, we can assume $\Gamma=A\times G$ and the action is given by

$$(a,g)\cdot(a',x)=\bigl(a+a'+\alpha(g;x),gx\bigr)$$

for some derivation $\alpha : G \times X \to A$. So, associated to an A-free $A \times G$-set Γ, there is a G-set X and a cohomology class $u = [\alpha] \in H^1_X(G, A)$.

Proposition 4.2. *Let Γ_1 and Γ_2 be two A-free $A \times G$-sets with corresponding G-sets X_1 and X_2 and cohomology classes $u_1 \in H^1_{X_1}(G, A)$ and $u_2 \in H^1_{X_2}(G, A)$. Then, Γ_1 and Γ_2 are isomorphic as $A \times G$-sets if and only if there is a G-set isomorphism $f : X_1 \to X_2$ such that $f^*(u_2) = u_1$.*

Proof. Let $F : \Gamma_1 \to \Gamma_2$ be an $A \times G$-set isomorphism. Passing to the orbit spaces, we obtain a G-set isomorphism $f : X_1 \to X_2$ such that the diagram

$$\begin{tikzcd} \Gamma_1 \arrow[r, "F"] \arrow[d, "\pi_1"] & \Gamma_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[r, "f"] & X_2 \end{tikzcd}$$

commutes. Choosing set theoretical splittings for both Γ_1 and Γ_2, we can assume that $\Gamma_i = A \times G$ with the actions given by

$$(a, g) \cdot (a', x) = \big(a + a' + \alpha_i(g; x), gx\big)$$

for some derivations $\alpha_i : G \times X_i \to A$ for $i = 1, 2$. Since $F : \Gamma_1 \to \Gamma_2$ is an A-map, we can write $F(a, x) = (a + t(x), f(x))$. Now, using the fact that F is an $A \times G$-map, we obtain

$$\alpha_1(g; x) - \alpha_2\big(g; f(x)\big) = t(x) - t(gx) = (\delta t)(g; x)$$

for $g \in G$ and $x \in X$. Thus $f^*(u_2) = u_1$ as desired.

Conversely, given a G-set isomorphism $f : X_1 \to X_2$ such that $f^*(u_2) = u_1$, we can pick representative derivations α_1 and α_2 such that $f^*(\alpha_2) = \alpha_1$, and assume that Γ_i is an $A \times G$-set with action determined by α_i for $i = 1, 2$. Then, the map $F(a, x) = (a, f(x))$ defines a $A \times G$-set isomorphism. $\square$

Let X be a G-set, and denote by $\mathrm{Aut}_G(X)$ the group of all G-set isomorphisms $f : X \to X$. For each $f \in \mathrm{Aut}_G(X)$, we have an isomorphism $f^* : H^1_X(G, A) \to H^1_X(G, A)$. So, $\mathrm{Aut}_G(X)$ acts on $H^1_X(G, A)$ as group automorphisms. We have the following:

Corollary 4.3. *Suppose that G is a finite group and A is an abelian group with trivial G-action. Then, the isomorphism classes of A-free $A \times G$-sets are in one-to-one correspondence with the pairs $([X], [\alpha])$ where $[X]$ runs through the isomorphism classes of G-sets and $[\alpha]$ is a representative of a class in $H^1_X(G, A)$ under the $\mathrm{Aut}_G(X)$-action.*

Proof. By Proposition 4.2, the isomorphism classes of A-free $A \times G$-sets are in one-to-one correspondence with the equivalence classes of pairs (X, u), where we say that (X, u) is equivalent to (Y, v) if there is a G-set isomorphism $f : X \to Y$ such that $f^*(v) = u$. Note that the set of equivalence classes of pairs (X, u) is the same as the set given in the corollary. $\square$

The category of A-free $A \times G$-sets admits two operations, called direct sum and direct product over A, which induce well defined addition and multiplication operations on isomorphism classes of A-free $A \times G$-sets. The Grothendieck ring of isomorphism classes of A-free $A \times G$-sets with these operations is called the monomial Burnside ring over A. Below we briefly describe how direct sum and direct product over A are defined. For more details on these operations and on the monomial Burnside ring, we refer the reader to Dress [5], Boltje [2], and Barker [1].

We assume that all the A-free $A \times G$-sets we consider have a fixed splitting, and hence they are sets of the form $A \times X$ with $A \times G$-action given by

$$(a, g) \cdot (a', x) = \bigl(a + a' + \alpha(g; x), gx\bigr)$$

for some derivation $\alpha : G \times X \to A$. We denote such an A-free $A \times G$-set briefly by $A_\alpha X$. Addition and multiplication of two A-free $A \times G$-sets $A_\alpha X$ and $A_\beta Y$ are defined as follows:

$$A_\alpha X \oplus_A A_\beta Y = A_\alpha X \amalg A_\beta Y$$

and

$$A_\alpha X \otimes_A A_\beta Y = A_\alpha X \times A_\beta Y/\sim$$

where the equivalence relation $\sim$ is defined by declaring

$$(a\zeta, \eta) \sim (\zeta, a\eta)$$

for all $a \in A$, $\zeta \in A_\alpha X$, and $\eta \in A_\beta Y$. The $A \times G$-action in the first case is defined in the obvious way, and in the second case by diagonal action. One can easily show that these give well defined addition and multiplication on the isomorphism classes of A-free $A \times G$-sets.

To see the effect of these operations on the derivations, observe that

$$A_\alpha X \oplus_A A_\beta Y = A_\theta(X \amalg Y)$$

and

$$A_\alpha X \otimes_A A_\beta Y = A_\gamma(X \times Y)$$

for some $\theta : G \times (X \amalg Y) \to A$ and $\gamma : G \times (X \times Y) \to A$. One can describe θ and γ in terms of α and β as follows:

$$\theta(g; z) = \begin{cases} \alpha(g; z) & \text{if } z \in X, \\ \beta(g; z) & \text{if } z \in Y \end{cases}$$

and

$$\gamma\bigl(g; (x, y)\bigr) = \alpha(g; x) + \beta(g; y).$$

In fact, one can verify that these define direct sum and tensor product on the one-dimensional cohomology classes. We do not give details of this here, since this will be done in the next section in greater generality.

Motivated by this example, in the next section we define cohomological Burnside rings $B^n(G, M)$ for each $n \geqslant 0$. In the case $n = 1$ and M is an abelian group A with trivial G-action, we obtain that $B^1(G, A)$ is equal to the monomial Burnside ring over A.

5. The cohomological Burnside ring

Throughout this section G is a finite group, M is a $\mathbb{Z}G$-module, and n is a non-negative integer.

Definition 5.1. A pair of the form (X, u), where X is a G-set and u is a class in $H^n_X(G, M)$, is called a positioned G-set of degree n with coefficients in M, or shortly a positioned G-set, when degree and coefficients are well understood. A map $f:(X,u) \to (Y,v)$ is called a positioned G-set map if $f: X \to Y$ is a G-set map such that $f^*(v) = u$.

We say that two positioned G-sets (X, u) and (Y, v) are isomorphic if there is a positioned G-set map $f:(X,u) \to (Y,v)$ such that $f: X \to Y$ is an isomorphism of G-sets. We denote the isomorphism class of a positioned G-set (X, u) simply by $[X, u]$.

The set of isomorphism classes of positioned G-sets is a semi-ring with addition and multiplication defined as follows: Given two positioned G-sets (X, u) and (Y, v), we define the cohomology class

$$u \oplus v \in H^n_{X \amalg Y}(G, M) \quad \text{by} \quad u \oplus v = (i_X)_*(u) + (i_Y)_*(v)$$

where $i_X : X \to X \amalg Y$ and $i_Y : Y \to X \amalg Y$ are the usual inclusion maps of X and Y. Note that if $u = [\alpha]$ and $v = [\gamma]$, then $u \oplus v = [\theta]$ where

$$\theta(g_1, \dots, g_n; z) = \begin{cases} \alpha(g_1, \dots, g_n; z) & \text{if } z \in X, \\ \gamma(g_1, \dots, g_n; z) & \text{if } z \in Y. \end{cases}$$

If $f_X : (X, u) \to (X', u')$ and $f_Y : (Y, v) \to (Y', v')$ are two positioned G-set isomorphisms, then

$$f_X \amalg f_Y : (X \amalg Y, u \oplus v) \to (X' \amalg Y', u' \oplus v')$$

is a positioned G-set isomorphism. To see this consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "f_X"'] & X \amalg Y \arrow[d, "f_X \amalg f_Y"'] & Y \arrow[l, "i_Y"'] \arrow[d, "f_Y"'] \\ X' \arrow[r, "i_{X'}"] & X' \amalg Y' & Y' \arrow[l, "i_{Y'}"'] \end{tikzcd}$$

where both of the diagrams are pullback diagrams. By Lemma 3.1, we have

$$\begin{aligned} (f_X \amalg f_Y)^*(u' \oplus v') &= (f_X \amalg f_Y)^*\big((i_{X'})_*(u') + (i_{Y'})_*(v')\big) \\ &= (i_X)_*(f_X)^*(u') + (i_Y)_*(f_Y)^*(v') \\ &= (i_X)_*(u) + (i_Y)_*(v) \\ &= u \oplus v. \end{aligned}$$

This shows that

$$[X, u] + [Y, v] = [X \amalg Y, u \oplus v]$$

gives a well defined addition on isomorphism classes.

To define the product of two positioned G-sets (X, u) and (Y, v), we first define the cohomology class

$$u \otimes v \in H^n_{X \times Y}(G, M) \quad \text{by} \quad u \otimes v = (\pi_X)^*(u) + (\pi_Y)^*(v)$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the projection maps. We can describe $u \otimes v$ on the chain level as follows: Let $u = [\alpha]$ and $v = [\gamma]$, then $u \otimes v = [\theta]$ where

$$\theta\big(g_1, \dots, g_n; (x, y)\big) = \alpha(g_1, \dots, g_n; x) + \gamma(g_1, \dots, g_n; y).$$

If $f_X : (X, u) \to (X', u')$ and $f_Y : (Y, v) \to (Y', v')$ are two positioned G-set isomorphisms, then

$$f_X \times f_Y : (X \times Y, u \otimes v) \to (X' \times Y', u' \otimes v')$$

is a positioned G-set isomorphism. One can verify this by forming appropriate commuting diagrams as in the case of addition. It follows that the formula

$$[X, u] \cdot [Y, v] = [X \times Y, u \otimes v]$$

defines a multiplication on isomorphism classes of positioned G-sets.

Definition 5.2. The cohomological Burnside ring $B^n(G, M)$ of degree n of the group G with coefficients in M is defined as the Grothendieck ring of the semi-ring of isomorphism classes of positioned G-sets (of degree n with coefficients in M) where addition and multiplication are defined by

$$[X, u] + [Y, v] = [X \amalg Y, u \oplus v],$$
$$[X, u] \cdot [Y, v] = [X \times Y, u \otimes v].$$

We have the following proposition, which is immediate from the discussion in Section 4.

Proposition 5.3. *If A is an abelian group with trivial G-action, then $B^1(G, A)$ is the same as the monomial Burnside ring over A.*

Note that for $n = 0$, the definition given in Definition 5.2 can be extended to non-abelian coefficients, even to a G-monoid. For this, first note that for a G-module M, the group $H^0_X(G, M)$ is the kernel of the first differential $\delta : C^0_X(G, M) \to C^1_X(G, M)$. So, an element in $H^0_X(G, M)$ is a map $f : X \to M$ such that

$$\delta(f)(g) = gf(x) - f(gx) = 0.$$

Thus, $H^0_X(G, M)$ can be identified with the group of G-maps from X to M.

Now, let M be a G-monoid, i.e., M is a semi-group with a unit element $1 \in M$ such that G-acts on M as monoid automorphisms. This means that there is a map $G \times M \to M$ denoted by $(g, m) \to {}^g m$ which satisfies

$${}^{gh}m = {}^g\big({}^h m\big), \qquad {}^1 m = m; \qquad {}^g(mn) = {}^g m \cdot {}^g n, \qquad {}^g 1 = 1$$

for $m, n \in M$, $g, h \in G$. We define $H^0_X(G, M)$ as the set of maps $f : X \to M$ such that ${}^g f(x) = f(gx)$. Such a map is usually called a weight function and a G-set X together with a weight function is called a crossed G-set (over M). The definition of isomorphisms for positioned G-sets (X, f) coincides with the notion of isomorphisms for crossed G-sets (see page 34 of [7]). Direct sums and tensor products of crossed G-sets are defined in the same way as we defined them for positioned G-sets. The Grothendieck ring of crossed G-sets is called the crossed Burnside ring and denoted by $B^c(G, M)$. So, we can conclude the following:

Proposition 5.4. *If M is a G-monoid, then $B^0(G, M)$ is isomorphic to the crossed Burnside ring $B^c(G, M)$.*

More details about crossed Burnside rings can be found in Bouc [3] and Yoshida–Oda [7]. For the construction of the Mackey functor $B^c(-, M)$ through the Dress construction see Bouc [4] and Yoshida–Oda [8].

Later in Section 8, we will give an interpretation of $B^2(G, M)$ in terms of twisted group rings when $M = k^\times$ is the unit group of a commutative ring k.

In the rest of this section, we calculate the rank of $B^n(G, M)$ as a free abelian group. We have the following.

Lemma 5.5. *Let $Cl(G)$ denote the set of conjugacy classes of subgroups of G, and let $\mathrm{Aut}_G(X)$ denote the group of G-set isomorphisms of X. For each $H \leqslant G$, let $\mathcal{U}_H(G, M)$ denote a set of orbit representatives for the elements in $H^n_{G/H}(G, M)$ under the $\mathrm{Aut}_G(G/H)$-action. Then,*

$$\mathcal{B} = \big\{[G/H, u] \bigm| H \in Cl(G),\ u \in \mathcal{U}_H(G, M)\big\}$$

is a basis for $B^n(G, M)$.

Proof. Given a positioned G-set (X, u) such that $X = X_1 \amalg X_2$, then there exist $u_i \in H^n_{X_i}(G, M)$ for $i = 1, 2$ such that $(X, u) = (X_1, u_1) + (X_2, u_2)$. In fact, for each $i = 1, 2$, we can take u_i as $(i_{X_i})^*(u)$. So, each element in $B^n(G, M)$ can be written as a linear combination of $[X, u]$'s with X a transitive G-set. From standard G-set theory, every transitive G-set is isomorphic to a left coset G-set G/H for some $H \leqslant G$. Moreover, two such G-sets G/H and G/K are isomorphic if and only if H and K are conjugate in G. So, every element $x \in B^n(G, M)$ can be written as

$$x = \sum_{[H] \in Cl(G)} \sum_{u \in \mathcal{U}_H(G,M)} a_{H,u} \cdot [G/H, u]$$

where $\mathcal{U}_H(G, M)$ is the set of representatives of $u \in H^n_{G/H}(G, M)$ under the equivalence relation defined by declaring $u_1 \sim u_2$ if and only if $[G/H, u_1] = [G/H, u_2]$. Note that $u_1 \sim u_2$ if and only if there is an G-set isomorphism $f : G/H \to G/H$ such that $u_1 = f^*(u_2)$. Hence $\mathcal{U}_H(G, M)$ is a set of orbit representatives for the elements in $H^n_{G/H}(G, M)$ under the $\mathrm{Aut}_G(G/H)$-action. This shows that $B^n(G, M)$ is generated by elements in

$$\mathcal{B} = \big\{[G/H, u] \bigm| H \in Cl(G),\ u \in \mathcal{U}_H(G, M)\big\}.$$

The linear independence of $\mathcal{B}$ is also clear since, first of all the $[G/H]$ for $H \in Cl(G)$ form a basis of the ordinary Burnside ring and, for a fixed H, the elements $(G/H, u)$ for $u \in \mathcal{U}_H(G, M)$ are non-isomorphic and irreducible (meaning that they cannot be written as a sum of two non-zero elements). □

The proof of Lemma 5.5 suggests that

$$B^n(G, M) \cong \bigoplus_{[H] \in Cl(G)} \big(\mathbb{Z}H^n_{G/H}(G, M)\big)_{\mathrm{Aut}_G(G/H)}$$

where $(\mathbb{Z}H^n_{G/H}(G, M))_{\mathrm{Aut}_G(G/H)}$ denotes the coinvariants of $\mathbb{Z}H^n_{G/H}(G, M)$ as a $\mathbb{Z}\,\mathrm{Aut}_G(G/H)$-module. A specific homomorphism Ψ inducing the above isomorphism can be given as follows: Let $[G/H, u]$ be a basis element in $B^n(G, M)$. Then, we define $\Psi([G/H, u])$ as a $Cl(G)$-tuple whose $[H]$th coordinate is the image of $u \in \mathbb{Z}H^n_{G/H}(G, M)$ under the coinvariants epimorphism

$$\mathbb{Z}H^n_{G/H}(G, M) \to \big(\mathbb{Z}H^n_{G/H}(G, M)\big)_{\mathrm{Aut}_G(G/H)}$$

and whose other coordinates are zero. To distinguish linear combinations from the addition of cohomology classes, we write $H^n_{G/H}(G, M)$ multiplicatively in the group ring $\mathbb{Z}H^n_{G/H}(G, M)$. Note that with this convention, then we have

$$\Psi\big([G/H \amalg G/H, u_1 \oplus u_2]\big)_{[H]} = u_1 + u_2$$

and

$$\Psi\big([G/H, u_1 + u_2]\big)_{[H]} = u_1 u_2,$$

where $u_1 + u_2$ denotes the addition in the group ring and $u_1 u_2$ denotes multiplication in the group ring, i.e., the group operation in the group $H^n_{G/H}(G, M)$.

Recall that, as a group, $\mathrm{Aut}_G(G/H)$ is isomorphic to $N_G(H)/H$ where the isomorphism is given by the map $N_G(H)/H \to \mathrm{Aut}_G(G/H)$ defined by $n \to f_n$, where $f_n(xH) = xn^{-1}H$. We denote $N_G(H)/H$ by $W_G(H)$ and call it the Weyl group of $H \leqslant G$. We can consider $H^n_{G/H}(G, M)$ as a $W_G(H)$-set via this isomorphism.

Note that $H^n(H, M)$ is a $W_G(H)$-set where the action of $W_G(H)$ is induced by the conjugation homomorphism $c^H_g : H^n(H, M) \to H^n(H, M)$. We have the following:

Lemma 5.6. *The isomorphism $H^n_{G/H}(G, M) \cong H^n(H, M)$ given in Corollary* 2.6 *is an isomorphism of $W_G(H)$-sets.*

Proof. Recall that $c^H_g : H^n_{G/H}(G, M) \to H^n_{G/{}^gH}(G, M)$ is defined as $(f_{H,g})_*$ where $f_{H,g} : G/H \to G/{}^gH$ is given by $f_{H,g}(xH) = xg^{-1}({}^gH)$. So, the action of $W_G(H)$ on $H^n_{G/H}(G, M)$ induced by the isomorphism $N_G(H)/H \to \mathrm{Aut}_G(G/H)$ is the same as the $W_G(H)$-action on $H^n_{G/H}(G, M)$ via the conjugation homomorphism $c_{H,g} : H^n_{G/H}(G, M) \to H^n_{G/H}(G, M)$. Since the isomorphism in Corollary 2.6 commutes with the conjugation homomorphisms, it also commutes with the $W_G(H)$-action. □

We conclude the following:

Proposition 5.7. *We have*

$$B^n(G,M) \cong \bigoplus_{[H]\in Cl(G)} \bigl(\mathbb{Z}H^n(H,M)\bigr)_{W_G(H)}$$

where the $W_G(H)$-action on $H^n(H,M)$ is the usual one, induced by conjugation.

6. The mark homomorphism

In this section, we will define the ghost ring and describe the mark homomorphism for cohomological Burnside rings.

Definition 6.1. The ghost ring of the cohomological Burnside ring is defined as

$$\beta^n(G,M) = \bigoplus_{[H]\in Cl(G)} \bigl(\mathbb{Z}H^n(H,M)\bigr)^{W_G(H)}$$

where $W_G(H)$ acts on $H^n(H,M)$ by conjugation.

Note that an alternative description for $\beta^n(G,M)$ can be given by using super class functions. Recall that a *super class function* for G is a function from the set of subgroups of G to the integers $\mathbb{Z}$ which is constant on the conjugacy classes of subgroups. We generalize this definition as follows.

Definition 6.2. We call a function

$$f : \{H \mid H \leqslant G\} \to \bigoplus_{H\leqslant G} \mathbb{Z}H^n(H,M)$$

a cohomological super class function if it satisfies the following two conditions:

(i) The Kth coordinate of $f(H)$ is zero if $K \neq H$, and
(ii) $f({}^gH) = c_g^H(f(H))$ for all $H \leqslant G$, and $g \in G$.

To describe the mark homomorphism for cohomological Burnside rings, we first define a family of ring homomorphisms

$$s_H : B^n(G,M) \to \mathbb{Z}H^n(H,M)$$

for every $H \leqslant G$.

Let (X,u) be a positioned G-set, where $u \in H^n_X(G,M)$. Let $\alpha : G^n \times X \to M$ be a cocycle representing u. For each $x \in X^H$, the map $\alpha_x : H^n \to M$ defined by

$$\alpha_x(h_1,\dots,h_n) = \alpha(h_1,\dots,h_n;x)$$

is a cocycle, giving a cohomology class $[\alpha_x] \in H^n(H,M)$.

Lemma 6.3. $[\alpha_x]$ *does not depend on the choice of* α *as a representing cocycle for* u.

Proof. Suppose $[\alpha] = [\beta] = u \in H^n_X(G, M)$. Then, $\beta = \alpha + \delta\lambda$ for some map $\lambda : G^{n-1} \times X \to M$. For any $x \in X^H$ we have $\beta_x = \alpha_x + (\delta\lambda)_x = \alpha_x + \delta(\lambda_x)$, where $\lambda_x : H^{n-1} \to M$ is defined by $\lambda_x(h_1, \dots, h_{n-1}) = \lambda(h_1, \dots, h_{n-1}; x)$. Hence $[\alpha_x] = [\beta_x]$. $\square$

The above lemma shows that we can denote $[\alpha_x]$ simply by u_x since it only depends on u. We define $s_H([X, u]) \in \mathbb{Z}H^n(H, M)$ as the linear combination

$$s_H\big([X, u]\big) = \sum_{x \in X^H} u_x.$$

Lemma 6.4. *The map* s_H *is a ring homomorphism.*

Proof. Let $[X, u], [Y, v] \in B^n(G, M)$. By definition,

$$s_H\big([X, u] \cdot [Y, v]\big) = s_H\big([X \times Y, u \otimes v]\big) = \sum_{(x,y) \in (X \times Y)^H} (u \otimes v)_{(x,y)}$$

and

$$s_H\big([X, u]\big) \cdot s_H\big([Y, v]\big) = \Big(\sum_{x \in X^H} u_x\Big)\Big(\sum_{y \in Y^H} v_y\Big) = \sum_{(x,y) \in (X \times Y)^H} u_x \otimes v_y.$$

Let $u = [\alpha]$ and $v = [\gamma]$, then

$$(\alpha \otimes \gamma)_{(x,y)}(h_1, \dots, h_n) = \alpha(h_1, \dots, h_n; x) + \gamma(h_1, \dots, h_n; y) = (\alpha_x \otimes \gamma_y)(h_1, \dots, h_n).$$

Hence $u_x \otimes v_y = (u \otimes v)_{(x,y)}$, concluding the proof. $\square$

We also have the following:

Lemma 6.5. *For every* $H \leqslant G$ *and* $g \in G$, *and for* $[X, u] \in B^n(G, M)$, *we have*

$$s_{{}^gH}\big([X, u]\big) = c_g^H\big(s_H\big([X, u]\big)\big).$$

Proof. Recall that there is a bijection between X^H and $X^{{}^gH}$ given by $x \to gx$. So,

$$s_{{}^gH}\big([X, u]\big) = \sum_{y \in X^{{}^gH}} u_y = \sum_{x \in X^H} u_{gx}.$$

It is easy to see that $u_{gx} \in H^n({}^gH, M)$ is the same as $c_g^H(u_x)$. $\square$

Now, we are ready to define the mark homomorphism.

Definition 6.6. The mark homomorphism

$$\varphi : B^n(G, M) \to \beta^n(G, M)$$

is defined as the group homomorphism which takes $[X, u]$ to a cohomological class function $f_{[X,u]}$ where

$$f_{[X,u]}(H) = s_H\big([X, u]\big) \in \mathbb{Z}H^n(H, M).$$

Note that the mark homomorphism is a ring homomorphism, since s_H is a ring homomorphism for all $H \leqslant G$.

7. The fundamental theorem for $B^n(G, M)$

The main purpose of this section is to prove the fundamental theorem for the cohomological Burnside rings. By the fundamental theorem we mean a theorem which explains the kernel and cokernel of the mark homomorphism of the cohomological Burnside ring. We first show that the mark homomorphism is injective.

Lemma 7.1. *The mark homomorphism*

$$\varphi : B^n(G, M) \to \beta^n(G, M)$$

is an injective ring homomorphism.

Proof. Note that

$$s_K\big([G/H, u]\big) = \sum_{gH \in (G/H)^K} u_{gH} = \sum_{gH \in G/H,\ K \leqslant {}^gH} \mathrm{res}^{{}^gH}_K c^H_g u$$

is zero if K is not conjugate to a subgroup of H. We can write the mark homomorphism φ as a family of homomorphisms $\{\varphi_{K,H} \mid [K], [H] \in Cl(G)\}$ where

$$\varphi_{K,H} : \big(\mathbb{Z}H^n(H, M)\big)_{W_G(H)} \to \big(\mathbb{Z}H^n(K, M)\big)^{W_G(K)}$$

is the restriction to corresponding summands. Ordering the conjugacy classes of subgroups in a way that the order respects inclusions, i.e., if K is conjugate to a subgroup of H, then $[K] \leqslant [H]$, it is easy to see from the above calculation that $\varphi_{K,H} = 0$ if $[H] < [K]$. Hence φ is (upper) triangular if expressed as a matrix of homomorphisms. To show that it is injective it is enough to prove that it is injective on the diagonal, i.e., we need to show

$$\varphi_{H,H} : \big(\mathbb{Z}H^n(H, M)\big)_{W_G(H)} \to \big(\mathbb{Z}H^n(H, M)\big)^{W_G(H)}$$

is injective for all $[H] \in Cl(G)$. For some $u \in H^n(H, M)$, we have

$$\varphi_{H,H}(u) = s_H\big([G/H, u]\big) = \sum_{gH \in N_G(H)/H} c^H_g(u) = \mathrm{Tr}_1^{W_G(H)} u$$

where $W_G(H) = N_G(H)/H$ as before, and $\mathrm{Tr}_1^{W_G(H)}$ is the usual trace map in representation theory, defined by $\mathrm{Tr}_1^W(x) = \sum_{w \in W} wx$. By standard homological algebra arguments, the kernel of the trace map is equal to the (-1)th Tate cohomology of $\mathbb{Z}H^n(H, M)$ as a $W_G(H)$-module. Hence the kernel of $\varphi_{H,H}$ is equal to $\widehat{H}^{-1}(W_G(H), \mathbb{Z}H^n(H, M))$ for all $[H] \in Cl(G)$. Note that as a $\mathbb{Z}W_G(H)$-module, $\mathbb{Z}H^n(H, M)$ is a permutation module isomorphic to

$$\bigoplus_{u \in H^n(H,M)/W_G(H)} \mathbb{Z}\uparrow_{W_G(H,u)}^{W_G(H)}$$

where u runs through a set of orbit representatives of the $W_G(H)$-action on the set $H^n(G, H)$ and $W_G(H, u)$ denotes the stabilizer of u in $W_G(H)$. So, we have

$$\begin{aligned} \widehat{H}^{-1}\big(W_G(H), \mathbb{Z}H^n(H, M)\big) &\cong \bigoplus_u \widehat{H}^{-1}\big(W_G(H), \mathbb{Z}\uparrow_{W_G(H,u)}^{W_G(H)}\big) \\ &\cong \bigoplus_u \widehat{H}^{-1}\big(W_G(H, u), \mathbb{Z}\big) = 0. \end{aligned}$$

The last equality follows from the fact that the trace map on the trivial module $\mathbb{Z}$ is equal to the map defined by multiplication by the order of the group, and over $\mathbb{Z}$ this is an injective map. This completes the proof of injectivity of the mark homomorphism. $\square$

In the case of Burnside rings, the mark homomorphism has the same cokernel as the diagonal homomorphism $\bigoplus_{[H] \in Cl(G)} \varphi_{H,H}$. In that case the cokernel is a direct sum of the form

$$\bigoplus_{[H] \in Cl(G)} \mathbb{Z}/\big|W_G(H)\big|\mathbb{Z}$$

which is equal to the 0th Tate cohomology group $\widehat{H}^0(W_G(H), \mathbb{Z})$. This group is usually denoted by $\mathrm{Obs}(G)$ since it is the group of obstructions for an element in the ghost ring to come from an element in the Burnside ring. Analogous to the usual Burnside ring case, we define the following:

Definition 7.2. The Obstruction group for the cohomological Burnside ring of degree n is defined as the group

$$\mathrm{Obs}^n(G, M) = \bigoplus_{[H] \in Cl(G)} \widehat{H}^0\big(W_G(H), \mathbb{Z}H^n(H, M)\big).$$

Now, we will define a map

$$\psi : \beta^n(G, M) \to \mathrm{Obs}^n(G, M)$$

which will appear in the fundamental theorem. First we define a function $\eta : \beta^n(G, M) \to \beta^n(G, M)$ such that for all $K \leqslant G$

$$\eta(f)(K) = \sum_{K \leqslant L} \mu(K, L)\, \mathrm{res}_K^L f(L)$$

where $\mu(K,L)$ denotes the Möbius function on the poset of subgroups of G. Note that $\eta(f)$ defines a cohomological super class function. To see this, observe that condition (i) in Definition 6.2 holds trivially, and condition (ii) follows from the following calculation:

$$\begin{aligned}\eta(f)\big({}^gK\big) &= \sum_{{}^gK\leqslant L} \mu\big({}^gK,L\big)\operatorname{res}^L_{{}^gK} f(L) = \sum_{K\leqslant L^g} \mu\big(K,L^g\big)\operatorname{res}^L_{{}^gK} f(L)\\ &= \sum_{K\leqslant J} \mu(K,J)\operatorname{res}^{{}^gJ}_{{}^gK} f\big({}^gJ\big) = \sum_{K\leqslant J} \mu(K,J)\operatorname{res}^{{}^gJ}_{{}^gK} c^J_g f(J)\\ &= c^K_g\bigg(\sum_{K\leqslant J} \mu(K,J)\operatorname{res}^J_K f(J)\bigg) = c^K_g\big(\eta(f)(K)\big).\end{aligned}$$

Note that

$$\beta^n(G,M) = \sum_{[H]\in Cl(G)} \big(\mathbb{Z}H^n(H,M)\big)^{W_G(H)}.$$

Hence we can define a surjective map $\pi:\beta^n(G,M)\to \mathrm{Obs}^n(G,M)$ as the projection onto the cokernel of the trace map on the each $[H]$-component. We define

$$\psi:\beta^n(G,M)\to \mathrm{Obs}^n(G,M)$$

as the composition $\pi\circ\eta$. We have the following:

Theorem 7.3 *(Fundamental theorem). Let G be a finite group, let M be a $\mathbb{Z}G$-module, and let n be a non-negative integer. Then the following sequence of abelian groups is exact*

$$0\to B^n(G,M)\xrightarrow{\varphi}\beta^n(G,M)\xrightarrow{\psi}\mathrm{Obs}^n(G,M)\to 0,$$

where φ is the mark homomorphism, and the $[K]$th component of ψ is defined by

$$\psi_K(f) = \sum_{K\leqslant L}\mu(K,L)\operatorname{res}^L_K f(L)$$

modulo the image of the trace map $\mathrm{Tr}_1^{W_G(K)}$.

Proof. We have already shown that φ is injective. Since η can be represented as an upper triangular matrix of homomorphisms with diagonal entries equal to the identity homomorphism, clearly ψ is surjective. So, the theorem follows from the following statement: The composition $\eta\circ\varphi: B^n(G,M)\to\beta^n(G,M)$ is a diagonal matrix with the diagonal entry at $[H]$ equal to the trace map $\mathrm{Tr}_1^{W_G(H)}$. To prove this statement we calculate:

$$\begin{aligned}(\eta\circ\varphi)\big([G/H,u]\big)(K) &= \sum_{K\leqslant L}\mu(K,L)\operatorname{res}^L_K s_L\big([G/H,u]\big)\\ &= \sum_{K\leqslant L}\mu(K,L)\sum_{gH\in G/H,\ L\leqslant {}^gH}\operatorname{res}^{{}^gH}_K c^H_g(u)\end{aligned}$$

$$= \sum_{gH \in G/H} \operatorname{res}^{{}^gH}_K c^H_g(u) \sum_{K \leqslant L \leqslant {}^gH} \mu(K, L)$$
$$= \sum_{gH \in G/H} \operatorname{res}^{{}^gH}_K c^H_g(u) \delta_{{}^gH, K}.$$

So, the sum is zero unless H and K are conjugate to each other. In the case that H and K are conjugate to each other, we can assume that $H = K$ in the above formula by replacing K with a conjugate if necessary. Putting $K = H$ in the last formula, we get

$$\sum_{gH \in G/H} \operatorname{res}^{{}^gH}_H c^H_g(u) \delta_{{}^gH, H} = \sum_{gH \in N_G(H)/H} c^H_g(u) = \operatorname{Tr}_1^{W_G(H)}(u).$$

This completes the proof of the fundamental theorem. □

Now, we calculate the obstruction group. Since $\mathbb{Z}H^n(H, M)$ is a permutation module isomorphic to

$$\bigoplus_{u \in H^n(H,M)/W_G(H)} \mathbb{Z}{\uparrow}^{W_G(H)}_{W_G(H,u)},$$

we have

$$\begin{aligned} \operatorname{Obs}^n(G, M) = \widehat{H}^0\big(W_G(H), \mathbb{Z}H^n(H, M)\big) &\cong \bigoplus_u \widehat{H}^0\big(W_G(H), \mathbb{Z}\uparrow^{W_G(H)}_{W_G(H,u)}\big) \\ &\cong \bigoplus_u \widehat{H}^0\big(W_G(H, u), \mathbb{Z}\big) \cong \bigoplus_u \mathbb{Z}/\big|W_G(H, u)\big|\mathbb{Z} \end{aligned}$$

where the second isomorphism is the Eckmann–Shapiro isomorphism. Note that $W_G(H, u) = N_G(H, u)/H$, so we conclude the following result.

Proposition 7.4. *We have*

$$\operatorname{Obs}^n(G, M) \cong \bigoplus_{[G/H,u]} \mathbb{Z}/\big|N_G(H, u) : H\big|\mathbb{Z}$$

where the sum is taken over the isomorphism classes of positioned G-sets with transitive G-sets.

Using this, one can easily see that the obstruction groups $\operatorname{Obs}^n(G, M)$ for $n = 0$ and $n = 1$ give the obstruction groups for the crossed Burnside ring $B^c(G, M)$ (see Corollary 5.3 in [7]), and the monomial Burnside ring (see Corollary 2.8 in [2]).

In [2], Boltje proves several fundamental theorems for the mark homomorphism $\varphi : A_+(H) \to A^+(H)$ from lower plus constructions to upper plus constructions (see [2] for the definition of lower and upper plus constructions). The obstruction group calculation we gave above shows that our fundamental theorem and the one given in [2] are related to each other. This relation is a consequence of the following observation.

Proposition 7.5. *Let $A(H)$ denote the restriction functor $H \to \mathbb{Z}H^n(H,M)$. Then, $B^n(G,M) \cong A_+(G)$ and $\beta^n(G,M) \cong A^+(G)$ as abelian groups.*

Proof. The lower plus construction $A_+(G)$ is defined as

$$\Bigl(\bigoplus_{H \leqslant G} \mathbb{Z}H^n(H,M)\Bigr)_G$$

where the G-action is by conjugation. So, it is easy to see that as abelian groups $A_+(G)$ and

$$B^n(G,M) = \bigoplus_{[H]\in Cl(G)} \mathbb{Z}H^n(H,M)_{W_G(H)}$$

are isomorphic. Similarly, the upper plus construction $A^+(G)$ is defined as

$$\Bigl(\bigoplus_{H \leqslant G} \mathbb{Z}H^n(H,M)\Bigr)^G$$

where the G action is by conjugation. So, it is easy to see that as an abelian group $A^+(G)$ is isomorphic to $\beta^n(G,M)$. □

One can also show that there is even an isomorphism of Mackey functors once the Mackey functor structure for $B^n(G,M)$ and $\beta^n(G,M)$ are defined in an appropriate way.

Remark 7.6. Under the identification given in Proposition 7.5, the map ψ in Theorem 7.3 is precisely the map π used in Proposition 2.4 of [2]. This means that Lemma 7.1 and Theorem 7.3 can also be deduced from the interpretation of $B^n(G,M)$ given in Proposition 7.5 and the results in Section 1.4 and Proposition 2.4 of [2].

8. Twisted group algebras

Let G be a finite group, let X be a finite G-set, and let k be any commutative ring. In this final section, we give an interpretation of $B^2(G,k^\times)$.

Let R be a commutative ring with a G-action, then it is well known that each cohomology class $[\alpha] \in H^2(G,R^\times)$ corresponds to an equivalence class of twisted group rings $R_\alpha G$, where $R_\alpha G$ has as R-basis the elements $\bar{g}$, for $g \in G$, and multiplication is given by

$$r_1\bar{g}_1 \cdot r_2\bar{g}_2 := \alpha(g_1,g_2)r_1\bigl({}^{g_1}r_2\bigr)\overline{g_1g_2}$$

for $r_1, r_2 \in R$ and $g_1, g_2 \in G$. Here two twisted group algebras are called equivalent, if they are isomorphic as G-graded R-algebras.

Now let $[\alpha] \in H^2_X(G,k^\times)$ for a finite G-set X. Via the isomorphism

$$H^2_X\bigl(G,k^\times\bigr) \cong H^2\bigl(G,\mathrm{Map}\bigl(X,k^\times\bigr)\bigr)$$

we can view α as a map $G \times G \to \mathrm{Map}(X, k^\times)$, where $\mathrm{Map}(X, k^\times)$ is viewed as a $\mathbb{Z}G$-module with $g \in G$ acting on $a \in \mathrm{Map}(X, k^\times)$ by $({}^g a)(x) = a(g^{-1}x)$, for $x \in X$. We can define multiplication in $\mathrm{Map}(X, k)$ pointwise, that is $(ab)(x) = a(x)b(x)$ for $a, b \in \mathrm{Map}(X, k)$ and $x \in X$. Then, for $R = \mathrm{Map}(X, k)$, the twisted group algebra $R_\alpha G$ is defined. To simplify notation, we will write $k_\alpha G$ for this algebra, the dependence on X being implicit in α. Then

$$k_\alpha G = \bigoplus_{g \in G} \mathrm{Map}(X, k)\bar{g}.$$

We will write just g for the element $\bar{g} \in k_\alpha G$. The multiplication in $k_\alpha G$ is given by

$$a_1 g_1 \cdot a_2 g_2 = \alpha(g_1, g_2) a_1 \big({}^{g_1} a_2\big)(g_1 g_2)$$

for $a_1, a_2 \in \mathrm{Map}(X, k)$ and $g_1, g_2 \in G$. Then $k_\alpha G$ is a k-algebra with identity $1_k \cdot 1_G$, where 1_k is the constant map with value $1 \in k$. It is clear that $k_\alpha G$ is associative, by the aforementioned general theory of twisted group rings.

An explicit k-basis for $k_\alpha G$ is given by the set $\{(x, g) \mid x \in X,\ g \in G\}$, where (x, g) represents $a_x g$ with $a_x \in \mathrm{Map}(X, k)$ defined by $a_x(y) = \delta_{x,y}$, for $y \in X$. Multiplication of two basis elements is then given by $(x, g) \cdot (y, h) = 0$, unless $x = gy$, and in this case $(gy, g) \cdot (y, h) = \alpha(g, h; gy)(gy, gh)$, now viewing α again as a map $G \times G \times X \to k^\times$. In terms of the basis, the identity element in $k_\alpha G$ is given by $\sum_{x \in X}(x, 1_G)$.

Now assume that $X = G/H$ for a subgroup H of G. By Corollary 2.6, we have an isomorphism

$$H^2_{G/H}\big(G, k^\times\big) \to H^2\big(H, k^\times\big), \quad \alpha \mapsto \hat{\alpha},$$

so one can expect a relationship between $k_\alpha G$ and the twisted group algebra $k_{\hat{\alpha}} H$.

Proposition 8.1. *The k-algebras $k_\alpha G$ and $k_{\hat{\alpha}} H$ are Morita equivalent.*

Proof. Let A be a k-algebra, and let $e \in A$ be an idempotent. By Theorem 9.9 in [9], the algebras A and eAe are Morita equivalent if and only if the two-sided ideal AeA equals A. In our situation, observe that $k_{\hat{\alpha}} H$ is isomorphic to the subalgebra B of $A = k_\alpha G$ with basis those pairs (x, g) with $g \in H$ and $x = H$ being the trivial coset. The identity in this subalgebra is the idempotent $e = (H, 1)$, so that $B = eAe$. By the above, we only need to show that $A = AeA$. So let (xH, g) denote an arbitrary basis element of A, with $g, x \in G$. One calculates easily that

$$(xH, g) = \alpha(x, 1; xH)^{-1} \alpha\big(x, x^{-1}g; xH\big)^{-1}(xH, x)(H, 1)\big(H, x^{-1}g\big),$$

so that $(xH, g) \in AeA$. $\square$

We can now view $B^2(G, k^\times)$ as the Grothendieck group of a category $Tw(G)$, defined as follows. The objects are $k_\alpha G$ for cocyles $\alpha : G \times G \times X \to k^\times$ and G-sets X. Morphisms $f : k_\alpha G \to k_\beta G$ are induced by G-set maps $f_0 : Y \to X$ satisfying $(f_0)^*[\alpha] = [\beta]$. Any such G-set map gives rise to a ring homomorphism

$$f_1 : \mathrm{Map}(X, k) \to \mathrm{Map}(Y, k), \quad a \mapsto a \circ f_0,$$

which in turn induces a G-graded k-algebra homomorphism

$$f: k_\alpha G \to k_\beta G, \quad ag \mapsto f_1(a)g.$$

Then $k_\alpha G$ and $k_\beta G$ are isomorphic in $Tw(G)$ if and only if $[X,[\alpha]] = [Y,[\beta]]$ in $B^2(G,k^\times)$. Direct sums in $Tw(G)$ are defined as

$$k_\alpha G \oplus_{Tw} k_\beta G = k_{\alpha \oplus \beta} G.$$

Note that $k_\alpha G \oplus_{Tw} k_\beta G$ is isomorphic to the direct sum of $k_\alpha G$ and $k_\beta G$ as algebras (with componentwise addition and multiplication) and also that the Morita equivalence in Proposition 8.1 respects direct sums.

The multiplication in $B^2(G,k^\times)$ corresponds to a product in$Tw(G)$, defined by

$$k_\alpha G \otimes_{Tw} k_\beta G = k_{\alpha \otimes \beta} G,$$

but this algebra is not isomorphic to the k-tensor product of the two algebras. In fact, $k_\alpha G \otimes_{Tw} k_\beta G$ is isomorphic to a diagonal subalgebra of $k_\alpha G \otimes_k k_\beta G$, that is

$$k_\alpha G \otimes_{Tw} k_\beta G \cong \left\{ \sum_{g \in G} a_g g \otimes b_g g \in k_\alpha G \otimes k_\beta G \;\middle|\; a_g \in \mathrm{Map}(X,k),\ b_g \in \mathrm{Map}(Y,k) \right\}$$

as G-graded algebras, the latter being equipped with componentwise addition and multiplication.

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