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Borel–Smith functions and the Dade group

Serge Bouc [a,*], Ergün Yalçın [b,1]

[a] *CNRS, LAMFA, Université de Picardie-Jules Verne, 33, rue Saint Leu 80039, Amiens cedex 1, France*
[b] *Department of Mathematics, Bilkent University, Ankara 06800, Turkey*

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Abstract

We show that there is an exact sequence of biset functors over p-groups

$$0 \to C_b \xrightarrow{j} B^* \xrightarrow{\Psi} D^{\Omega} \to 0$$

where C_b is the biset functor for the group of Borel–Smith functions, B^* is the dual of the Burnside ring functor, D^{Ω} is the functor for the subgroup of the Dade group generated by relative syzygies, and the natural transformation Ψ is the transformation recently introduced by the first author in [S. Bouc, A remark on the Dade group and the Burnside group, J. Algebra 279 (2004) 180–190]. We also show that the kernel of mod 2 reduction of Ψ is naturally equivalent to the functor $B^{\times}$ of units of the Burnside ring and obtain exact sequences involving the torsion part of D^{Ω}, mod 2 reduction of C_b, and $B^{\times}$.

Keywords: Burnside ring; Borel–Smith functions; Dade group; Representation rings

1. Introduction and statement of results

Let $\mathcal{C}$ denote the biset category for finite groups: it is defined as the category whose objects are finite groups, and where the morphism set $\mathrm{Map}_{\mathcal{C}}(G, H)$ from the group G to the group H is

* Corresponding author.
E-mail addresses: serge.bouc@u-picardie.fr (S. Bouc), yalcine@fen.bilkent.edu.tr (E. Yalçın).
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the Grothendieck group $\Gamma(H,G)$ of finite (H,G)-bisets, i.e. the free abelian group on the set of isomorphism classes of finite (H,G)-bisets, quotiented by the subgroup generated by elements of the form $[U \sqcup V] - [U] - [V]$, where $U \sqcup V$ is the disjoint union of the (H,G)-bisets U and V, and $[U]$ denotes the isomorphism class of U. The composition of two morphisms is given by the bilinear map $\Gamma(K,H) \times \Gamma(H,G) \to \Gamma(K,G)$ defined as the linear extension of the assignment $(V,U) \mapsto V \times_H U$. A *biset functor* is an additive functor from $\mathcal{C}$ to the category $\mathcal{A}b$ of abelian groups.

If p is a prime number, then the biset category over p-groups is the full subcategory of $\mathcal{C}$ whose objects are finite p-groups. It is denoted by $\mathcal{C}_p$. An additive functor $F : \mathcal{C}_p \to \mathcal{A}b$ is called a biset functor over p-groups or briefly a *p-biset functor*. More details about biset functors can be found in [2] (see also [5–7], or [9]).

An important example of a biset functor is the Burnside group functor which sends each finite group G to its Burnside group $B(G)$, where the Burnside group $B(G)$ is defined as the Grothendieck group of isomorphism classes of finite left G-sets. For each finite (H,G)-biset U, the group homomorphism $B(U) : B(G) \to B(H)$ is defined as the linear map sending the isomorphism class of the left G-set X to the isomorphism class of the left H-set $U \times_G X$. The other well-known examples are the representation ring functors R_k over a field k of characteristic 0 (equal to $\mathbb{Q}$ or $\mathbb{R}$ in this paper). The biset functor structure for R_k is defined in a similar way to the Burnside ring using tensor product instead of cartesian product. We usually use the same notation for biset functors and for their associated p-biset functors obtained by restriction to the subcategory of p-groups.

Another interesting biset functor is the functor $B^\times$ of units of the Burnside ring. This is a functor which assigns to each group G, the unit group $B(G)^\times$ of the Burnside ring $B(G)$. The biset functor structure for $B^\times$ is rather complicated, involving multiplicative induction instead of the usual induction. The details of this structure can be found in Section 5 of [8] and Section 2 of [14].

In this paper, we are particularly interested in two other p-biset functors. One is the dual of the Burnside ring functor B^* which assigns to each p-group P the dual group $B^*(P) = \mathrm{Hom}(B(P), \mathbb{Z})$ and to each (Q,P)-biset U, the transpose of the linear map $B(U^{\mathrm{op}}) : B(Q) \to B(P)$. Here U^{op} denotes the (P,Q)-biset which is isomorphic to U as a set and whose (P,Q)-action is given by $g \cdot u \cdot h = h^{-1} u g^{-1}$. The second functor is the functor D^Ω which sends each p-group P to the subgroup $D^\Omega(P)$ of the Dade group $D_k(P)$ generated by relative syzygies. Here k can be taken as any field of characteristic p, and the choice does not affect the structure of the group $D^\Omega(P)$. The biset functor structure of D^Ω is described in [3]. The definition of tensor induction and verification of composition rule is particularly difficult. The following theorem is proved in [5].

Theorem 1.1. (Bouc [5]) *There is a unique natural transformation $\Psi : B^* \to D^\Omega$ of p-biset functors with the property that*

$$\Psi_P(\omega_X) = \Omega_X$$

for any finite p-group P and any finite P-set X.

The element $\omega_X \in B^*(P)$ denotes the homomorphism $B(P) \to \mathbb{Z}$ such that $\omega_X(P/H)$ is equal to 1 if H fixes a point on X and equal to 0 if it does not. The element $\Omega_X \in D^\Omega(P)$ is the equivalence class of the endo-permutation module $\Delta_k(X)$, where $\Delta_k(X)$ is the kernel of the augmentation map $\varepsilon : kX \to k$ (in the cases where $\Delta_k(X)$ is not a capped endo-permutation module, we take $\Omega_X = 0$). The fact that Ψ_P is a well-defined homomorphism is a nontrivial fact requiring verification that both ω_X and Ω_X are subject to the same set of relations. We refer the reader to [3] and [5] for details.

Note that the dual of the Burnside ring $B^*(P)$ can be naturally identified with the group of super class functions $C(P)$ where a super class function is a function from the set of subgroups of P to the integers which is constant on the conjugacy classes. The identification comes from the duality pairing $C(P) \times B(P) \to \mathbb{Z}$ defined by $(f, [G/H]) = f(H)$. Under the identification $B^* \cong C$, the kernel of Ψ_P can be described as a subset of super class functions formed by super class functions satisfying a certain set of conditions. We observe that these conditions are exactly the same as the conditions known as the Borel–Smith conditions (see Definition 3.1). We obtain the following:

Theorem 1.2. *The kernel of $\Psi : B^* \to D^\Omega$ is naturally equivalent to the biset functor C_b of Borel–Smith functions under the identification of B^* with the functor C of super class functions. Hence, there is an exact sequence of p-biset functors of the form*

$$0 \to C_b \xrightarrow{j} B^* \xrightarrow{\Psi} D^\Omega \to 0.$$

The Borel–Smith conditions are the conditions which the dimension function of a homotopy representation satisfies. This suggests that the exact sequence given in Theorem 1.2 has some geometric meaning. One probably needs to extend the concept of homotopy representation in a suitable way so that it includes G-CW-complexes which are homotopy equivalent to a wedge of spheres. If this can be done, then it may lead to a more natural description of the transformation Ψ. At this point we do not know how to do this and we leave it as an open problem.

In [5], the first author considers another subfunctor of B^*, namely the dual of rational representations functor $R^*_{\mathbb{Q}}$. Note that by the Ritter–Segal theorem, the linearization map $\mathrm{Lin}_{\mathbb{Q}} : B(P) \to R_{\mathbb{Q}}(P)$ is surjective for every p-group P, so the dual of the natural transformation $\mathrm{Lin}_{\mathbb{Q}}$ gives an injective natural transformation $i : R^*_{\mathbb{Q}} \to B^*$. In [5], it is shown that the image of the natural transformation $\Psi \circ i$ is equal to the torsion part D^Ω_{tors} of D^Ω. This gives an exact sequence of the form

$$0 \to R^*_{\mathbb{Q}} \xrightarrow{i} B^* \xrightarrow{\overline{\Psi}} D^\Omega / D^\Omega_{\mathrm{tors}} \to 0$$

where $\overline{\Psi}$ is the composition of Ψ with the quotient transformation $D^\Omega \to D^\Omega / D^\Omega_{\mathrm{tors}}$.

The first step in our proof of Theorem 1.2 will be to show that $j(C_b)$ is a subfunctor of $i(R^*_{\mathbb{Q}})$, so that after some identifications Theorem 1.2 is equivalent to the following:

Theorem 1.3. *There is an exact sequence of p-biset functors of the following form*

$$0 \to C_b \xrightarrow{j} R^*_{\mathbb{Q}} \xrightarrow{\widetilde{\Psi}} D^\Omega_{\mathrm{tors}} \to 0$$

where $\widetilde{\Psi}$ is the composition $\Psi \circ i$.

To prove Theorem 1.3, we first observe that all the p-biset functors involved in the sequence are rational in the sense of Section 7 of [6]. Thus to show the exactness of this sequence, it is enough to check the exactness only at p-groups of normal p-rank one. Then, the proof follows from an inspection of this sequence in the case of these groups.

In the rest of the paper, we consider the mod 2 reduction of the exact sequence given in Theorem 1.2. Let us denote the mod 2 reduction $\mathbb{F}_2 \otimes_{\mathbb{Z}} F$ of a biset functor F by $\mathbb{F}_2 F$ and the torsion group $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{F}_2, F)$ by $\mathrm{Tor}_{\mathbb{F}_2} F$. Applying the mod 2 reduction to the sequence given in Theorem 1.2, we obtain a long exact sequence of the form

$$0 \to \mathrm{Tor}_{\mathbb{F}_2} D^{\Omega} \longrightarrow \mathbb{F}_2 C_b \longrightarrow \mathbb{F}_2 B^* \xrightarrow{\Psi_2} \mathbb{F}_2 D^{\Omega} \to 0$$

which gives us two short exact sequences involving the kernel of Ψ_2.

Theorem 1.4. *The kernel of $\Psi_2 : \mathbb{F}_2 B^* \to \mathbb{F}_2 D^{\Omega}$ is naturally equivalent to the functor $B^{\times}$ of units of the Burnside ring.*

As an immediate consequence, we obtain

Corollary 1.5. *The following sequences of p-biset functors are exact*:

$$0 \to B^{\times} \to \mathbb{F}_2 B^* \xrightarrow{\Psi_2} \mathbb{F}_2 D^{\Omega} \to 0,$$
$$0 \to \mathrm{Tor}_{\mathbb{F}_2} D^{\Omega} \to \mathbb{F}_2 C_b \to B^{\times} \to 0.$$

The second sequence is closely related to the following exact sequence of p-biset functors recently given by the first author in [8]:

$$0 \to B^{\times} \to \mathbb{F}_2 R_{\mathbb{Q}}^* \to \mathbb{F}_2 D_{\mathrm{tors}}^{\Omega} \to 0.$$

By taking Yoneda splice, we can view these sequences as parts of a long exact sequence of the form

$$0 \to \mathrm{Tor}_{\mathbb{F}_2} D^{\Omega} \to \mathbb{F}_2 C_b \to \mathbb{F}_2 R_{\mathbb{Q}}^* \xrightarrow{\widetilde{\Psi}_2} \mathbb{F}_2 D_{\mathrm{tors}}^{\Omega} \to 0$$

where the kernel of the last natural transformation is equal to $B^{\times}$. We prove in Section 5 that this long exact sequence is nothing but the mod 2 reduction of the exact sequence in Theorem 1.3 (see Proposition 5.3).

In [9], Bouc and Thévenaz gave an exact sequence of p-biset functors

$$0 \to \overline{D}_{\mathrm{tors}} \to \mathbb{F}_2 R_{\mathbb{Q}} \to \Gamma_{\mathbb{F}_2} \to 0$$

for odd primes, where $\Gamma_{\mathbb{F}_2}$ is the constant functor with values $\mathbb{F}_2$ and $\overline{D}_{\mathrm{tors}}$ is a quotient of the torsion part of the Dade group. Later Carlson and Thévenaz [11, Theorem 13.3] proved that actually $\overline{D}_{\mathrm{tors}} = D_{\mathrm{tors}}$. These two results together with Theorem 6.2 of [9] show that $D_{\mathrm{tors}} = D_{\mathrm{tors}}^{\Omega}$ when p is an odd prime. So $\mathrm{Tor}_{\mathbb{F}_2} D^{\Omega}$ can be identified with the functor D_{tors}. In this case also $B^{\times}$ is naturally equivalent to the constant functor $\Gamma_{\mathbb{F}_2}$, and it is interesting to ask whether $\mathbb{F}_2 C_b$ can be identified with $\mathbb{F}_2 R_{\mathbb{Q}}$ when p is an odd prime. We show that this is true if and only if p is congruent to 3 modulo 4 (Theorem 5.4 and Remark 5.5). Thus, in this case, the second exact sequence given in Corollary 1.5 is the same as the exact sequence given by Bouc–Thévenaz in [9].

The paper is organized as follows: Section 2 is a quick exposition of the notion of rational p-biset functor. In Section 3, we give the definition of Borel–Smith functions and explain their basic properties. Section 4 is devoted to the proof of Theorems 1.2 and 1.3, and in Section 5, we prove Theorem 1.4 and discuss its consequences.

2. Rational p-biset functors

Since the notion of *rational biset functor* is an essential tool in the present paper, we will quickly recall the basic definitions and properties related to this particular class of p-biset functors.

2.1. Some particular bisets

Recall that the formalism of bisets allows for a unified description of the operations of induction, restriction, inflation, deflation, and transport by isomorphism:

• If H is a subgroup of the finite group G, the *induction* biset Ind_H^G is the (G,H)-biset equal to G as a set, with biset structure given by left multiplication by elements of G and right multiplication by elements of H. The *restriction* biset Res_H^G is the (H,G)-biset G, with biset structure given by left multiplication by elements of H and right multiplication by elements of G.

• If N is a normal subgroup of G, then the *inflation* biset $\mathrm{Inf}_{G/N}^G$ is the $(G,G/N)$-biset equal to G/N as a set, with right G/N-action by multiplication, and left G-action by projection onto G/N, and next multiplication in G/N. The *deflation* biset $\mathrm{Def}_{G/N}^G$ is the $(G/N,G)$-biset equal to G/N as a set, with left G/N-action by multiplication, and right G-action by projection and multiplication.

• If $\varphi : G \to G'$ is a group isomorphism, then the *transport by isomorphism* biset $\mathrm{Iso}(\varphi)$ or $\mathrm{Iso}_G^{G'}$ is the (G',G)-biset equal to G' as a set, with left action of G' by multiplication, and right action of G by first taking image by φ, and then multiplying in G'.

• If (T,S) is a section of G, i.e. if $S \trianglelefteq T \leqslant G$, we denote by $\mathrm{Indinf}_{T/S}^G$ the composition $\mathrm{Ind}_T^G \times_T \mathrm{Inf}_{T/S}^T$. As a $(G,T/S)$ biset, it is isomorphic to the set G/S, with the obvious biset structure. Similarly, we denote by $\mathrm{Defres}_{T/S}^G$ the composition $\mathrm{Def}_{T/S}^T \times_T \mathrm{Res}_T^G$. As a $(T/S,G)$-biset, it is isomorphic to $S\backslash G$, with the obvious biset structure.

• When F is a biset functor, and U is one of the above (H,G)-bisets, we will also denote by U the map $F(U) : F(G) \to F(H)$: e.g., when (T,S) is a section of G, we will write $\mathrm{Indinf}_{T/S}^G$ for the map $F(T/S) \to F(G)$ obtained by composition of the maps $\mathrm{Inf}_{T/S}^T : F(T/S) \to F(T)$ and $\mathrm{Ind}_T^G : F(T) \to F(G)$.

• If G and H are finite groups, and U is an (H,G)-biset, then U^{op} denotes the (G,H)-biset equal to U as a set, with biset structure defined by

$$\forall (g,u,h) \in G \times U \times H, \quad g.u.h\big(\text{in } U^{\mathrm{op}}\big) = h^{-1}ug^{-1}(\text{in } U).$$

For example, one checks easily that if (T,S) is a section of the finite group G, then the $(G,T/S)$-bisets $(\mathrm{Defres}_{T/S}^G)^{\mathrm{op}}$ and $\mathrm{Indinf}_{T/S}^G$ are isomorphic.

• If F is a biset functor, the *dual biset functor* F^* is the biset functor defined by $F^*(G) = \mathrm{Hom}_{\mathbb{Z}}(F(G),\mathbb{Z})$ for any finite group G, and by $F^*(U) = {}^tF(U^{\mathrm{op}})$, for any finite (H,G)-biset U, where ${}^tF(U^{\mathrm{op}})$ denotes the transposed map of $F(U^{\mathrm{op}})$.

In particular, if (T, S) is a section of the finite group G, if $f \in F^*(G)$, then $\mathrm{Defres}^G_{T/S} f$ is the element of $F^*(T/S)$ defined by

$$\forall u \in F(T/S), \quad \left(\mathrm{Defres}^G_{T/S} f\right)(u) = f\left(\mathrm{Indinf}^G_{T/S} u\right).$$

2.2. Idempotents

Let G be a finite group, and N be a normal subgroup of G. It is easy to check that $\mathrm{Def}^G_{G/N} \times_G \mathrm{Inf}^G_{G/N}$ is isomorphic to the identity $(G/N, G/N)$-biset G/N. It follows that the composition

$$j^G_N = \mathrm{Inf}^G_{G/N} \times_{G/N} \mathrm{Def}^G_{G/N}$$

is an idempotent endomorphism of G in the category $\mathcal{C}$. Moreover, $j^G_{\mathbf{1}}$ is the identity of $\mathrm{End}_{\mathcal{C}}(G)$, and if N and M are normal subgroups of G, then one checks easily that $j^G_M \circ j^G_N = j^G_{MN}$ in $\mathrm{End}_{\mathcal{C}}(G)$. A classical inversion procedure now shows that if for $N \trianglelefteq G$, we set

$$f^G_N = \sum_{\substack{M \trianglelefteq G \\ M \supseteq N}} \mu_{\trianglelefteq G}(N, M) j^G_M,$$

we get a complete set of orthogonal (nonprimitive in general) idempotents in $\mathrm{End}_{\mathcal{C}}(G)$, as N runs through the set of normal subgroups of G. Here $\mu_{\trianglelefteq G}$ denotes the Möbius function of the poset of normal subgroups of G.

It follows in particular that for any biset functor F and any finite group G, the subgroup

$$\partial F(G) = F\left(f^G_{\mathbf{1}}\right)\left(F(G)\right)$$

is a direct summand of $F(G)$. It is called the set of *faithful elements* of $F(G)$. It is also the set of elements of $F(G)$ mapping to zero by any proper deflation.

2.3. Genetic subgroups

Let p be a prime, and P be a finite p-group. A subgroup Q of P is called *genetic* if the following two conditions hold:

- The group $N_P(Q)/Q$ has *normal p-rank* 1, i.e. all its abelian normal subgroups are cyclic.
- Let $Z_P(Q)$ be the subgroup of $N_P(Q)$ defined by

$$Z_P(Q)/Q = Z\left(N_P(Q)/Q\right).$$

  Then for any $x \in P$, the intersection $Q^x \cap Z_P(Q)$ is contained in Q if and only if $Q^x = Q$.

Two genetic subgroups Q and R are said to be *linked modulo P* (notation $Q \frac{\quad}{}_P R$) if there exist elements x and y in P such that $Q^x \cap Z_P(R) \subseteq R$ and $R^y \cap Z_P(Q) \subseteq Q$. It was shown in [4] that this is an equivalence relation on the set of genetic subgroups of P, and that the equivalence classes are in one to one correspondence with the isomorphism classes of rational irreducible representations of P.

A *genetic basis* is a set of representatives of genetic subgroups of P for the relation $\frac{\quad}{}_P$.

2.4. Rational p-biset functors

Let p be a prime, and let F be a p-biset functor. If P is a finite p-group, and $\mathcal{G}$ is a genetic basis of P, one can show ([6, Theorem 3.2], see also [8, Remark 4.6]) that the map

$$\mathcal{I}_{\mathcal{G}} = \bigoplus_{Q \in \mathcal{G}} \mathrm{Indinf}_{N_P(Q)/Q}^{P} : \bigoplus_{Q \in \mathcal{G}} \partial F\big(N_P(Q)/Q\big) \to F(P)$$

is split injective.

The p-biset functor F is called *rational* if for any p-group P, there exists a genetic basis $\mathcal{G}$ of P such that the map $\mathcal{I}_{\mathcal{G}}$ is an isomorphism. Equivalently, for any genetic basis $\mathcal{I}_{\mathcal{G}}$ of P, the map $\mathcal{I}_{\mathcal{G}}$ is an isomorphism.

The use of the word *rational* here comes from the fact that $R_{\mathbb{Q}}$ is a rational p-biset functor (Example 7.2 of [6]). Rational p-biset functors have two important properties: the first one is that by definition, proving a result involving only rational p-biset functors and morphisms between them, generally amounts to checking that the desired result holds when evaluated at p-groups of normal p-rank 1. The second one is that the full subcategory of the abelian category of p-biset functors, whose objects are rational p-biset functors, is a *Serre subcategory*, i.e. if $F' \subseteq F$ are p-biset functors, then F is rational if and only if F' and F/F' are. Moreover, any dual functor of a rational p-biset functor is rational (see Proposition 7.4 of [6] for details).

3. Borel–Smith functions

Let G be a finite group, and let $C(G)$ denote the set of super class functions. Recall that a super class function is a function from the set of subgroups of G to integers which is constant on conjugacy classes. Borel–Smith functions are defined as follows:

Definition 3.1. A function $f \in C(G)$ is called a *Borel–Smith function* if it satisfies the following conditions:

(i) If $H \trianglelefteq L \leqslant G$, $L/H \cong \mathbb{Z}/p\mathbb{Z}$, and p is odd, then $f(H) - f(L)$ is even.
(ii) If $H \trianglelefteq L \leqslant G$, $L/H \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, H_i/H the subgroups of order p in L/H, then

$$f(H) - f(L) = \sum_{i=0}^{p}\big(f(H_i) - f(L)\big).$$

(iii) If $H \trianglelefteq L \trianglelefteq N \leqslant N_G(H)$ and $L/H \cong \mathbb{Z}/2\mathbb{Z}$, then $f(H) - f(L)$ is even if $N/H \cong \mathbb{Z}/4\mathbb{Z}$, and $f(H) - f(L)$ is divisible by 4 if N/H is the quaternion group of order 8.

These conditions are usually referred as *Borel–Smith conditions*. They were first discovered as the conditions satisfied by the dimension function of a homology mod p sphere with a G-action. The set of Borel–Smith functions is an additive subgroup of $C(G)$ which we denote by $C_b(G)$ (see p. 210 in [12] for more details).

Remark 3.2. Condition (iii) is usually stated in stronger terms, since one requires that $f(H) - f(L)$ should be divisible by 4 for any $H \trianglelefteq L \trianglelefteq N \leqslant N_G(H)$ such that N/H is a *generalized quaternion group* of order 2^k, for $k \geqslant 3$. But clearly, it is enough to consider the quaternion group of order 8, since any larger quaternion group contains such a group of order 8, which contains its unique subgroup of order 2.

Given a real representation V, we define the super class function $\mathrm{Dim}\, V$ as the function with values $\mathrm{Dim}\, V(H) = \dim_{\mathbb{R}} V^H$ for all $H \leqslant G$. It is an easy exercise to show that the dimension function of a real representation satisfies the Borel–Smith conditions, and hence $\mathrm{Dim}\, V$ is a Borel–Smith function. The key result on Borel–Smith functions is that when G is a nilpotent group, for every $f \in C_b(G)$, there exist real representations V and W such that $f = \mathrm{Dim}\, V - \mathrm{Dim}\, W$. In particular, we have the following:

Theorem 3.3. (Theorem 5.4 on p. 211 of [12]) *Let G be a nilpotent group, and let $R_{\mathbb{R}}(G)$ denote the real representation ring of G. Consider the group homomorphism* $\mathrm{Dim}: R_{\mathbb{R}}(G) \to C(G)$ *defined as the linear extension of the assignment $V \mapsto \mathrm{Dim}\, V$. Then, the image of* Dim *is exactly equal to the group of Borel–Smith functions $C_b(G)$.*

The assignment $G \mapsto C_b(G)$ together with appropriate action of bisets is a biset functor, and the assignment $V \mapsto \mathrm{Dim}\, V$ is a morphism of biset functors. To show this, we first need to describe the biset functor structure of the group of super class functions, and for this we will identify $C(G)$ with $B^*(G)$.

Recall that the biset functor B^*, the dual of the Burnside group functor, is defined as the functor which sends every finite group G to $B^*(G) = \mathrm{Hom}(B(G), \mathbb{Z})$, and every morphism $U \in \Gamma(H, G)$, to the homomorphism $B^*(U): B^*(G) \to B^*(H)$ where $B^*(U)$ is the transpose of the linear map $B(U^{\mathrm{op}}): B(H) \to B(G)$. Now the group $C(G)$ of super class functions can be identified with $B^*(G)$ via the duality pairing $C(G) \times B(G) \to \mathbb{Z}$ defined by $(f, [G/L]) = f(L)$, and we will freely use this identification throughout the paper. In particular, we will use it for considering the assignment $G \mapsto C(G)$ as a biset functor. It is easy to check that if G and H are finite groups, if U is a finite (H, G)-biset, and if $f \in C(G)$, then the value of the superclass function $C(U)(f)$ at the subgroup K of H is equal to

$$C(U)(f)(K) = \sum_{u \in [K \backslash U/G]} f\bigl(K^u\bigr), \tag{3.4}$$

where $[K\backslash U/G]$ is a set of representatives of (K, G)-orbits on U, and K^u is the subgroup of G defined by

$$K^u = \{g \in G \mid \exists k \in K,\ ku = ug\}.$$

Note that for a real representation V of G, the value of the element $\mathrm{Dim}\, V$ of $B^*(G)$ on $X \in B(G)$ is equal to $\dim_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}G}(\mathbb{Q}X, V)$. It follows easily that the assignment $V \mapsto \mathrm{Dim}\, V$ is a morphism of biset functors from $R_{\mathbb{R}}$ to B^*.

Notation 3.5. If H is a finite p-group, define an element ε_H of $B(H)$ by

$$\varepsilon_H = \sum_{E \leqslant \Omega_1 Z(H)} \mu(\mathbf{1}, E)[H/E],$$

where $\Omega_1 Z(H)$ is the largest elementary abelian subgroup in the center of H, and $\mu(\mathbf{1}, E)$ is the value of the Möbius function of the poset of subgroups of E.

It follows easily from this definition that $\mathrm{Def}^H_{H/N}\,\varepsilon_H = 0$ for any nontrivial normal subgroup N of H (for details, note that with the notation of Lemma 3.12 and Remark 3.13 of [7], one has that $\varepsilon_H = f_{\mathbf{1}}^H[H/\mathbf{1}]$ in $B(H)$).

Notation 3.6. Let Ξ denote the class of p-groups which are cyclic of order p with $p > 2$, cyclic of order 4, quaternion of order 8, or elementary abelian of rank 2. If $H \in \Xi$, define the integer m_H by $m_H = 2$ if H is cyclic, by $m_H = 4$ if H is quaternion, and by $m_H = 0$ if H is elementary abelian of rank 2.

With this notation, we can rephrase the Borel–Smith conditions in the following way: observe first that $\varepsilon_H = H/\mathbf{1} - H/Z$, if H is a nontrivial cyclic p-group or a generalized quaternion 2-group, where Z is the unique central subgroup of order of p, and that

$$\varepsilon_H = H/\mathbf{1} - \sum_{|H:K|=p} H/K + pH/H = (H/\mathbf{1} - H/H) - \sum_{|H:K|=p} (H/K - H/H),$$

if $H \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Now the Borel–Smith conditions can be expressed by saying that an element $f \in B^*(G)$ is in $C_b(G)$ if and only if $(\mathrm{Defres}^G_{T/S}\, f)(\varepsilon_{T/S}) \in m_{T/S}\mathbb{Z}$, whenever (T, S) is a section of G such that $T/S \in \Xi$. This leads to the following:

Proposition 3.7. *For a finite group G, let $C_b(G)$ denote the group of Borel–Smith functions. Set*

$$\beta(G) = \bigl\{ f \in B^*(G) \bigm| \forall H \in \Xi,\ \forall \psi \in \mathrm{Map}_{\mathcal{C}}(G, H),\ B^*(\psi)(f)(\varepsilon_H) \in m_H\mathbb{Z} \bigr\}.$$

Then $\beta(G) = C_b(G)$, up to the identification $B^*(G) = C(G)$. In particular, the assignment $G \mapsto C_b(G) \subseteq C(G)$ defines a subfunctor of C.

Proof. First $\beta(G) \subseteq C_b(G)$, because if (T, S) is a section of G with $T/S \in \Xi$, then the set $S\backslash G$ is a $(T/S, G)$-biset, i.e. an element of $\mathrm{Map}_{\mathcal{C}}(G, T/S)$, whose action is precisely $\mathrm{Defres}^G_{T/S}$.

Conversely, let G be any finite group, and let $f \in C_b(G)$. We prove that for any $H \in \Xi$ and any $\psi \in \mathrm{Map}_{\mathcal{C}}(G, H)$, the value $B^*(\psi)(f)(\varepsilon_H)$ is a multiple of m_H.

We can assume that ψ is some transitive (H, G)-biset, so that, by [2, Lemme 3], there exists a section (Y, X) of H and a section (T, S) of G, and a group isomorphism $\theta : T/S \to Y/X$ with

$$B^*(\psi)(f) = \mathrm{Indinf}^H_{Y/X}\,\mathrm{Iso}(\theta)\,\mathrm{Defres}^G_{T/S}\, f,$$

thus

$$B^*(\psi)(f)(\varepsilon_H) = \bigl(\mathrm{Iso}(\theta)\,\mathrm{Defres}^G_{T/S}\, f\bigr)\bigl(\mathrm{Defres}^H_{Y/X}\,\varepsilon_H\bigr).$$

If the section (Y, X) is the section $(H, \mathbf{1})$, then

$$\begin{aligned} B^*(\psi)(f)(\varepsilon_H) &= \bigl(\mathrm{Iso}(\theta)\,\mathrm{Defres}^G_{T/S}\, f\bigr)\bigl(\mathrm{Defres}^H_{Y/X}\,\varepsilon_H\bigr) = \bigl(\mathrm{Defres}^G_{T/S}\, f\bigr)\bigl(\mathrm{Iso}\bigl(\theta^{-1}\bigr)\varepsilon_H\bigr) \\ &= \bigl(\mathrm{Defres}^G_{T/S}\, f\bigr)(\varepsilon_{T/S}) \in m_H\mathbb{Z}, \end{aligned}$$

since $f \in C_b(G)$ and $T/S \cong H$. So we can assume that (Y, X) is a proper section of H.

Suppose first that $H \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Then it is easily checked that any proper restriction and any proper deflation of ε_H is equal to 0. So $B^*(\psi)(f)(\varepsilon_H) = 0$ as was to be shown in this case.

Now, if H is cyclic, then $m_H = 2$. If H has odd prime order p, then the proper deflation $\mathrm{Def}^H_{H/H}\,\varepsilon_H$ is zero, and the proper restriction $\mathrm{Res}^H_{\mathbf{1}}\,\varepsilon_H$ is $(p-1)\mathbf{1}/\mathbf{1}$, which is a multiple of 2. So $B^*(\psi)(f)(\varepsilon_H)$ is even. If H has order 4, then any proper deflation of ε_H is zero, and the restriction of ε_H to its subgroup K of order 2 is $2(K/\mathbf{1} - K/K)$, again a multiple of m_H. It follows that the restriction of ε_H to the trivial group is also a multiple of m_H. Hence $B^*(\psi)(f)(\varepsilon_H)$ is a multiple of m_H in this case.

Finally, suppose H is the quaternion group of order 8. Since any proper deflation of ε_H is zero and every subgroup of H is normal,

$$\mathrm{Defres}^H_{Y/X}\,\varepsilon_H = \mathrm{Res}^{H/X}_{Y/X}\,\mathrm{Def}^H_{H/X}\,\varepsilon_H = 0$$

for every section (Y, X) with $X \neq \mathbf{1}$. So, we can assume $X = \mathbf{1}$. Note that for every subgroup Y of H, we have $\mathrm{Res}^H_Y\,\varepsilon_H = 2^{|H:Y|}\varepsilon_Y$ which is obviously a multiple of $m_H = 4$ if $|H:Y| \geqslant 4$. In the case $|H:Y| = 2$, we have $Y \cong \mathbb{Z}/4\mathbb{Z}$, and hence

$$B^*(\psi)(f)(\varepsilon_H) = 2\big(\mathrm{Defres}^G_{T/S}\,f\big)(\varepsilon_{T/S})$$

is multiple of 4 since $(\mathrm{Defres}^G_{T/S}\,f)(\varepsilon_{T/S})$ is even, for $f \in C_b(G)$ and $T/S \cong \mathbb{Z}/4\mathbb{Z}$. Hence $\beta(G) = C_b(G)$.

Now if G and G' are two finite groups, and if $\varphi \in \mathrm{Map}_{\mathcal{C}}(G, G')$, and if $f \in \beta(G)$, then for any $H \in \Xi$ and any $\psi \in \mathrm{Map}_{\mathcal{C}}(G', H)$, one has that

$$B^*(\psi)B^*(\varphi)(f)(\varepsilon_H) = B^*(\psi\varphi)(f)(\varepsilon_H) \in m_H\mathbb{Z},$$

since $\psi\varphi \in \mathrm{Map}_{\mathcal{C}}(G, H)$ and $f \in \beta(G)$. Thus $B^*(\varphi)(f) \in \beta(G')$, and β is a biset subfunctor of B^*. This completes the proof. □

4. Proof of Theorems 1.2 and 1.3

Let P be a p-group, and k be a field of characteristic p. Given a nonempty finite P-set X, consider the kG-module $\Delta_k(X) = \ker\{\varepsilon : kX \to k\}$ where ε is the k-linear homomorphism which takes every element $x \in X$ to $1 \in k$. It has been shown by Alperin [1] that $\Delta_k(X)$ is an endo-permutation module which is capped in most of the cases (when P does not have a single fixed point on X). Recall that a kP-module M is called an endo-permutation module if $\mathrm{End}_k(M)$ is a permutation module, and it is called capped if it has an indecomposable summand with vertex P. When $\Delta_k(X)$ is a capped module, then we define the relative syzygy Ω_X as the equivalence class of the endo-permutation module $\Delta_k(X)$ in the Dade group $D_k(P)$ (see [7] for the definition of the Dade group). When $\Delta_k(X)$ is not capped, or when $X = \emptyset$, we take $\Omega_X = 0$. The group $D^\Omega(P)$ is defined to be the subgroup of the Dade group $D_k(P)$ generated by relative syzygies.

By Theorem 1.1, there is a surjective natural transformation $\Psi : B^* \to D^\Omega$ of bisets functors which is defined by

$$\Psi_P(\omega_X) = \Omega_X$$

for any finite p-group P and any finite P-set X. There is also an injective natural transformation $j : C_b \to B^*$ defined as the composition of the inclusion of the subfunctor C_b into C with the identification $C \cong B^*$.

Recall that the transpose of the linearization map $B \to R_{\mathbb{Q}}$ is an injection $i : R_{\mathbb{Q}}^* \to B^*$. It was shown in Theorem 1.8 of [5] that the image of $R_{\mathbb{Q}}^*(P)$ by the map Ψ_P is precisely the torsion part $D^{\Omega}_{\mathrm{tors}}(P)$ of $D^{\Omega}(P)$.

Our proof of Theorem 1.2 is as follows: we first show that C_b is a subfunctor of the image of $R_{\mathbb{Q}}^*$ in B^*. In other words, the injection $j : C_b \to B^*$ factors through an injection we also denote by $j : C_b \to R_{\mathbb{Q}}^*$, so Theorem 1.2 is equivalent to Theorem 1.3, i.e. to the exactness of the following sequence of biset functors

$$0 \to C_b \xrightarrow{j} R_{\mathbb{Q}}^* \xrightarrow{\widetilde{\Psi}} D^{\Omega}_{\mathrm{tors}} \to 0 \tag{4.1}$$

where $\widetilde{\Psi} = \Psi \circ i$.

Now all functors in this sequence are rational p-biset functors in the sense of Section 7 of [6]. In particular, the evaluation of this sequence at some p-group P is determined in a precise way by its evaluations at p-groups of normal p-rank one, and the proof of its exactness comes down to an inspection of this sequence in the case of these groups.

Lemma 4.2. *Let P be a p-group, and $f \in B^*(P)$. Then $f \in R_{\mathbb{Q}}^*(P)$ if and only if for any section (T,S) of P with $T/S \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, one has that* $(\mathrm{Defres}^P_{T/S} f)(\varepsilon_{T/S}) = 0$*, i.e.*

$$f(P/S) - f(P/T) = \sum_{S<X<T} \big(f(P/X) - f(P/T)\big).$$

Proof. The proof is similar to the proof of Lemma 3.2 of [5] (which gives another more complicated criterion for f to belong to $R_{\mathbb{Q}}^*(P)$), and we refer to this lemma for details. Since $R_{\mathbb{Q}}(P)$ and $B(P)$ are free abelian groups, the commutative diagram

$$\begin{tikzcd} R_{\mathbb{Q}}^*(P) \arrow[r] \arrow[d] & B^*(P) \arrow[d] \\ \mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}^*(P) \arrow[r] & \mathbb{Q} \otimes_{\mathbb{Z}} B^*(P) \end{tikzcd}$$

is a pullback diagram, where all the maps are injective. Moreover, $\mathbb{Q} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}^*(P) = \mathbb{Q}R_{\mathbb{Q}}^*(P)$ identifies with $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}R_{\mathbb{Q}}(P), \mathbb{Q})$, and $\mathbb{Q}B^*(P)$ identifies with $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}B(P), \mathbb{Q})$. So the assertion of the lemma is equivalent to the same assertion with $R_{\mathbb{Q}}^*(P)$ replaced by $\mathbb{Q}R_{\mathbb{Q}}^*(P)$ and $B^*(P)$ replaced by $\mathbb{Q}B^*(P)$.

Now $\mathbb{Q}B(P)$ has a basis over $\mathbb{Q}$ consisting of its primitive idempotents e_Q^P, which are indexed by subgroups Q of P up to P-conjugation. The kernel of the linearization map $\mathbb{Q}B(P) \to \mathbb{Q}R_{\mathbb{Q}}(P)$ consists of the linear combinations of idempotents e_Q^P indexed by noncyclic subgroups Q of P. Thus an element f of $\mathbb{Q}B^*(P)$ lies in $\mathbb{Q}R_{\mathbb{Q}}^*(P)$ if and only if $f(e_Q^P) = 0$ for any noncyclic subgroup Q of P.

Suppose first that $f \in \mathbb{Q}R_{\mathbb{Q}}^*(P)$. Then since $\mathbb{Q}R_{\mathbb{Q}}^*$ is a biset subfunctor of $\mathbb{Q}B^*$, it follows that $\mathrm{Defres}^P_{T/S} f \in \mathbb{Q}R_{\mathbb{Q}}^*(T/S)$ for any section (T,S) of P. In particular if T/S is elementary abelian of rank 2, this amounts to saying that $(\mathrm{Defres}^P_{T/S} f)(e^{T/S}_{T/S}) = 0$, since the only noncyclic subgroup of T/S is T/S itself. Moreover, one checks easily that $\varepsilon_{T/S} = pe^{T/S}_{T/S}$. Hence $(\mathrm{Defres}^P_{T/S} f)(\varepsilon_{T/S}) = 0$.

Conversely, suppose that this condition holds for any section (T, S) of P such that T/S is elementary abelian of rank 2. Then $(\mathrm{Defres}^P_{T/S} f)(e^{T/S}_{T/S}) = 0$. We prove that $f(e^P_Q) = 0$ for any noncyclic subgroup Q of P by induction on $|Q|$. If $|Q| = p^2$, then Q is elementary abelian of rank 2, so $(\mathrm{Res}^P_Q f)(e^Q_Q) = 0$, using the hypothesis for the section $(Q, \mathbf{1})$ of P. But $(\mathrm{Res}^P_Q f)(e^Q_Q) = f(\mathrm{Ind}^P_Q e^Q_Q)$, and $\mathrm{Ind}^P_Q e^Q_Q$ is a nonzero multiple of e^P_Q. Thus $f(e^P_Q) = 0$ in this case, and this starts induction.

Suppose that Q is a noncyclic subgroup of P, such that $f(e^P_X) = 0$ for any noncyclic subgroup X of P with $|X| < |Q|$. Choose a normal subgroup S of Q such that Q/S is elementary abelian of rank 2. Such a subgroup exists since Q is noncyclic. Then

$$0 = \big(\mathrm{Defres}^P_{Q/S} f\big)\big(e^{Q/S}_{Q/S}\big) = f\big(\mathrm{Ind}^P_Q \mathrm{Inf}^Q_{Q/S} e^{Q/S}_{Q/S}\big).$$

Moreover, $\mathrm{Inf}^Q_{Q/S} e^{Q/S}_{Q/S} = \sum_X e^Q_X$, where the summation is over all subgroups X of Q for which $XS = Q$, up to Q-conjugation. Such subgroups are noncyclic, since $X/X \cap S \cong Q/S$ is noncyclic, and all of them except Q itself have order less than $|Q|$. Now $\mathrm{Ind}^P_Q e^Q_X$ is a nonzero multiple of e^P_X, thus $f(\mathrm{Ind}^P_Q e^Q_X) = 0$ for $X \neq Q$. It follows that $f(\mathrm{Ind}^P_Q e^Q_Q) = 0$, hence $f(e^P_Q) = 0$, completing the inductive step of the proof. □

Corollary 4.3. *Let P be a p-group. Then $C_b(P) \subseteq R^*_{\mathbb{Q}}(P)$. So C_b is a p-biset subfunctor of $R^*_{\mathbb{Q}}$. In particular, the functor C_b is a rational p-biset functor.*

Proof. The inclusion $C_b(P) \subseteq R^*_{\mathbb{Q}}(P)$ follows from the lemma, and from the second Borel–Smith condition. So C_b is a p-biset subfunctor of $R^*_{\mathbb{Q}}$, hence it is rational, since $R_{\mathbb{Q}}$ is rational, and since the dual as well as any subfunctor of a rational p-biset functor are rational. □

Proof of Theorem 1.3. Proving that the sequence (4.1) is exact amounts to showing that for each p-group P, the sequence

$$0 \to C_b(P) \xrightarrow{j_P} R^*_{\mathbb{Q}}(P) \xrightarrow{\widetilde{\Psi}_P} D^{\Omega}_{\mathrm{tors}}(P) \to 0$$

is an exact sequence of abelian groups. Choose a genetic basis $\mathcal{G}$ of P. We have a diagram

$$\begin{tikzcd}
0 \arrow[r] & C_b(P) \arrow[r, "j_P"] & R^*_{\mathbb{Q}}(P) \arrow[r, "\widetilde{\Psi}_P"] & D^{\Omega}_{\mathrm{tors}}(P) \arrow[r] & 0 \\
0 \arrow[r] & \bigoplus_{Q\in\mathcal{G}} \partial C_b(N_P(Q)/Q) \arrow[r] \arrow[u] & \bigoplus_{Q\in\mathcal{G}} \partial R^*_{\mathbb{Q}}(N_P(Q)/Q) \arrow[r] \arrow[u] & \bigoplus_{Q\in\mathcal{G}} \partial D^{\Omega}_{\mathrm{tors}}(N_P(Q)/Q) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where the vertical arrows are the maps $\bigoplus_{Q\in\mathcal{G}} \mathrm{Indinf}^P_{N_P(Q)/Q}$, and where for any p-biset functor F and any p-group P, the group $\partial F(R)$ is the subgroup of faithful elements of $F(R)$, i.e. the image of the map $F(f^R_{\mathbf{1}})$ associated to the idempotent $f^R_{\mathbf{1}} \in \mathrm{End}_{\mathcal{C}}(R)$.

The existence of the bottom horizontal maps in this diagram, and the fact that this diagram is commutative, follow from the fact that the maps j and $\widetilde{\Psi}$ are maps of p-biset functors. The vertical maps are, moreover, isomorphisms, because all three p-biset functors are rational: the functor $R_{\mathbb{Q}}^*$ is dual to a rational p-biset functor, the functor C_b is a subfunctor of $R_{\mathbb{Q}}^*$, and the functor $D_{\mathrm{tors}}^{\Omega}$ is a quotient of $R_{\mathbb{Q}}^*$, since $\widetilde{\Psi}: R_{\mathbb{Q}}^* \to D_{\mathrm{tors}}^{\Omega}$ is surjective.

In other words, the above diagram is an isomorphism from the bottom line to the top one. Moreover, the bottom line is the direct sum of sequences

$$0 \to \partial C_b(S) \xrightarrow{j_S} \partial R_{\mathbb{Q}}^*(S) \xrightarrow{\widetilde{\Psi}_S} \partial D_{\mathrm{tors}}^{\Omega}(S) \to 0, \tag{4.4}$$

where $S = N_P(Q)/Q$ for $Q \in \mathcal{G}$. So all we have to do is to check that this sequence is an exact sequence when S is a group of normal p-rank one, and we do this by a case by case inspection. Let S be a p-group of normal p-rank one. Recall that:

- The group S is cyclic if $p \neq 2$, or cyclic, generalized quaternion, dihedral of order at least 16, or semi-dihedral if $p = 2$.
- (See [14] or [4].) If S is nontrivial, then there is a unique subgroup Z of order p in the center of S. If Q is a subgroup of S not containing Z, then $Q = \mathbf{1}$ if S is cyclic or generalized quaternion, or $|Q| \leqslant 2$ if S is dihedral or semi-dihedral. The noncentral subgroups of order 2 form a single conjugacy class of subgroups of S if S is semi-dihedral, and two conjugacy classes of subgroups if S is dihedral.
- (Proposition 3.7 of [4]) The group S has a unique faithful irreducible rational representation Φ_S.
- (Theorem 10.3 of [10]) The group $\partial D_{\mathrm{tors}}^{\Omega}(S)$ is:
  - trivial if $|S| \leqslant 2$, or if S is dihedral,
  - of order 2 if S is cyclic of order at least 3, generated by $\Omega_{S/\mathbf{1}}$,
  - cyclic of order 4 if S is generalized quaternion, generated by $\Omega_{S/\mathbf{1}}$,
  - of order 2 if S is semi-dihedral, generated by $\Omega_{S/\mathbf{1}} + \Omega_{S/I}$, where I is a noncentral subgroup of order 2 of S.

Now we observe that for any p-group S, the group $\partial R_{\mathbb{Q}}^*(S)$ is the subgroup of $R_{\mathbb{Q}}^*(S)$ with basis the elements V^*, where V is a faithful rational irreducible representation of S, and V^* is the element of $B^*(S)$ defined by setting $V^*(S/R)$ to be equal to the multiplicity $m(V, \mathbb{Q}S/R)$ of V as a summand of $\mathbb{Q}S/R$, for any subgroup R of S. It follows that if S has normal p-rank one, then $\partial R_{\mathbb{Q}}^*(S) = \mathbb{Z}\Phi_S^*$.

Now for any finite p-group S, the group $\partial B^*(S)$ is the group of linear forms $B(S) \to \mathbb{Z}$ which map to 0 by any proper deflation. But if $N \trianglelefteq S$ and $f \in B^*(S)$, then for any subgroup X/N of S/N one has that

$$\left(\mathrm{Def}_{S/N}^{P} f\right)\left((S/N)/(X/N)\right) = f(S/X).$$

This means that $f \in \partial B^*(S)$ if and only if $f(S/X) = 0$ whenever X contains a nontrivial normal subgroup of S, or equivalently since S is a p-group, if X intersects the center of S nontrivially. The group $\partial C_b(S)$ consists of linear forms which satisfy this condition, together with the Borel–Smith conditions.

Suppose first that S is cyclic of order at most 2. In this case the Borel–Smith conditions are void, so an element f of $\partial C_b(S)$ has only one possibly nonzero value, namely $f(S/\mathbf{1})$, and this value is arbitrary. In other words $\partial C_b(S) = \partial R^*_{\mathbb{Q}}(S)$ in this case. But $D^{\Omega}_{\mathrm{tors}}(S) = 0$ in this case, so the sequence (4.4) is exact.

Now suppose that S is cyclic of order at least 3 or generalized quaternion. Then any nontrivial subgroup of S intersects the center of S nontrivially. An element f in $\partial B^*(S)$ has only one possibly nonzero value, namely $f(S/\mathbf{1})$. Thus $\partial B^*(S) = \mathbb{Z}\omega_{S/\mathbf{1}}$. This is also equal to $\partial R^*_{\mathbb{Q}}(S)$, since the conditions of Lemma 4.2 are trivially true for $f = \omega_{S/\mathbf{1}}$ in this case.

Now $f \in \partial C_b(S)$ if and only if the additional Borel–Smith condition coming from a section $(T, 1)$ is fulfilled, where T is cyclic of prime order if S is cyclic of odd order, or cyclic of order 4 if S is a cyclic 2-group, or quaternion of order 8 if S is generalized quaternion, i.e. if $f(S/\mathbf{1})$ is a multiple of 2 if S is cyclic, or 4 if S is generalized quaternion.

This shows that $\partial C_b(S)$ is generated by $m_S\omega_{S/\mathbf{1}}$, where $m_S = 2$ if S is cyclic, or 4 if S is generalized quaternion. But $\widetilde{\Psi}_S(\omega_{S/\mathbf{1}}) = \Omega_{S/\mathbf{1}}$, and the order of $\Omega_{S/\mathbf{1}}$ is precisely 2 if S is cyclic of order at least 3, or 4 if S is generalized quaternion. Hence the sequence (4.4) is again exact in this case.

Now if S is dihedral of order at least 16, and $f \in \partial B^*(S)$, the only possibly nonzero values of f are $f(S/\mathbf{1})$, $f(S/I)$, and $f(S/J)$, where I and J are noncentral subgroups of order 2 of S, not conjugate in S. The Borel–Smith condition coming from the section $(E, \mathbf{1})$, where E is an elementary abelian subgroup of rank 2 of S containing I, gives $f(S/\mathbf{1}) = 2f(S/I)$, because E contains exactly 2 conjugates of I in S. Similarly $f(S/\mathbf{1}) = 2f(S/J)$, hence $f(S/I) = f(S/J)$. Thus $f \in R^*_{\mathbb{Q}}(S)$ if and only if $f(S/\mathbf{1}) = 2f(S/I) = 2f(S/J)$. The only other nontrivial Borel–Smith condition comes from the section $(T, \mathbf{1})$, where T is the subgroup of order 4 in S. This condition gives that $f(S/\mathbf{1})$ is even, but this follows from the previous conditions. This shows that $\partial C_b(S) = \partial R_{\mathbb{Q}}(S)$ in this case, generated by the linear form whose nonzero values are 1 at S/I and S/J, and 2 at $S/\mathbf{1}$. But $\partial D^{\Omega}_{\mathrm{tors}}(S) = 0$ if S is dihedral, and the sequence (4.4) is exact in this case.

Finally if S is semi-dihedral, and $f \in B^*(S)$, then the only possibly nonzero values of f are $f(S/\mathbf{1})$ and $f(S/I)$, where I is a noncentral subgroup of order 2 in S. Now $f \in \partial R^*_{\mathbb{Q}}(P)$ if and only it satisfies the Borel–Smith condition obtained for the section $(E, \mathbf{1})$, where E is the elementary abelian subgroup of rank 2 containing I. This condition gives $f(S/\mathbf{1}) = 2f(S/I)$ as in the dihedral case. Hence the generator Φ^*_S of $\partial R^*_{\mathbb{Q}}(S)$ has value 1 at S/I, 2 at $S/\mathbf{1}$, and zero anywhere else. Now $f \in \partial C_b(S)$ if and only if the additional Borel–Smith condition coming from the section $(T, \mathbf{1})$ is fulfilled, where T is a quaternion subgroup of order 8 in S. This condition on f is that $f(S/\mathbf{1})$ is a multiple of 4. So $\partial C_b(S)$ is generated by $2\Phi^*_S$. But also $\Phi^*_S = \omega_{S/I} + \omega_{S/\mathbf{1}}$ in this case, since the multiplicity of Φ_S as a summand of $\mathbb{Q}P/I$ and $\mathbb{Q}P/\mathbf{1}$ is equal to 1 and 2, respectively (Lemma 4.1 of [6]). Thus $\widetilde{\Psi}_S(\Phi^*_S) = \Omega_{S/I} + \Omega_{S/\mathbf{1}}$, which has precisely order 2 in $D^{\Omega}(S)$. Hence the sequence (4.4) is again exact in this case, and this completes the proof of Theorem 1.3, hence also of Theorem 1.2. □

Remark 4.5. It follows from Theorem 1.2 that for any p-group P, the map

$$\prod_{(T,S)\in \Xi(P)} \mathrm{Defres}^P_{T/S} : D^{\Omega}(P) \to \prod_{(T,S)\in \Xi(P)} D^{\Omega}(T/S)$$

is injective, where $\Xi(P)$ is the set of sections (T, S) of P such that T/S is in Ξ. This is a weak form of the detection theorem proved by Carlson and Thévenaz for the whole Dade group (Theorem 13.1 of [11]). Conversely, one can give an alternative proof of Theorem 1.2 based on this detection theorem, which comes down to examining the cases of groups in Ξ.

5. Proof of Theorem 1.4

Let G be a finite group. In our identification $C(G) \cong B^*(G)$, we have until now only considered the additive group structure on $C(G)$. But $C(G)$ has also a natural ring structure, given by pointwise multiplication of superclass functions. We will now consider the group $C(G)^\times$ of units of the ring $C(G)$.

Of course, since the units group of the ring $\mathbb{Z}$ is $\{\pm 1\}$, an element f of $C(G)$ is in $C(G)^\times$ if and only if f takes values in $\{\pm 1\}$. It follows that $C(G)^\times$ is an elementary abelian 2-group of rank equal to the number of conjugacy classes of subgroups of G. Hence if we define the exponential map $\gamma : C(G) \to C(G)^\times$ by

$$\gamma(f)(K) = (-1)^{f(K)}$$

for any $f \in C(G)$ and any subgroup K of G, we get a surjective group homomorphism $C(G) \to C(G)^\times$, which factors as

$$\gamma : C(G) \to \mathbb{F}_2 C(G) \xrightarrow{\bar{\gamma}} C(G)^\times$$

where the left map is the mod 2 reduction $C(G) \to \mathbb{F}_2 C(G)$, and the map $\bar{\gamma}$ is a group isomorphism $\mathbb{F}_2 C(G) \to C(G)^\times$.

We can now use this isomorphism to endow the assignment $G \mapsto C(G)^\times$ with a biset functor structure, since $\mathbb{F}_2 C \cong \mathbb{F}_2 B^*$ is a biset functor. We denote this biset functor by $C^\times$: in other words, for any finite group G, we set $C^\times(G) = C(G)^\times$. If G and H are finite groups, if U is a finite (H, G)-biset, and if $f \in C^\times(G)$, then the value at the subgroup K of H of the superclass function $C^\times(U)(f)$ is obtained by the following formula, similar to 3.4:

$$C^\times(U)(f)(K) = \prod_{u \in [K \backslash U / G]} f\big(K^u\big).$$

This formula shows in particular that the embedding of $B^\times(G)$ into $C^\times(G)$ given by the Mark homomorphism (also called *the ghost map*) is a morphism of biset functors (see Section 5 of [8] for details).

Now composing $\gamma : C(G) \to C^\times(G)$ with the dimension function $\mathrm{Dim} : R_{\mathbb{R}}(G) \to C(G)$, we obtain the homomorphism

$$\Theta = \gamma \circ \mathrm{Dim} : R_{\mathbb{R}}(G) \to C^\times(G)$$

which is known as tom Dieck's homomorphism. It is shown by tom Dieck that the image of Θ lies in $B^\times(G)$. Moreover, by a result of Tornehave [13] it is known that tom Dieck's homomorphism is surjective onto $B^\times(G)$ when G is a p-group (see also [14] for an alternative proof). One of the consequences of Tornehave's result is the following:

Proposition 5.1. *Let P be a p-group. Then, $\gamma(C_b(P)) = B^\times(P)$.*

Proof. First we show that $\gamma(C_b(P))$ lies in $B^\times(P)$. Take $u = \gamma(f)$ with $f \in C_b(P)$. Since the image of Dim is equal to $C_b(P)$ when P is a p-group, there is a virtual real representation $\xi \in R_{\mathbb{R}}(P)$ such that $\mathrm{Dim}(\xi) = f$. This gives $u = \gamma(f) = \Theta(\xi)$, so u is in $B^\times(P)$ as desired. Now, the equality $\operatorname{im}\gamma = B^\times(P)$ follows from Tornehave's result that Θ is surjective onto $B^\times(P)$ for a p-group. □

Now, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let P be a p-group. Consider the following commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & C_b(P) \arrow[r, "j_P"] \arrow[d, "\gamma|_{C_b(P)}"] & C(P) \arrow[r, "\Psi_P"] \arrow[d, "\gamma"] & D^\Omega(P) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B^\times(P) \arrow[r] & C^\times(P) \arrow[r] & Q(P) \arrow[r] & 0
\end{tikzcd}$$

where the first exact sequence is the one given in Theorem 1.2 and $Q(P)$ denotes the quotient group $C^\times(P)/B^\times(P)$. Taking the mod 2 reduction of the top sequence, we obtain

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{F}_2C_b(P)/\operatorname{Tor}_{\mathbb{F}_2} D^\Omega(P) \arrow[r] \arrow[d, "\bar\gamma|_{C_b(P)}"] & \mathbb{F}_2C(P) \arrow[r, "\Psi_2"] \arrow[d, "\bar\gamma"] & \mathbb{F}_2D^\Omega(P) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B^\times(P) \arrow[r] & C^\times(P) \arrow[r] & Q(P) \arrow[r] & 0.
\end{tikzcd}$$

Since $\bar\gamma|_{C_b(P)}$ is surjective and $\bar\gamma$ is an isomorphism, all the vertical maps are isomorphisms. Hence,

$$\ker\Psi_2 \cong \mathbb{F}_2C_b(P)/\operatorname{Tor}_{\mathbb{F}_2} D^\Omega(P) \cong B^\times(P).$$

This completes the proof. □

Remark 5.2. One of the consequences of Theorem 1.4 is that for every p-group P, there is an exact sequence of elementary abelian groups ($\mathbb{F}_2$-vector spaces) of the form

$$0 \to B^\times(P) \to C^\times(P) \to \mathbb{F}_2D^\Omega(P) \to 0$$

where the first map is the ghost map. In [14], it has been shown that $B^\times(P)$ can be characterized as the subspace of $C^\times(P)$ satisfying certain conditions called Yoshida conditions (see Corollary 2.3 of [14]) and that these conditions can be viewed as a set of $\mathbb{F}_2$-linear forms coming from certain subquotients of P. The exact sequence above gives that $\mathbb{F}_2D^\Omega(P)$ has a presentation by Yoshida conditions as an $\mathbb{F}_2$-vector space.

In the rest of the section, we study the mod 2 reduction of the exact sequence in Theorem 1.3. We prove the following:

Proposition 5.3. *Let*

$$0 \to \operatorname{Tor}_{\mathbb{F}_2} D^\Omega \to \mathbb{F}_2C_b \to \mathbb{F}_2R_{\mathbb{Q}}^* \xrightarrow{\widetilde{\Psi}_2} \mathbb{F}_2D^\Omega_{\mathrm{tors}} \to 0$$

be the mod 2-reduction of the exact sequence given in Theorem 1.3. *Then, the kernel of* $\widetilde{\Psi}_2$ *is naturally equivalent to the functor* $B^\times$ *of units of the Burnside ring.*

Proof. Consider the commuting diagram of p-biset functors

$$\begin{tikzcd} 0 \arrow[r] & \ker \widetilde{\Psi}_2 \arrow[r] \arrow[d] & \mathbb{F}_2 R_{\mathbb{Q}}^* \arrow[r, "\widetilde{\Psi}_2"] \arrow[d, "i"] & \mathbb{F}_2 D_{\mathrm{tors}}^{\Omega} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B^\times \arrow[r] & \mathbb{F}_2 B^* \arrow[r, "\Psi_2"] & \mathbb{F}_2 D^{\Omega} \arrow[r] & 0. \end{tikzcd}$$

Note that by Theorem 1.8 of [5] there is an exact sequence of the form

$$0 \to R_{\mathbb{Q}}^* \to B^* \to D^{\Omega}/D_{\mathrm{tors}}^{\Omega} \to 0.$$

Since the cokernel $D^{\Omega}/D_{\mathrm{tors}}^{\Omega}$ is torsion free, this still gives us an exact sequence after tensoring with $\mathbb{F}_2$. Thus the second vertical map in the above diagram is injective with cokernel equal to $\mathbb{F}_2 \otimes_{\mathbb{Z}} (D^{\Omega}/D_{\mathrm{tors}}^{\Omega})$. By a similar argument one can see easily that the third vertical transformation is also injective with the same cokernel. So, the first vertical map is an isomorphism by the Snake lemma. □

Note that as a consequence of Proposition 5.3, we obtain two short exact sequences of p-biset functors:

$$0 \to B^\times \to \mathbb{F}_2 R_{\mathbb{Q}}^* \to \mathbb{F}_2 D_{\mathrm{tors}}^{\Omega} \to 0,$$
$$0 \to \mathrm{Tor}_{\mathbb{F}_2} D^{\Omega} \to \mathbb{F}_2 C_b \to B^\times \to 0.$$

From the proof of Proposition 5.3, it is easy to see that the second exact sequence is the same as the second exact sequence given in Corollary 1.5. We also observe that the first exact sequence above is the same as the exact sequence given in Proposition 9.11 of [8]. Thus our arguments provide a more natural way to see the exactness of the sequence given there.

Note that when p is an odd prime, the unit group $B^\times(P)$ is equal to $\{\pm 1\}$ for every p-group P, so in this case the second exact sequence above reduces to a sequence of the form

$$0 \to \mathrm{Tor}_{\mathbb{F}_2} D^{\Omega} \to \mathbb{F}_2 C_b \to \Gamma_{\mathbb{F}_2} \to 0$$

where $\Gamma_{\mathbb{F}_2}$ denotes the constant functor with values $\mathbb{F}_2$. This is closely related to an exact sequence given by Bouc and Thévenaz in [9]. The exact sequence given there is of the form

$$0 \to \overline{D}_{\mathrm{tors}} \to \mathbb{F}_2 R_{\mathbb{Q}} \to \Gamma_{\mathbb{F}_2} \to 0$$

where $\overline{D}_{\mathrm{tors}}$ is a quotient of the torsion part of the Dade group. It has been shown by Carlson and Thévenaz [11, Theorem 13.3] that actually $\overline{D}_{\mathrm{tors}} = D_{\mathrm{tors}}$, and by Theorem 6.2 and Theorem 11.2 of [9], it follows that $D_{\mathrm{tors}} = D_{\mathrm{tors}}^{\Omega}$ when p is an odd prime. It is clear from the second sequence above that D_{tors} is an $\mathbb{F}_2$-vector space. So, we can identify $\mathrm{Tor}_{\mathbb{F}_2} D^{\Omega}$ with D_{tors}. Thus, it makes sense to ask whether the two above sequences are the same. We have the following:

Theorem 5.4. *The p-biset functors $\mathbb{F}_2C_b$ and $\mathbb{F}_2R_{\mathbb{Q}}$ are naturally equivalent when p is congruent to* 3 *modulo* 4.

Proof. Let P be a p-group with $p > 2$. Consider the $\mathbb{F}_2$-linear map $\varphi:\mathbb{F}_2R_{\mathbb{Q}}(P) \to \mathbb{F}_2C_b(P)$ defined as the linear extension of the assignment $V \mapsto f_V$ where V is a rational representation of P and f_V is the Borel–Smith function given by

$$f_V(Q) = \frac{2}{p-1}\bigl(\dim_{\mathbb{Q}} V^Q - \dim_{\mathbb{Q}} V^P\bigr) + \dim_{\mathbb{Q}} V^P$$

for every subgroup $Q \leqslant P$. To see that $f_V(Q)$ is a Borel–Smith function, we first note that when $Q \leqslant R$ are two subgroups in P, then $p-1$ divides $\dim_{\mathbb{Q}} V^Q - \dim_{\mathbb{Q}} V^R$. This can be shown by an easy induction and by noting that it is true when Q has index p in R (for a cyclic group H of order p, it is clear that $\dim_{\mathbb{Q}} W - \dim_{\mathbb{Q}} W^H$ is divisible by $p-1$ for every $\mathbb{Q}H$-module W). This shows in particular that $f_V(Q)$ is an integer for all $Q \leqslant P$. It also shows that for every $Q \leqslant R$ with $|R:Q| = p$, we have $f_V(Q) \equiv f_V(R) \bmod 2$. To show that f_V also satisfies the Borel–Smith conditions coming from $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ sections, we just notice that f_V is a linear combination (with rational coefficients) of two Borel–Smith functions, namely $\operatorname{Dim} V$ and a constant function. So, it satisfies these Borel–Smith conditions as well.

Now, we need to verify that φ commutes with biset action. For this, it is enough to show that for any (P,R)-biset U and an $\mathbb{Q}R$-module V, the equality

$$C(U)f_V(Q) = f_{\mathbb{Q}U \otimes_{\mathbb{Q}R} V}(Q)$$

holds for every $Q \leqslant P$. Since the assignment $V \to \operatorname{Dim} V$ commutes with the biset action, we have

$$\begin{aligned} f_{\mathbb{Q}U\otimes_{\mathbb{Q}R}V}(Q) - C(U)f_V(Q) &= \frac{p-3}{p-1}\Biggl(\dim_{\mathbb{Q}}(\mathbb{Q}U \otimes_{\mathbb{Q}R} V)^P - \sum_{u \in Q\backslash U/R} \dim_{\mathbb{Q}} V^R\Biggr) \\ &= \frac{p-3}{p-1}\Bigl(\operatorname{Dim}(\mathbb{Q}U \otimes_{\mathbb{Q}R} V)(P) - |Q\backslash U/R| \dim_{\mathbb{Q}} V^R\Bigr). \end{aligned}$$

Since $p-3$ is a multiple of 4, and since $4R^*_{\mathbb{Q}} \subseteq 2C_b$, it is enough to show that the difference

$$D = \operatorname{Dim}(\mathbb{Q}U \otimes_{\mathbb{Q}R} V)(P) - |Q\backslash U/R| \dim_{\mathbb{Q}} V^R$$

is divisible by $p-1$. But this follows from the fact that the function $\operatorname{Dim} V$ is constant modulo $(p-1)$, and that there exists a constant biset functor modulo $p-1$. In other words,

$$\begin{aligned} \operatorname{Dim}(\mathbb{Q}U \otimes_{\mathbb{Q}R} V)(P) = \bigl(C(U)\operatorname{Dim} V\bigr)(P) &= \sum_{u \in P\backslash U/R} \dim_{\mathbb{Q}} V^{P^u} \\ &\equiv |P\backslash U/R| \dim_{\mathbb{Q}} V \quad \bigl(\mathrm{mod}\ (p-1)\bigr). \end{aligned}$$

Since $\dim_{\mathbb{Q}} V^R$ is also equal to $\dim_{\mathbb{Q}} V$ modulo $p-1$, we have that

$$D \equiv \bigl(|P\backslash U/R| - |Q\backslash U/R|\bigr) \dim_{\mathbb{Q}} V$$

modulo $p-1$. But $|P\backslash U/R|-|Q\backslash U/R|$ is divisible by $p-1$: this easily follows from the fact that for a cyclic group H of order p, the difference $|W|-|W/H|$ is divisible by $p-1$ for every H-set W. Thus $D \equiv 0$ modulo $p-1$, as was to be shown.

Finally, the fact that φ is an isomorphism follows from the general theory for rational biset functors. It is easy to see that both of these functors are rational and so it is enough to check the isomorphism on genetic sections, i.e. on p-groups of normal p-rank one, which are cyclic, in this case. For cyclic groups it is very easy to verify that φ is an isomorphism by direct calculation. □

Remark 5.5. In the case $p \equiv 1 \pmod 4$, then it is easy to check that the constant functions form a subfunctor of $\mathbb{F}_2 C_b$, isomorphic to $\Gamma_{\mathbb{F}_2}$. Since $\mathbb{F}_2 R_{\mathbb{Q}}$ has no such subfunctor [9, Corollary 8.4], it follows that the functors $\mathbb{F}_2 R_{\mathbb{Q}}$ and $\mathbb{F}_2 C_b$ are not isomorphic in this case.

We conclude the following

Corollary 5.6. *If* $p \equiv 3 \pmod 4$*, then the following two exact sequences are isomorphic*:

$$0 \to \mathrm{Tor}_{\mathbb{F}_2} D^{\Omega} \to \mathbb{F}_2 C_b \to B^{\times} \to 0,$$
$$0 \to D_{\mathrm{tors}} \to \mathbb{F}_2 R_{\mathbb{Q}} \to \Gamma_{\mathbb{F}_2} \to 0.$$

Proof. Since $2 \mid p-1$, the functor $\mathbb{F}_2 R_{\mathbb{Q}}$ has a unique proper nonzero subfunctor, by Corollary 8.4 of [9]. Since each of the functors $\mathrm{Tor}_{\mathbb{F}_2} D^{\Omega}$, $B^{\times}$, D_{tors} and $\Gamma_{\mathbb{F}_2}$ is nonzero, it follows that the isomorphism $\mathbb{F}_2 R_{\mathbb{Q}} \to \mathbb{F}_2 C_b$ of Theorem 5.4 maps the subfunctor D_{tors} of $\mathbb{F}_2 R_{\mathbb{Q}}$ to the subfunctor $\mathrm{Tor}_{\mathbb{F}_2} D^{\Omega}$ of $\mathbb{F}_2 C_b$, and induces the isomorphism $\Gamma_{\mathbb{F}_2} \to B^{\times}$. □

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