Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 315 (2007) 224–248
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Mackey functors, induction from restriction functors and coinduction from transfer functors

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Received 17 June 2006
Available online 18 May 2007
Communicated by Michel Broué

Abstract

Boltje's plus constructions extend two well-known constructions on Mackey functors, the fixed-point functor and the fixed-quotient functor. In this paper, we show that the plus constructions are induction and coinduction functors of general module theory. As an application, we construct simple Mackey functors from simple restriction functors and simple transfer functors. We also give new proofs for the classification theorem for simple Mackey functors and semisimplicity theorem of Mackey functors.

Keywords: Mackey functors; Restriction functors; Transfer functors; Plus constructions; Fixed-point functor; Induction; Coinduction; Simple Mackey functors

1. Introduction

The theory of Mackey functors was introduced by Green to provide a unified treatment of group representation theoretic constructions involving restriction, conjugation and transfer. Thévenaz and Webb improved Green's definition of a Mackey functor, and they realized Mackey functors as representations of the *Mackey algebra* $\mu_R(G)$. Using this identification, Thévenaz and Webb applied methods of module theory to classify the simple Mackey functors [11] and to describe the structure of Mackey functors [12]. Their description of simple Mackey functors

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[1] The author is partially supported by TÜBİTAK through PhD award program.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.04.026

used induction and inflation from subgroups and two dual constructions, known as the fixed-point functor and the fixed-quotient functor.

Applying the notion of Mackey functors to the problem of finding an explicit version of Brauer's induction theorem, Boltje introduced the theory of canonical induction [6,7]. In order to solve the problem in this general context, Boltje considered not only the category $\mathrm{Mack}_R(G)$ of Mackey functors, but also two more categories, namely the category $\mathrm{Con}_R(G)$ of conjugation functors and the category $\mathrm{Res}_R(G)$ of restriction functors. His main tools were the lower-plus and the upper-plus constructions, which extend the fixed-quotient and the fixed-point functors, respectively.

The lower-plus construction, denoted by $-_+$, is defined as a functor $\mathrm{Res}_R(G) \to \mathrm{Mack}_R(G)$. By introducing the *restriction algebra* $\rho_R(G)$, written ρ when R and G are understood, we realize the restriction functors as representations of the restriction algebra. This leads us to

Theorem 5.1. *The functors* $-_+$ *and* ind_ρ^μ *are naturally equivalent.*

On the other hand, the upper-plus construction, denoted by $-^+$, is defined as a functor $\mathrm{Con}_R(G) \to \mathrm{Mack}_R(G)$. By introducing the *transfer algebra* $\tau_R(G)$, written τ, and its representations, called *transfer functors* and realizing conjugation functors as representations of the *conjugation algebra* $\gamma_R(G)$, written γ, we prove

Theorem 5.2. *The functors* $-^+$ *and* $\mathrm{coind}_\tau^\mu \mathrm{inf}_\gamma^\tau$ *are naturally equivalent.*

As a consequence of these identifications, we realize the fixed-point and fixed-quotient functors as coinduced and induced modules, respectively. Given an RG-module V, we denote by FQ_V the fixed-quotient functor and by FP_V the fixed-point functor.

Proposition 5.4. *Let V be an RG-module. Then, the following isomorphisms hold.*

$$\text{(i)} \quad \mathrm{FQ}_V \simeq \mathrm{ind}_\rho^\mu \mathrm{inf}_\gamma^\rho D_V \quad \textit{and} \quad \text{(ii)} \quad \mathrm{FP}_V \simeq \mathrm{coind}_\tau^\mu \mathrm{inf}_\gamma^\tau D_V$$

where D_V denotes the γ-module which is non-zero only at the trivial group and $D_V(1) = V$.

We also prove that the Brauer quotient (also known as the bar construction) is the composition of certain restriction and deflation functors (see Corollary 5.7). Via this identification, we see that Thévenaz' twin functor is the composition of coinduction, inflation, deflation and restriction functors.

The plus constructions are also used by Bouc [4] and Symonds [9]. To obtain information about projective Mackey functors, Bouc considered restriction functors defined only on p-subgroups and also the functor $-_+$ (which is denoted by $\mathcal{I}$ in [4]). In [9], Symonds constructed induction formulae using the plus constructions described in terms of the zero degree group homology and group cohomology functors.

The subalgebra structure of the Mackey algebra, we describe above, leads us to

Theorem 3.2 *(Mackey structure theorem). The τ–ρ-bimodule ${}_\tau\mu_\rho$ is isomorphic to $\tau \otimes_\gamma \rho$.*

As a consequence of this theorem, we obtain several equivalences relating the functors between the algebras μ, τ, ρ and γ. Using some of these equivalences, we show that the well-known mark homomorphism corresponds to the identity map on conjugation functors (see Proposition 5.10).

Our module theoretic approach not only unearths the nature of some known constructions for Mackey functors, but also allows us to understand the classification of simple Mackey functors better. The classification theorem of Thévenaz and Webb [11] asserts that the simple Mackey functors are parameterized by the G-classes of *simple pairs* (H, V) where H is a subgroup of G and V is a simple $RN_G(H)/H$-module. It is easy to see that the simple conjugation functors are also parameterized by the G-classes of simple pairs (H, V). It is almost as easy that the simple restriction functors and the simple transfer functors are parameterized in the same way. As an application of our characterization of the plus constructions, we show how the classification theorem for simple Mackey functors follows quickly from the classification of the simple restriction functors. Moreover, we obtain two new descriptions of the simple Mackey functors. In the case where $|G|$ is invertible in the base field R, we see that induction from the restriction algebra and coinduction from the transfer algebra respect simple modules. We also give a new proof of the semisimplicity theorem [11], which states that the Mackey functors are semisimple when R is a field of characteristic coprime to $|G|$.

Let us mention that, in a sequel to this paper, we shall be adapting some of these methods and results to the content of biset functors.

The organization of the paper is as follows. In Section 2, we collect together necessary facts concerning the Mackey functors. In Section 3 we prove the Mackey structure theorem and its consequences. Section 4 contains the duality theorems. Our main results, the description of plus constructions via induction, coinduction and restriction are proved in Section 5. Also in this section, we give alternative descriptions of the fixed-point functor, the fixed-quotient functor, the twin functor and the mark homomorphism. The applications to the classification of simple Mackey functors and to the semisimplicity of Mackey functors are the contents of Sections 6 and 7, respectively.

2. Preliminaries

Let G be a finite group and R be a commutative ring with unity. Consider the free algebra on variables c_g^H, r_K^H, t_K^H where $K \leqslant H \leqslant G$ and $g \in G$. We define the *Mackey algebra* $\mu_R(G)$ for G over R as the quotient of this algebra by the ideal generated by the following six relations, where $L \leqslant K \leqslant H \leqslant G$ and $h \in H$ and $g, g' \in G$:

(1) $c_h^H = r_H^H = t_H^H$,
(2) $c_{g'}^{{}^gH} c_g^H = c_{g'g}^H$ and $r_L^K r_K^H = r_L^H$ and $t_K^H t_L^K = t_L^H$,
(3) $c_g^K r_K^H = r_{{}^gK}^{{}^gH} c_g^H$ and $c_g^H t_K^H = t_{{}^gK}^{{}^gH} c_g^K$,
(4) $r_J^H t_K^H = \sum_{x \in J\backslash H/K} t_{J \cap {}^xK}^J c_x r_{J^x \cap K}^K$ for $J \leqslant H$ (Mackey relation),
(5) $\sum_{H \leqslant G} r_H^H = 1$,
(6) all other products of generators are zero.

It is known that, letting H and K run over the subgroups of G and letting g run over the double coset representatives $HgK \subset G$ and letting L run over representatives of the subgroups of $H^g \cap K$ up to conjugacy, the elements $t_{{}^gL}^H c_g^L r_L^K$ run (without repetitions) over the elements of an R-basis for the Mackey algebra $\mu_R(G)$ (cf. [12, Section 3]).

We denote by $\rho_R(G)$, called the *restriction algebra* for G over R, the subalgebra of the Mackey algebra generated by c_g^H and r_K^H for $K \leqslant H \leqslant G$ and $g \in G$. We denote by $\tau_R(G)$ the *transfer algebra* for G over R the subalgebra generated by c_g^H and t_K^H for $K \leqslant H \leqslant G$ and $g \in G$. The *conjugation algebra*, denoted $\gamma_R(G)$, is the subalgebra generated by the elements c_g^H. When there is no ambiguity, we write $\mu = \mu_R(G)$, and $\rho = \rho_R(G)$ and $\tau = \tau_R(G)$ and $\gamma = \gamma_R(G)$. Evidently, the restriction algebra ρ has generators $c_g^J r_J^K$, the transfer algebra τ has generators $c_g^K t_J^K$ and the conjugation algebra γ has generators c_g^J.

We define a *Mackey functor* for G over R to be a $\mu_R(G)$-module. Similarly, we define a *restriction functor*, a *transfer functor* and a *conjugation functor* as a $\rho_R(G)$-module, a $\tau_R(G)$-module and a $\gamma_R(G)$-module, respectively.

We can also define a Mackey functor as a quadruple (M, c, r, t) consisting of a family of R-modules $M(K)$ for each $K \leqslant G$ and families of three types of maps:

(i) *conjugation maps*, $c_g^K : M(K) \to M({}^gK)$ for each $g \in G$ and $K \leqslant G$,
(ii) *restriction maps*, $r_L^K : M(K) \to M(L)$ for each $L \leqslant K \leqslant G$, and
(iii) *transfer maps*, $t_L^K : M(L) \to M(K)$ for each $L \leqslant K \leqslant G$.

These maps have to satisfy the relations (2), (3) and (4), above and the following relation

(1′) $c_h^H = r_H^H = t_H^H = \mathrm{id}_H$ for all $h \in H \leqslant G$.

We write M for the quadruple (M, c, r, t). Then to pass from the first definition to the second one, we put $M(K) = c_1^K M$ for each $K \leqslant G$ and conversely, we take $M = \bigoplus_{K \leqslant G} M(K)$. Similar comments apply to restriction and transfer and conjugation functors (cf. [8,12]).

Defining a *morphism* of Mackey functors to be an R-module homomorphism compatible with conjugation, restriction and transfer maps, we obtain the category $\mathrm{Mack}_R(G)$ of Mackey functors for G over R. Similarly, we have the category $\mathrm{Res}_R(G)$ of restriction functors, the category $\mathrm{Tran}_R(G)$ of transfer functors and $\mathrm{Con}_R(G)$ of conjugation functors.

Remark 2.1. In [4], Bouc introduced an algebra, denoted $r\mu_R(G)$, which is generated by c_g^G and r_K^H where $K \leqslant H \leqslant G$, $g \in G$ and H is a p-subgroup. He also introduced $t\mu_R(G)$ as the dual of $r\mu_R(G)$. Upper and lower plus constructions are also introduced in this settings.

In [7,8], Boltje introduced two functors $-^+ : \mathrm{Con}_R(G) \to \mathrm{Mack}_R(G)$ and $-_+ : \mathrm{Res}_R(G) \to \mathrm{Mack}_R(G)$, called upper-plus and lower-plus constructions, respectively. In Section 5, we show that these functors have descriptions as induction and coinduction functors. We review the constructions of these functors.

To a conjugation functor C, we associate a Mackey functor C^+ where for $H \leqslant G$, we define the modules as

$$C^+(H) = \Bigl(\prod_{L \leqslant H} C(L)\Bigr)^H.$$

Here H acts on the product by coordinate-wise conjugation. We define the maps for $K \leqslant H \leqslant G$ and $g \in G$ and $x_L \in C(L)$ as follows:

Conjugation:

$$c_g^{+H} : C^+(H) \to C^+\left({}^gH\right) \quad \text{where } (x_L)_{L \leqslant H} \mapsto \left({}^g x_{L^g}\right)_{L \leqslant {}^gH}.$$

Restriction:

$$r_K^{+H} : C^+(H) \to C^+(K) \quad \text{where } (x_L)_{L \leqslant H} \mapsto (x_L)_{L \leqslant K}.$$

Transfer:

$$t_K^{+H} : C^+(K) \to C^+(H) \quad \text{where } (x_L)_{L \leqslant K} \mapsto \sum_{h \in H/K} c_h^{+K}\left((x_L)_{L \leqslant K}\right).$$

The functor $-^+$ is defined on morphisms, in the obvious way, that is, if $f : B \to C$ is a morphism of conjugation functors, then $f^+ : B^+ \to C^+$ is defined by $f_H^+((x_L)_{L \leqslant H}) = (f_L(x_L))_{L \leqslant H}$.

To a restriction functor D, we associate a Mackey functor D_+ where for $H \leqslant G$, the modules are

$$D_+(H) = \left(\bigoplus_{L \leqslant H} D(L)\right)_H.$$

Here, for an RH-module M, we write M_H for the (maximal) H-fixed quotient, that is to say, $M_H = M/I(RH)M$ where $I(RH)$ denotes the augmentation ideal of RH. For $K \leqslant H$ and $a \in D(K)$, we write the image of a in $D_+(H)$ as $[K, a]_H$. Clearly, $[K, a]_H = [{}^hK, {}^ha]_H$ for $h \in H$ and as an R-module, $D_+(H)$ is generated by the elements $[K, a]_H$ for $K \leqslant H$ and $a \in D(K)$.

The maps are defined for $L \leqslant H \leqslant G$ and $g \in G$ as follows:

Conjugation:

$$c_{+g}^H : D_+(H) \to D_+\left({}^gH\right) \quad \text{where } [K, a]_H \mapsto \left[{}^gK, {}^ga\right]_{{}^gH}.$$

Restriction:

$$r_{+L}^H : D_+(H) \to D_+(L) \quad \text{where } [K, a]_H \mapsto \sum_{h \in L \backslash H/K} \left[L \cap {}^hK, r_{L \cap {}^hK}^{{}^hK}\left({}^ha\right)\right]_L.$$

Transfer:

$$t_{+L}^H : D_+(L) \to D_+(H) \quad \text{where } [N, b]_L \mapsto [N, b]_H.$$

For a morphism $f : D \to E$ of restriction functors, we define $f_+ : D_+ \to E_+$ by $f_{+H}([K, a]_H) = [K, f_K(a)]_H$ for $K \leqslant H$ and $a \in D(K)$.

The plus constructions are related to each other by a morphism, called the mark homomorphism, denoted by ρ in [7,8]. We write β for the mark homomorphism. It is defined as follows: Let D be a restriction functor and $H \leqslant G$. Then

$$\beta_H := \left(\pi_K \circ r_{+K}^H\right)_{K \leqslant H} : D_+(H) \to (\mathcal{F}D)^+(H)$$

where $\mathcal{F}:\mathrm{Res}_R(G)\to \mathrm{Con}_R(G)$ is the forgetful functor and π_K is the projection

$$\pi_K[L,a]_K = a$$

if $L=K$ and equal to zero otherwise. The mark homomorphism is an isomorphism if $|G|$ is invertible in R and is injective if $D_+(H)$ has trivial $|H|$-torsion for all $H\leqslant G$ (cf. [7, Proposition 1.3.2]).

The functors $-^+$ and $-_+$ have crucial use in constructing canonical induction formulae for Mackey functors. For further details, see [7,8], for applications see [4,9].

Two other constructions in the theory of Mackey functors that are used frequently are the bar construction and the twin functor. We review the definitions of these constructions.

Definition 2.2. Let M be a Mackey functor. The *bar construction* of M is the conjugation functor $\overline{M}$ where for $K\leqslant G$, we have

$$\overline{M}(K) = M(K)\Big/\sum_{L<K}\mathrm{Im}\big(t_L^K\big)$$

and the conjugation maps are inherited from those of M.

The bar construction composed with the functor $-^+$ gives the *twin functor* TM of M (cf. [7, Section 1.1.2]). We have the following morphism between a Mackey functor and its twin. For $K\leqslant G$ and $m\in M(K)$, we define

$$\beta_K : M(K)\to TM(K)$$

where $\beta_K(m) = (\pi_L(r_L^K m))_{L\leqslant K}$ and

$$\pi_K : M(K)\to \overline{M}(K)$$

is the quotient map. Note that the mark homomorphism is a special case of the morphism $\beta : M\to TM$ where we put $M=D_+$ for a restriction functor D.

Let $\mathcal{E}$ and $\mathcal{G}$ be rings and $\alpha:\mathcal{E}\to\mathcal{G}$ be a unital ring homomorphism. We can regard any $\mathcal{G}$-module as an $\mathcal{E}$-module by α. This induces a functor

$$\mathrm{res}_\alpha : \mathcal{G}\text{-mod}\to\mathcal{E}\text{-mod}$$

called the *generalized restriction*. There are two functors in the opposite direction.

Induction: We regard $\mathcal{G}$ as a right $\mathcal{E}$-module by $fe = f\alpha(e)$ for $e\in\mathcal{E}$ and $f\in\mathcal{G}$. Then, for any (left) $\mathcal{E}$-module M, we make $\mathcal{G}\otimes_{\mathcal{E}} M$ a (left) $\mathcal{G}$-module by $f(f'\otimes m) = ff'\otimes m$ for $m\in M$. Note that, the action is well-defined as the natural action of $\mathcal{G}$ on itself commutes with the action of $\mathcal{E}$ on $\mathcal{G}$. We call $\mathcal{G}\otimes_{\mathcal{E}} M$ the *induced module*, written $\mathrm{ind}_\alpha M$, and obtain the *generalized induction functor*

$$\mathrm{ind}_\alpha - := \mathcal{G}\otimes_{\mathcal{E}} - : \mathcal{E}\text{-mod}\to\mathcal{G}\text{-mod}.$$

Coinduction: Now we regard $\mathcal{G}$ as a left $\mathcal{E}$-module by $ef = \alpha(e)f$ for $e\in\mathcal{E}$ and $f\in\mathcal{G}$. Then, for any (left) $\mathcal{E}$-module M, we make $\mathrm{Hom}_{\mathcal{E}}(\mathcal{G}, M)$ a (left) $\mathcal{G}$-module by $(f\phi)(f') = \phi(f'f)$ for

$f, f' \in \mathcal{G}$ and $\phi \in \mathrm{Hom}_{\mathcal{E}}(\mathcal{G}, M)$. Note that, the natural action of $\mathcal{G}$ on itself commutes with the action of $\mathcal{E}$ on $\mathcal{G}$. We call $\mathrm{Hom}_{\mathcal{E}}(\mathcal{G}, M)$ the *coinduced module*, written $\mathrm{coind}_\alpha M$, and obtain the *generalized coinduction functor*

$$\mathrm{coind}_\alpha := \mathrm{Hom}_{\mathcal{E}}(\mathcal{G}, -) : \mathcal{E}\text{-mod} \to \mathcal{G}\text{-mod}.$$

We recall the adjointness properties of these three functors:

Proposition 2.3. *The induction functor* ind_α *is right adjoint of the restriction* res_α. *The coinduction functor* coind_α *is the left adjoint of the restriction* res_α.

The proof of the proposition and further details can be found in [2, Section 3.3]. In all our applications, α will be an inclusion $\mathcal{E} \hookrightarrow \mathcal{G}$ or a projection $\mathcal{E} \to \mathcal{G} = \mathcal{E}/\Delta$ for some ideal Δ of $\mathcal{E}$. For the first case, we write the induction and coinduction functors as $\mathrm{ind}_{\mathcal{E}}^{\mathcal{G}}$ and $\mathrm{coind}_{\mathcal{E}}^{\mathcal{G}}$, respectively. For the second case, we write induction and restriction as $\mathrm{def}_{\mathcal{G}}^{\mathcal{E}}$ and $\mathrm{inf}_{\mathcal{G}}^{\mathcal{E}}$, respectively.

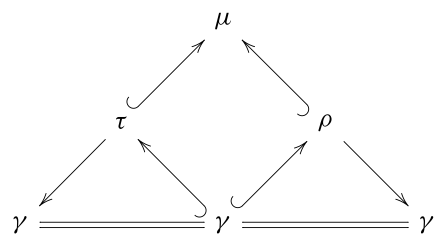
Finally, we recall the following well-known proposition.

Proposition 2.4. *(See [1, Section 2.8].) Let* $\mathcal{E}$ *and* $\mathcal{G}$ *be rings. Let* M *be a left* $\mathcal{G}$*-module and let* A *be a* $\mathcal{G}$*–*$\mathcal{E}$*-bimodule and let* N *be a left* $\mathcal{E}$*-module. Then, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{E}}\big(N, \mathrm{Hom}_{\mathcal{G}}(A, M)\big) \cong \mathrm{Hom}_{\mathcal{G}}(A \otimes_{\mathcal{E}} N, M).$$

3. The Mackey triangle

In this section, we examine the relations between the algebras μ, τ, ρ and γ. Mainly, we explain the following triangle, which we call the *Mackey triangle*.



Here the arrows $\rho \hookrightarrow \mu$ and $\tau \hookrightarrow \mu$ denote the inclusions of algebras and so are $\gamma \to \rho$ and $\gamma \to \tau$. The arrows $\rho \twoheadrightarrow \gamma$ and $\tau \twoheadrightarrow \gamma$ denote surjections explained in the next lemma, which also describes the identifications at the bottom of the triangle.

Lemma 3.1. *Let* $\mathcal{J}(\rho)$ *be the two-sided ideal of the restriction algebra* ρ *generated by all non-trivial restriction maps. Then, there is an evident identification* $\gamma = \rho/\mathcal{J}(\rho)$. *Similarly, we make the identification* $\gamma = \tau/\mathcal{J}(\tau)$ *where* $\mathcal{J}(\tau)$ *is generated by all non-trivial transfer maps.*

Proof. Recall that the restriction algebra (respectively transfer algebra) is generated by $c_g r_H^K$ where $H \leqslant K \leqslant G$ and $g \in G$. As an R-module, $\mathcal{J}(\rho)$ is spanned by the elements $c_g r_H^K$ where $K < H$. It is now clear that the quotient is isomorphic to the conjugation algebra. The last part can be proved similarly. □

The main property of the Mackey triangle is the following.

Theorem 3.2 *(Mackey structure theorem). The τ–ρ-bimodule ${}_{\tau}\mu_{\rho}$ is isomorphic to $\tau \otimes_{\gamma} \rho$.*

Proof. It is clear that $\tau \otimes_\gamma \rho$ is generated by the elements $t^H_{{}^gJ} \otimes c^J_g r^K_J$. Now we show that $\tau \otimes_\gamma \rho$ is freely generated by these elements. To this aim, we decompose the left γ-module ρ as

$$ {}_\gamma\rho = \bigoplus_{K \leqslant G,\, J \leqslant_K K} \gamma r^K_J $$

and the right γ-module τ as

$$ \tau_\gamma = \bigoplus_{H \leqslant G,\, I \leqslant_H H} t^H_I \gamma. $$

Then, the tensor product becomes

$$ \begin{aligned} \tau \otimes_\gamma \rho &= \bigoplus_{I \leqslant_H H \leqslant G,\, J \leqslant_K K \leqslant G} t^H_I \gamma \otimes_\gamma \gamma r^K_J \\ &= \bigoplus_{I \leqslant_H H \leqslant G,\, J \leqslant_K K \leqslant G,\, L \leqslant_G G} R t^H_I \gamma c^{[L]} \otimes_\gamma c^{[L]} \gamma r^K_J \end{aligned} $$

where $c^{[L]} = \sum_{L' =_G L} c^{L'}$. Here c^L is the generator c^L_1.

To focus on each summand separately, fix $H, K, L \leqslant G$. Then,

$$ c^{[L]} \gamma r^K_J = \bigoplus_{x \in G/K,\, L^x =_K J} R c^L c^J_x r^K_J. $$

Indeed, the equality holds since J is taken up to K-conjugacy and $c^L_x c^J = 0$ unless $L^x =_K J$. Similarly,

$$ t^H_I \gamma c^{[L]} = \bigoplus_{y \in H\backslash G,\, {}^yL =_H I} R t^H_I c^L_y c^L. $$

Hence,

$$ t^H_I \gamma c^{[L]} \otimes_\gamma c^{[L]} \gamma r^K_J = \bigoplus_{x,y} R t^H_I c^L_y \otimes c^J_x r^K_J. $$

Therefore,

$$ \begin{aligned} \tau \otimes_\gamma \rho &= \bigoplus_{H,K,I,J,L,x,y} R t^H_I c^L_y \otimes c^J_x r^K_J \\ &= \bigoplus_{H,K,I,J} \bigoplus_{g \in H\backslash G/K,\, I^g = J} R t^H_I \otimes c^J_g r^K_J. \end{aligned} $$

Hence, we see that $\tau \otimes_\gamma \rho$ is freely generated over R by the elements $t_{{}^gJ}^H \otimes c_g^J r_J^K$. It is also clear from the last equation that given $t_{{}^gJ}^H \otimes c_g^J r_J^K$ and $t_{{}^fI}^H \otimes c_f^I r_I^K$ then

$$t_{{}^gJ}^H \otimes c_g^J r_J^K = t_{{}^fI}^H \otimes c_f^I r_I^K$$

if and only if $HgK = HfK$ and J and I are $H^g \cap K$-conjugate. But this is equivalent to saying that $t_{{}^gJ}^H c_g^J r_J^K$ is equal to $t_{{}^fI}^H c_f^I r_I^K$ as elements of the Mackey algebra (cf. [12, Proposition 3.2]). Hence the correspondence

$$\Gamma : \tau \otimes_\gamma \rho \to \mu$$

given by $\Gamma(t_{{}^gJ}^H \otimes c_g^J r_J^K) = t_{{}^gJ}^H c_g^J r_J^K$ extends linearly to an isomorphism of R-modules. Evidently, the map Γ is compatible with the left action of the transfer algebra τ and the right action of the restriction algebra ρ. Thus Γ is an isomorphism of τ–ρ-bimodules from $\tau \otimes_\gamma \rho$ to ${}_\tau\mu_\rho$. □

Now as a result of these relations we obtain several induction, coinduction and restriction functors and some equivalences between them. As we shall see in the next section, some of these functors are also naturally equivalent to some well-known constructions. For the rest of this section, we prove some equivalences as consequences of Theorem 3.2. In the next lemma, which we state without proof, we collect some trivial but necessary observations about some of these functors:

Lemma 3.3. *In the Mackey triangle, there are two inflation functors,* inf_γ^τ *and* inf_γ^ρ. *For a* γ-*module* C, *the* τ-*module* $\mathrm{inf}_\gamma^\tau C$ *is the module* C *regarded as a* τ-*module by letting all non-trivial transfer maps* t_L^K *for* $L < K \leqslant G$ *act as zero maps. A similar result holds for the* ρ-*module* $\mathrm{inf}_\gamma^\rho C$. *Moreover, the compositions* $\mathrm{res}_\gamma^\tau \mathrm{inf}_\gamma^\tau$ *and* $\mathrm{res}_\gamma^\rho \mathrm{inf}_\gamma^\rho$ *are both naturally equivalent to the identity functor on* γ-*mod.*

For the rest of this section, we prove more equivalences. Most of the equivalences are consequences of the Mackey structure theorem.

Theorem 3.4. *The following natural equivalences hold.*

(i) $\mathrm{ind}_\gamma^\tau \mathrm{res}_\gamma^\rho \cong \mathrm{res}_\tau^\mu \mathrm{ind}_\rho^\mu$.
(ii) $\mathrm{coind}_\gamma^\rho \mathrm{res}_\gamma^\tau \cong \mathrm{res}_\rho^\mu \mathrm{coind}_\tau^\mu$.

Proof. The first equivalence is induced by the isomorphism Γ of $\tau - \rho$-bimodules μ and $\tau \otimes_\gamma \rho$ defined in the proof of Theorem 3.2. Indeed

$$\mathrm{ind}_\gamma^\tau \mathrm{res}_\gamma^\rho \cong \tau \otimes_\gamma \rho \otimes_\rho \quad \text{and} \quad \mathrm{res}_\tau^\mu \mathrm{ind}_\rho^\mu \cong {}_\tau\mu_\rho \otimes_\rho.$$

The induced equivalence is clearly natural. To prove the second equivalence, note that by the definition of coinduction,

$$\mathrm{coind}_\gamma^\rho \mathrm{res}_\gamma^\tau = \mathrm{Hom}_\gamma\big(\rho, \mathrm{Hom}_\tau(\tau, -)\big).$$

Now applying Proposition 2.4, we obtain a natural equivalence

$$\Upsilon : \mathrm{Hom}_\gamma\big(\rho, \mathrm{Hom}_\tau(\tau, -)\big) \cong \mathrm{Hom}_\tau(\tau \otimes_\gamma \rho, -)$$

of functors with values in R-mod. It is easy to check that for any τ-module E, the isomorphism Υ_E is compatible with conjugation and restriction maps. But, in that case the right-hand side of the last equation becomes

$$\mathrm{Hom}_\tau(\tau \otimes_\gamma \rho, -) \cong \mathrm{res}^\mu_\rho \, \mathrm{coind}^\mu_\tau$$

since $\tau \otimes_\gamma \rho \cong \mu$ as left τ-modules. □

Corollary 3.5. *The following equivalences hold.*

(i) $\mathrm{ind}^\tau_\gamma \cong \mathrm{res}^\mu_\tau \, \mathrm{ind}^\mu_\rho \, \mathrm{inf}^\rho_\gamma$.
(ii) $\mathrm{coind}^\rho_\gamma \cong \mathrm{res}^\mu_\rho \, \mathrm{coind}^\mu_\tau \, \mathrm{inf}^\tau_\gamma$.

Proof. This follows from Theorem 3.4 and Lemma 3.3 by composing with the corresponding inflations. □

Finally, we have two more functors that are naturally equivalent to the identity functor on γ-mod. Let us write $\mathrm{codef}^\rho_\gamma$ for the left adjoint of the inflation inf^ρ_γ. Explicitly, for a ρ-module D and for $K \leqslant G$, we have

$$\mathrm{codef}^\rho_\gamma \, D(K) = \bigcap_{L<K} \mathrm{Ker}\big(r^K_L : D(K) \to D(L)\big)$$

and the conjugation maps are obtained from those for the ρ-module D. The other functor is the deflation functor def^τ_γ induced by the map of Lemma 3.1. Note also that we have a deflation functor def^ρ_γ and a codeflation functor $\mathrm{codef}^\tau_\gamma$, but we shall not introduce these as we will not use them.

Proposition 3.6. *The following equivalences hold*:

$$\mathrm{def}^\tau_\gamma \, \mathrm{ind}^\tau_\gamma \cong \mathrm{id}_\gamma \cong \mathrm{codef}^\rho_\gamma \, \mathrm{coind}^\rho_\gamma .$$

Proof. The equivalences follows easily from Lemma 3.3, since a left and a right adjoint of the identity functor and the identity functor are naturally equivalent to each other. □

4. Duality theorems

Theorem 3.4 suggests a duality in the Mackey triangle. In this section, we clarify this duality. Following [11], we denote by $-^{\mathrm{op}}$, the *opposite functor*, defined by

$$-^{\mathrm{op}} : \mu\text{-mod} \to \text{mod-}\mu$$

where for a left μ-module M, the right μ-module M^{op} is the same R-module M with the right Mackey functor structure given by

$$m\big(t^H_{{}^gJ}c_g r^K_J\big) = \big(t^K_J c_{g^{-1}} r^H_{{}^gJ}\big)m$$

where $t^H_{{}^gJ}c_g r^K_J \in \mu$ and $m \in M(H)$.

We have another *duality* (cf. [11])

$$\mathcal{D}_\mu : \mu\text{-mod} \to \text{mod-}\mu$$

where for a left μ-module M, we let $\mathcal{D}_\mu M$ to be the right μ-module $\mathrm{Hom}_R(M, R)$ where μ acts on the right as usual. Note that $\mathcal{D}_\mu M$ is the usual duality D^* in module theory. Clearly, these functors can be defined in the reverse direction, and we can compose one with the other to obtain

$$\mathcal{D}^{\mathrm{op}}_\mu : \mu\text{-mod} \to \mu\text{-mod}.$$

Note that there is no ambiguity writing $\mathcal{D}^{\mathrm{op}}_\mu$ since the functors commute.

The functors $\mathcal{D}_\mu M$ and $-^{\mathrm{op}}$ also induce functors on the modules of the subalgebras ρ and τ. Since $-^{\mathrm{op}}$ interchanges restriction and transfer maps, we obtain dualities

$$-^{\mathrm{op}} : \rho\text{-mod} \to \text{mod-}\tau \quad \text{and} \quad -^{\mathrm{op}} : \tau\text{-mod} \to \text{mod-}\rho.$$

On the other hand, the functor $\mathcal{D}_\mu$ induces

$$\mathcal{D}_\rho : \rho\text{-mod} \to \text{mod-}\rho \quad \text{and} \quad \mathcal{D}_\tau : \tau\text{-mod} \to \text{mod-}\tau.$$

The following theorem describes induction from right τ-modules and coinduction from right ρ-modules to right μ-modules.

Theorem 4.1 *(The first duality theorem). Let D be a ρ-module and E be a τ-module. Then*

(i) $(\mathrm{ind}^\mu_\rho D)^{\mathrm{op}} \cong \mathrm{ind}^\mu_\tau(D^{\mathrm{op}})$ *where* $\mathrm{ind}^\mu_\tau : \textit{mod-}\tau \to \textit{mod-}\mu$.
(ii) $(\mathrm{coind}^\mu_\tau E)^{\mathrm{op}} \cong \mathrm{coind}^\mu_\rho(E^{\mathrm{op}})$ *where* $\mathrm{coind}^\mu_\rho : \textit{mod-}\rho \to \textit{mod-}\mu$.

Proof. The first part is clear, since we have

$$\big(\mathrm{ind}^\mu_\rho D\big)^{\mathrm{op}} = (\mu \otimes_\rho D)^{\mathrm{op}} \cong D^{\mathrm{op}} \otimes_\tau \mu = \mathrm{ind}^\mu_\tau\big(D^{\mathrm{op}}\big).$$

The second part can be proved similarly. □

Combining the above functors, we can define

Definition 4.2. The *transfer-restriction duality* is the equivalence

$$\mathcal{D}^{\mathrm{op}}_\rho := \mathcal{D}^{-1}_\rho \circ -^{\mathrm{op}} : \tau\text{-mod} \to \rho\text{-mod}$$

of categories $\tau\text{-mod} \cong \rho\text{-mod}$. We call the inverse equivalence

$$\mathcal{D}^{\mathrm{op}}_\tau := \mathcal{D}^{-1}_\tau \circ -^{\mathrm{op}} : \rho\text{-mod} \to \tau\text{-mod}$$

the *restriction-transfer duality*.

Finally, note that $\mathcal{D}_\mu^{\mathrm{op}}$ induces a duality $\mathcal{D}_\gamma^{\mathrm{op}}$ on γ-modules. The following theorem describes the duality we promised earlier.

Theorem 4.3 *(Restriction-transfer duality). Let D be a ρ-module and E be a τ-module. Then*

(i) $\mathcal{D}_\mu^{\mathrm{op}}(\mathrm{ind}_\rho^\mu D) \cong \mathrm{coind}_\tau^\mu(\mathcal{D}_\tau^{\mathrm{op}} D)$.

(ii) $\mathcal{D}_\mu^{\mathrm{op}}(\mathrm{coind}_\tau^\mu E) \cong \mathrm{ind}_\rho^\mu(\mathcal{D}_\rho^{\mathrm{op}} E)$.

Proof. The first part follows from Proposition 2.3 as we have

$$\begin{aligned}\mathcal{D}_\mu^{\mathrm{op}}\big(\mathrm{ind}_\rho^\mu D\big) &= \mathrm{Hom}_R\big(\big(\mathrm{ind}_\rho^\mu D\big)^{\mathrm{op}}, R\big)\\ &= \mathrm{Hom}_R\big(\mathrm{ind}_\tau^\mu\big(D^{\mathrm{op}}\big), R\big)\\ &\cong \mathrm{Hom}_\tau\big(\mu, \mathrm{Hom}_R\big(D^{\mathrm{op}}, R\big)\big)\\ &= \mathrm{coind}_\tau^\mu\big(\mathcal{D}_\tau^{\mathrm{op}} D\big).\end{aligned}$$

Note that although the above isomorphism is an isomorphism of R-modules, it is easily checked that it is an isomorphism of left Mackey functors. The second statement can be proved similarly. □

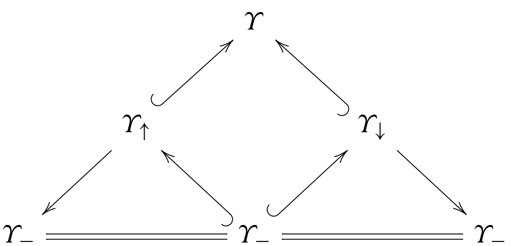
In the next theorem, we collect together some more dualities relating induction, coinduction and restriction. The theorem and any other duality can be proved in the same way.

Theorem 4.4. *Let M be a μ-module, E be a τ-module and C be a γ-module. Then*

(i) $\mathcal{D}_\tau^{\mathrm{op}}(\mathrm{res}_\rho^\mu M) \cong \mathrm{res}_\tau^\mu(\mathcal{D}_\mu^{\mathrm{op}} M)$.
(ii) $\mathcal{D}_\rho^{\mathrm{op}}(\mathrm{res}_\tau^\mu M) \cong \mathrm{res}_\rho^\mu(\mathcal{D}_\mu^{\mathrm{op}} M)$.
(iii) $\mathcal{D}_\tau^{\mathrm{op}}(\mathrm{inf}_\gamma^\rho C) \cong \mathrm{inf}_\gamma^\tau(\mathcal{D}_\gamma^{\mathrm{op}} C)$.
(iv) $\mathcal{D}_\rho^{\mathrm{op}}(\mathrm{inf}_\gamma^\tau C) \cong \mathrm{inf}_\gamma^\rho(\mathcal{D}_\gamma^{\mathrm{op}} C)$.
(v) $\mathcal{D}_\gamma^{\mathrm{op}}(\mathrm{def}_\gamma^\tau E) \cong \mathrm{codef}_\gamma^\rho(\mathcal{D}_\rho^{\mathrm{op}} E)$.

Let us end with an abstraction of the above situation. Let Υ be a finitely generated R-algebra such that it has subalgebras $\Upsilon_\uparrow$, $\Upsilon_\downarrow$ and Υ_- with the following two properties.

(i) The subalgebras together with Υ form the following triangle:



where the maps are as explained in the previous section.

(ii) The structure theorem, ${}_{\Upsilon_\uparrow}\Upsilon_{\Upsilon_\downarrow} \simeq \Upsilon_\uparrow \otimes_{\Upsilon_-} \Upsilon_\downarrow$, holds.

Then the results in Sections 3 and 4 hold for the modules of the algebras Υ, $\Upsilon_\uparrow$, $\Upsilon_\downarrow$ and Υ_- and for induction, coinduction and restriction functors. Moreover our classification and description of simple Mackey functors can be modified for the simple modules of the algebra Υ.

There are at least two more algebras having this structure. The first example is the algebra μ_A associated to a Green functor A (see [3] for the definition). Note that the Mackey algebra is obtained by taking $A = B^G$, the Burnside Mackey functor [3].

Another occurrence of this structure is in the biset functors, introduced by Bouc [5]. As mentioned in the introduction we shall adopt the methods of this paper to the analogous algebra for biset functors.

5. Plus constructions via induction and coinduction

In this section, we show that under the equivalence of categories $\mu_R(G) \cong \mathrm{Mack}_R(G)$, the plus constructions $-_+$ and $-^+$ are realizable in terms of generalized restriction, generalized induction and generalized coinduction. Moreover the well-known fixed-point functor and the fixed-quotient functor [11], and the twin functor [10] have similar descriptions. We begin by proving our first identification.

Theorem 5.1. *The functors $-_+$ and ind_ρ^μ are naturally equivalent.*

Proof. To specify a natural equivalence $\Phi : \mathrm{ind}_\rho^\mu \to -_+$, we must specify a map of μ-modules

$$\phi_D : \mathrm{ind}_\rho^\mu D \to D_+$$

for any ρ-module D and show that it is natural in D. To do that, we must specify an isomorphism of R-modules

$$\phi_{D,H} : \mathrm{ind}_\rho^\mu D(H) \to D_+(H)$$

for any subgroup $H \leqslant G$ which is compatible with the actions of transfer, restriction and conjugation. Now

$$\mathrm{ind}_\rho^\mu D(H) = \bigoplus_{K \leqslant_H H} \left\{ t_K^H \otimes_\rho a \colon a \in D(K) \right\}$$

where the notation indicates that K runs over representatives of the conjugacy classes of subgroups of H. Also,

$$D_+(H) = \bigoplus_{K \leqslant_H H} \left\{ [K, a]_H \colon a \in D(K) \right\}.$$

We have $t_K^H \otimes a_K = 0$ if and only if $a_K \in I(N_H(K))D(K)$, where $I(N_G(H))$ is the augmentation ideal as before. But this is equivalent to the condition that $[K, a]_H = 0$. So we can define $\Phi_{D,H}$ by

$$\Phi_{D,H}\left(t_K^H \otimes_\rho a\right) = [K, a]_H.$$

Thus, we have defined an R-module isomorphism $\Phi_D = (\Phi_{D,H})_{H \leqslant G}$ from $\operatorname{ind}_\rho^\mu D$ to D_+. Now we show that Φ_D is compatible with the actions of conjugation, restriction and transfer. We must also check that Φ is natural.

Given $L \leqslant G$ and $a \in D(K)$, then

$$\begin{aligned}
\Phi_{D,L}\big(r_L^H\big(t_K^H \otimes a\big)\big) &= \Phi_{D,L}\left(\sum_{h \in L\backslash H/K} t^L_{L\cap^h K} r^{^hK}_{L\cap^h K} c_h^K \otimes a\right) \\
&= \sum_{h \in L\backslash H/K} \Phi_{D,L}\big(t^L_{L\cap^h K} \otimes r^{^hK}_{L\cap^h K} c_h^K a\big) \\
&= \sum_{h \in L\backslash H/K} \big[L \cap {}^hK, r^{^hK}_{L\cap^h K} c_h^K\, a\big]_L \\
&= r^H_{+L}[K,a]_H \\
&= r^H_{+L}\Phi_{D,H}\big(t_K^H \otimes a\big).
\end{aligned}$$

We have established compatibility with restriction, $\Phi_D r_L^H = R_L^H \Phi_D$. Compatibility with conjugation and transfer can be shown similarly (and more easily).

Finally, for the naturality, consider a map of ρ-modules $f : D \to D'$. The maps of μ-modules

$$\operatorname{ind}_\rho^\mu f : \operatorname{ind}_\rho^\mu D \to \operatorname{ind}_\rho^\mu D', \qquad f_+ : D_+ \to D'_+$$

are given by $(\operatorname{ind}_\rho^\mu f)_H(t_K^H \otimes a) = t_K^H \otimes f_K(a)$ and $(f_+)_H([K,a]_H) = [K, f_K(a)]_H$. Hence,

$$\begin{aligned}
\Phi_{D'}\big(\operatorname{ind}_\rho^\mu f\big(t_K^H \otimes a\big)\big) &= \phi_{D'}\big(t_K^H \otimes f_K(a)\big) = \big[K, f_K(a)\big]_H = f_+\big([K,a]_H\big) \\
&= f_+\big(\Phi_D\big(t_K^H \otimes a\big)\big).
\end{aligned}$$

So $\Phi_{D'} \circ \operatorname{ind}_\rho^\mu f = f_+ \circ \Phi_D$, in other words, Φ is natural. □

Theorem 5.2. *The functors* $-^+$ *and* $\operatorname{coind}_\tau^\mu \inf_\gamma^\tau$ *are naturally equivalent.*

Proof. As in the previous proof, to specify a natural equivalence $\Psi : \operatorname{coind}_\tau^\mu \inf_\gamma^\tau \to -^+$, we must specify a map of μ-modules

$$\Psi_C : \operatorname{coind}_\tau^\mu \inf_\gamma^\tau C \to C^+$$

for any γ-module C and show that it is natural in C. In order to do that, we must specify an R-module isomorphism

$$\Psi_{C,H} : \operatorname{coind}_\tau^\mu \inf_\gamma^\tau C(H) \to C^+(H)$$

for any subgroup $H \leqslant G$ and show that it is compatible with the action of transfer, restriction and conjugation. Now

$$\operatorname{coind}_\tau^\mu \inf_\gamma^\tau C(H) = \operatorname{Hom}_\tau\big(\mu, \inf_\gamma^\tau C\big)(H) = \operatorname{Hom}_\tau\big(\mu c^H, \inf_\gamma^\tau\big)$$

where $\inf_\gamma^\tau C = \bigoplus_{J \leqslant H} C(J)$. Recall that any element of μc^H is a linear combination of elements of the form $t_{{}^gJ}^K c_g r_J^H$ where $g \in G$ and $J \leqslant K^g \cap H$. But, for such an element and for a map $\phi : \mu c^H \to \inf_\gamma^\tau C$ of τ-modules, we have

$$\phi\bigl(t_{{}^gJ}^K c_g r_J^H\bigr) = t_{{}^gJ}^K \phi\bigl(c_g r_J^H\bigr) = 0$$

unless $K = {}^gJ$. Indeed, t_L^K annihilates the τ-module $\inf_\gamma^\tau C$ if $L \neq K$. Also, if $K = {}^gJ$, then

$$\phi\bigl(c_g r_J^H\bigr) = c_g \phi\bigl(r_J^H\bigr)$$

that is, the value of ϕ at $c_g r_J^H$ is determined by the value of ϕ at r_J^H. Moreover, for any $h \in H$, we have

$$\phi\bigl(r_{{}^hJ}^H\bigr) = \phi\bigl(r_{{}^hJ}^H c_h^H\bigr) = \phi\bigl(c_h^J r_J^H\bigr) = c_h^J\bigl(\phi\bigl(r_J^H\bigr)\bigr).$$

Now recall that

$$C^+(H) = \Bigl(\prod_{J \leqslant H} C(J)\Bigr)^H = \Bigl\{(x_J)_{J \leqslant H} \in \prod_{J \leqslant H} C(J)\colon\ {}^h(x_J) = x_{{}^hJ} \text{ for } J \leqslant H,\ h \in H\Bigr\}.$$

So, we can define

$$\Psi_{C,H}(\phi) = \bigl(\phi\bigl(r_J^H\bigr)\bigr)_{J \leqslant H}.$$

The map $\Psi_{C,H}$ is an isomorphism of R-modules from $\operatorname{coind}_\tau^\mu \inf_\gamma^\tau C(H)$ to $C^+(H)$ with the inverse given by

$$\Psi_{C,H}^{-1}(X) = \phi_X.$$

Here, $X = (x_J)_{J \leqslant H}$ and ϕ_X is the map defined by $\phi_X(c_g r_J^H) = {}^g(x_J)$. Thus, we have defined an R-module isomorphism $\Psi_C : \operatorname{coind}_\tau^\mu \inf_\gamma^\tau C \to C^+$. We must show that Ψ_C is compatible with the actions of conjugation, restriction and transfer. Also, we must check that Ψ is natural in C.

Given $J \leqslant H \leqslant K \leqslant G$ and $\phi \in \operatorname{coind}_\tau^\mu \inf_\gamma^\tau C(H)$, then

$$\begin{aligned}
\Psi_{C,H}\bigl(t_H^K\phi\bigr) &= \bigl(\bigl(t_H^K\phi\bigr)\bigl(r_J^K\bigr)\bigr)_{J \leqslant K} \\
&= \bigl(\phi\bigl(r_J^K t_H^K\bigr)\bigr)_{J \leqslant K} \\
&= \Bigl(\sum_{x \in J\backslash K/H} \bigl(t_{J \cap {}^xH}^J c_x\bigr)\bigl(\phi\bigl(r_{J^x \cap H}^H\bigr)\bigr)\Bigr)_{J \leqslant K} \\
&= \Bigl(\sum_{x \in J\backslash K/H,\ J \cap {}^xH = J} c_x\bigl(\phi\bigl(r_{J^x}^H\bigr)\bigr)\Bigr)_{J \leqslant K} \\
&= \Bigl(\sum_{x \in K/H,\ J^x \leqslant H} c_x\bigl(\phi\bigl(r_{J^x}^H\bigr)\bigr)\Bigr)_{J \leqslant K} \\
&= t_H^{+K}\bigl(\phi\bigl(r_J^H\bigr)\bigr)_{J \leqslant H} = t_H^{+K}\Psi_{C,H}(\phi).
\end{aligned}$$

We have established compatibility with transfer, $\Psi_{C,K} \circ t_H^K = t_H^K \Psi_{C,H}$. Compatibility with restriction and conjugation can be proved similarly. Finally, one can check that the transformation Ψ is natural as above. □

By Theorems 3.4 and 5.2, we obtain an explicit description of the functor coind_τ^μ.

Theorem 5.3. *Let E be a transfer functor. Then for $H \leqslant G$, we have*

$$\mathrm{coind}_\tau^\mu E(H) \cong \Big(\prod_{L \leqslant H} E(L)\Big)^H.$$

The actions of conjugation and restriction are the same as the actions of conjugation and restriction for the functor E^+, respectively, and the transfer map is defined for $\phi \in \mathrm{coind}_\tau^\mu E(H)$ and $K \geqslant H$ as

$$\big(t_H^K \phi\big)\big(r_J^K\big) = \sum_{k \in J\backslash K/H} \big(t_{J \cap {}^k H}^J c_k\big)\big(\phi\big(r_{J^k \cap H}^H\big)\big).$$

Proof. By Theorem 3.4, there is an isomorphism

$$\mathrm{res}_\rho^\mu \mathrm{coind}_\tau^\mu E \cong \mathrm{coind}_\gamma^\rho \mathrm{res}_\gamma^\tau E$$

of ρ-modules. Now by Corollary 3.5, we obtain

$$\mathrm{res}_\rho^\mu \mathrm{coind}_\tau^\mu E \cong \mathrm{res}_\rho^\mu \mathrm{coind}_\tau^\mu \inf_\gamma^\tau \mathrm{res}_\gamma^\tau E.$$

Now by Theorem 5.2, the right-hand side is $(\mathrm{res}_\gamma^\tau E)^+$ regarded as a restriction functor. Hence the isomorphism

$$\mathrm{coind}_\tau^\mu E(H) \cong \Big(\prod_{L \leqslant H} E(L)\Big)^H$$

holds. Evidently, the actions of conjugation and restriction are the same as those for the right-hand side. Finally it is clear that the action of transfer is given as above. □

Given an RG-module V, we denote by D_V the conjugation functor where $D_V(1) = V$ and $D_V(H) = 0$ for $1 \neq H \leqslant G$.

Proposition 5.4. *The following isomorphisms hold.*

(i) $\mathrm{FQ}_V \cong \mathrm{ind}_\rho^\mu \inf_\gamma^\rho D_V$.
(ii) $\mathrm{FP}_V \cong \mathrm{coind}_\tau^\mu \inf_\gamma^\tau D_V$.

Proof. It is clear from the construction of the fixed-point functor and the fixed-quotient functor that we have the following isomorphisms (cf. [11, Section 6]):

$$\mathrm{FP}_V \cong (D_V)^+ \quad \text{and} \quad \mathrm{FQ}_V \cong \big(\inf_\gamma^\rho D_V\big)_+.$$

Now the result follows from Theorem 5.1 and Theorem 5.2. □

Corollary 5.5. *(See [11, Proposition 6.1].) The functor* $\operatorname{ind}_\rho^\mu \operatorname{inf}_\gamma^\rho D_V$ *is left adjoint to the functor* $F : \mu\text{-mod} \to RG\text{-mod}$ *which sends a Mackey functor* M *to the RG-module* $c_1^1 M = M(1)$. *The right adjoint of* F *is* $\operatorname{coind}_\tau^\mu \operatorname{inf}_\gamma^\tau D_V$.

Proof. We have $\operatorname{inf}_\gamma^\rho D_V(K) = 0$ for each subgroup $1 < K \leqslant G$ and $\operatorname{inf}_\gamma^\rho D_V(1) = V$. Therefore

$$\begin{aligned}
\operatorname{Hom}_\mu\bigl(\operatorname{ind}_\rho^\mu \operatorname{inf}_\gamma^\rho D_V, M\bigr) &\cong \operatorname{Hom}_\rho\bigl(\operatorname{inf}_\gamma^\rho D_V, \operatorname{res}_\rho^\mu M\bigr)\\
&\cong \operatorname{Hom}_{RG}\bigl(\operatorname{inf}_\gamma^\rho D_V(1), \operatorname{res}_\rho^\mu M(1)\bigr)\\
&\cong \operatorname{Hom}_{RG}\bigl(V, M(1)\bigr)\\
&\cong \operatorname{Hom}_{RG}(V, FM).
\end{aligned}$$

The second statement is proved similarly. □

Remark 5.6. It is possible to define the fixed-point functor and the fixed-quotient functor for the right μ-modules, as well as the other constructions. For example, by the Duality Theorem 4.1, we see that

$$\mathcal{D}_\mu^{\mathrm{op}}\bigl(\operatorname{ind}_\rho^\mu \operatorname{inf}_\gamma^\rho D_V\bigr) = \operatorname{coind}_\tau^\mu \operatorname{inf}_\gamma^\tau \bigl(\mathcal{D}_\gamma^{\mathrm{op}} D_V\bigr)$$

which is the part (iii) of Proposition 4.1 in [12]. Also, note that we can define a fixed-point functor and a fixed-quotient functor for the right μ-modules using the functor $-^{\mathrm{op}}$. In that case, for a right RG-module V, we define

$${}_V\mathrm{FQ} := \operatorname{ind}_\tau^\mu \operatorname{inf}_\gamma^\tau {}_V D.$$

By the Duality Theorem 4.1, we obtain

$${}_V\mathrm{FQ} = \bigl(\operatorname{ind}_\rho^\mu \operatorname{inf}_\gamma^\rho D_{V^{\mathrm{op}}}\bigr)^{\mathrm{op}} = \operatorname{ind}_\tau^\mu \operatorname{inf}_\gamma^\tau D_V.$$

We can define ${}_V\mathrm{FP}$ similarly.

Finally, we have the following proposition.

Proposition 5.7. *The bar construction* $\bar{?}$, *defined in Definition* 2.2 *is naturally equivalent to* $\operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu$.

Proof. This is immediate from the equality

$$\bigl(\mathcal{J}(\tau)M\bigr)(H) = \sum_{L<H} t_L^H M(L)$$

for $H \leqslant G$. □

Corollary 5.8. *The twin functor* T *is naturally equivalent to* $\operatorname{coind}_\tau^\mu \operatorname{inf}_\gamma^\tau \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu$.

The morphism β between a Mackey functor and its twin can be expressed in terms of the above equivalence.

Proposition 5.9. *Let M be a Mackey functor. The morphism*

$$\beta : M \to \operatorname{coind}_\tau^\mu \operatorname{inf}_\gamma^\tau \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M$$

as an element in $\operatorname{Hom}_\mu(M, \operatorname{coind}_\tau^\mu \operatorname{inf}_\gamma^\tau \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M)$ *is induced by the identity endomorphism* $\operatorname{id}_{\operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M}$ *in* $\operatorname{Hom}_\gamma(\operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M, \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M)$.

Proof. By Proposition 2.3

$$\begin{aligned}\operatorname{Hom}_\mu\big(M, \operatorname{coind}_\tau^\mu \operatorname{inf}_\gamma^\tau \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M\big) &\cong \operatorname{Hom}_\tau\big(\operatorname{res}_\tau^\mu M, \operatorname{inf}_\gamma^\tau \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M\big)\\ &\cong \operatorname{Hom}_\gamma\big(\operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M, \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M\big).\end{aligned}$$

Now the counit of the adjunction

$$\operatorname{Hom}_\tau\big(\operatorname{res}_\tau^\mu M, \operatorname{inf}_\gamma^\tau \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M\big) \cong \operatorname{Hom}_\gamma\big(\operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M, \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M\big)$$

is given by composition with the quotient map. That is, for $\phi : \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M \to \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M$, the corresponding morphism $\overline{\phi} : \operatorname{res}_\tau^\mu M \to \operatorname{inf}_\gamma^\tau \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M$ is given by

$$\overline{\phi}_H(m) = \phi_H\big(\pi_H(m)\big)$$

where $m \in M(H)$ and $\pi : \operatorname{res}_\tau^\mu M \to \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M$ is the quotient map.

On the other hand, the counit of the adjunction

$$\operatorname{Hom}_\mu\big(M, \operatorname{coind}_\tau^\mu \operatorname{inf}_\gamma^\tau \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M\big) \cong \operatorname{Hom}_\tau\big(\operatorname{res}_\tau^\mu M, \operatorname{inf}_\gamma^\tau \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M\big)$$

is given by composition with the restriction maps. Explicitly, for $\psi : \operatorname{res}_\tau^\mu M \to \operatorname{inf}_\gamma^\tau \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M$, the corresponding morphism $\overline{\psi} : M \to \operatorname{coind}_\tau^\mu \operatorname{inf}_\gamma^\tau \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M$ is given by

$$\overline{\psi}_H(m) = \big(\psi_K\big(r_K^H m\big)\big)_{K \leqslant H}$$

where $m \in M(H)$.

Now put $\phi = \operatorname{id}$. Then $\overline{\phi}_H(m) = \pi_H(m)$ is the quotient map. Then, put $\psi = \overline{\phi}$ and get $\overline{\psi}_H(m) = (\psi_K(r_K^H m))_{K \leqslant H} = (\pi_K(r_K^H m))_{K \leqslant H}$, which coincides with the definition of the morphism β defined in Section 2. □

Since the mark homomorphism is a special case of the morphism β, we have the following corollary.

Corollary 5.10. *Let D be a ρ-module. The mark homomorphism*

$$\beta : \operatorname{ind}_\rho^\mu D \to \operatorname{coind}_\tau^\mu \operatorname{inf}_\gamma^\tau \operatorname{res}_\gamma^\rho D$$

is induced by the identity endomorphism $\operatorname{id}_{\operatorname{res}_\gamma^\rho D}$ *of the γ-module* $\operatorname{res}_\gamma^\rho D$.

Proof. Let us put $M = \operatorname{ind}_\rho^\mu D$ for some ρ-module D. Then by part (i) of Theorem 3.4 and by Proposition 3.6

$$\operatorname{coind}_\tau^\mu \inf_\gamma^\tau \operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu \operatorname{ind}_\rho^\mu D \cong \operatorname{coind}_\tau^\mu \inf_\gamma^\tau \operatorname{res}_\gamma^\rho D.$$

Also the quotient map π above coincides with the projection map π since $\operatorname{def}_\gamma^\tau \operatorname{res}_\tau^\mu M = D$. □

6. Simple Mackey functors

Throughout this section, we assume that R is a field. In [11], Thévenaz and Webb established a bijective correspondence between the G-classes of the simple pairs (H, V) where $H \leqslant G$ and V a simple $R\overline{N}_G(H)$-module where $\overline{N}_G(H) := N_G(H)/H$ and the isomorphism classes of the simple Mackey functors. We denote by $S_{H,V}$ the simple Mackey functor corresponding to the pair (H, V), under this correspondence.

To illustrate the usefulness of our module-theoretic approach we give an alternative proof to this result by realizing simple Mackey functors as quotients of induced simple restriction functors. As we shall see below, the classification of simple restriction functors is trivial. Then we give another new description of the simple Mackey functor $S_{H,V}$ as the unique minimal subfunctor of the Mackey functor $\operatorname{coind}_\tau^\mu \mathcal{S}_{H,V}^\tau$, where $\mathcal{S}_{H,V}^\tau$ is a simple transfer functor, introduced below.

Throughout this section, let $H \leqslant G$ and V be a simple $R\overline{N}_G(H)$-module. We write $\mathcal{S}_{H,V}^\gamma$ for the conjugation functor defined for $K \leqslant G$ by

$$\mathcal{S}_{H,V}^\gamma(K) = {}^gV \quad \text{if} \quad K = {}^gH$$

and zero otherwise. We also write $\mathcal{S}_{H,V}^\rho = \inf_\gamma^\rho \mathcal{S}_{H,V}^\gamma$ and $\mathcal{S}_{H,V}^\tau = \inf_\gamma^\tau \mathcal{S}_{H,V}^\gamma$.

Proposition 6.1. *(See [7, Remark 1.6.6], [4, Proposition 3.2].) The followings hold.*

(i) *The conjugation functor $\mathcal{S}_{H,V}^\gamma$ is simple. Moreover, any simple conjugation functor is isomorphic to $\mathcal{S}_{H,V}^\gamma$ for some simple pair (H, V).*
(ii) *The restriction functor $\mathcal{S}_{H,V}^\rho$ is simple. Moreover, any simple restriction functor is isomorphic to $\mathcal{S}_{H,V}^\rho$ for some simple pair (H, V).*
(iii) *The transfer functor $\mathcal{S}_{H,V}^\tau$ is simple. Moreover, any simple transfer functor is isomorphic to $\mathcal{S}_{H,V}^\tau$ for some simple pair (H, V).*

We recall, without proof, the description of the simple Mackey functors $S_{H,V} := S_{H,V}^G$ from [11]. Since the Mackey algebra $\mu_R(H)$ is a (non-unital) subalgebra of the Mackey algebra $\mu_R(G)$ for $H \leqslant G$ and the Mackey algebra $\mu_R(G/H)$ is a quotient of the Mackey algebra $\mu_R(G)$ for $H \trianglelefteq G$, we obtain an induction functor $\operatorname{ind}_{\mu_R(H)}^{\mu_R(G)}$ and an inflation functor $\inf_{\mu_R(G/H)}^{\mu_R(G)}$. Explicit descriptions of these functors are given in [11, Section 4,5].

Lemma 6.2. *(See [11, Lemma 8.1].) Let H be a subgroup of G and V be a simple $R\overline{N}_G(H)$. Then*

(i) *The functor* $M = \mathrm{ind}_{N_G(H)}^G \mathrm{inf}_{\overline{N}_G(H)}^{N_G(H)} FP_V$ *has a unique minimal subfunctor* $S_{H,V}^G$ *generated by* $M(H) = V$.

(ii) *The functor* $\mathrm{ind}_{N_G(H)}^G \mathrm{inf}_{\overline{N}_G(H)}^{N_G(H)} FQ_V$ *has a unique maximal subfunctor. Moreover, the quotient is isomorphic to* $S_{H,V}^G$.

Now we want to state the main result of this section. For this we need the following notation. Let M be a Mackey functor and $H \leqslant G$ be a minimal subgroup for M, that is, $M(L) = 0$ for $L < H$ and $M(H) \neq 0$. After [11], we define two subfunctors of M as follows:

$$\mathcal{I}_{M(H)}(K) = \sum_{L \leqslant K:\ L =_G H} \mathrm{Im}\big(t_L^K : M(L) \to M(K)\big)$$

and

$$\mathcal{K}_{M(H)}(K) = \bigcap_{L \leqslant K:\ L =_G H} \mathrm{Ker}\big(r_L^K : M(K) \to M(L)\big).$$

Theorem 6.3. *We have the following isomorphisms of Mackey functors*

$$S_{H,V}^G \simeq \mathrm{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho / \mathcal{K}_{\mathrm{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho(H)} \simeq \mathcal{I}_{\mathrm{coind}_\tau^\mu \mathcal{S}_{H,V}^\tau(H)}.$$

We prove the theorem in several steps. The first step is the following lemma.

Lemma 6.4. *The subfunctor* $\mathcal{K} = \mathcal{K}_{\mathrm{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho(H)}$ *of the Mackey functor* $\mathrm{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho$ *is the unique maximal subfunctor of* $\mathrm{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho$.

Proof. Let T be a proper subfunctor of $\mathrm{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho$. We are to show that $T \leqslant \mathcal{K}$, that is

$$T(K) \subset \bigcap_{L \leqslant K:\ L =_G H} \mathrm{Ker}\big(r_L^K\big)$$

for any $K \leqslant G$. So, we must show that for each $K \leqslant G$ and any $x \in T(K)$, we have $r_L^K x = 0$ for all $H =_G L \leqslant K$. But since $\mathrm{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho(H) = V$, it is evident that $T(L) = 0$ for any $L =_G H$. Indeed, otherwise $T(H) = V$ as V is a simple $R\overline{N}_G(H)$-module. But, by definition of the action of t_L^K, the functor $\mathrm{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho$ is generated by the images of the transfer maps t_L^K for $H =_G L \leqslant K$, that is, we have $\mathcal{I}_{\mathrm{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho(H)} = \mathrm{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho$. Hence the subfunctor T containing the subfunctor generated by $T(H) = V$ is not proper, contradicting our assumption. Thus, $T \leqslant \mathcal{K}$ as required. $\square$

We denote the simple quotient of $\mathrm{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho$ by

$$\tilde{S}_{H,V} = \mathrm{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho / \mathcal{K}.$$

Note that if (K, W) is another simple pair, then $\tilde{S}_{K,W}$ is not isomorphic to $\tilde{S}_{H,V}$. Indeed, since $\mathcal{K}(H) = 0$, the subgroup H is still a minimal subgroup of the quotient $\tilde{S}_{H,V}$ and similarly, K is a minimal subgroup of $\tilde{S}_{K,W}$. Hence for $K \neq H$, the simple modules $\tilde{S}_{H,V}$ and $\tilde{S}_{K,W}$ are non-isomorphic. Also, for $K = H$, any morphism $\tilde{S}_{H,V} \to \tilde{S}_{K,W}$ of Mackey functors induces a map $V \to W$ of $R\overline{N}_G(H)$-modules. But, by the Schur's lemma, any such map is either an isomorphism or the zero map. Thus $\tilde{S}_{H,V}$ is not isomorphic to $\tilde{S}_{K,W}$ unless $H = K$ and $V \cong W$.

Having the above description, we get another proof of Thévenaz and Webb's classification theorem:

Theorem 6.5. *Any simple Mackey functor is isomorphic to $\tilde{S}_{H,V}$ for some simple pair (H, V).*

Proof. Let S be a simple Mackey functor with a minimal subgroup H and $S(H) = V$. It suffices to show that there is a non-zero morphism of Mackey functors $\tilde{S}_{H,V} \to S$. We show that there is a morphism of Mackey functors $F : \operatorname{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho \to S$ such that $F_H \neq 0$.

By Proposition 2.4, we have

$$\operatorname{Hom}_\mu\big(\operatorname{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho, S\big) \simeq \operatorname{Hom}_\rho\big(\mathcal{S}_{H,V}^\rho, \operatorname{res}_\rho^\mu S\big).$$

But, $\mathcal{S}_{H,V}^\rho(K) = 0$ unless $K =_G H$. So the identity map $\mathrm{id}_V : V \to V$ of $R\overline{N}_G(H)$-modules induces a non-zero map $f : \mathcal{S}_{H,V}^\rho \to \operatorname{res}_\rho^\mu S$ of ρ-modules. Hence the corresponding map $F \in \operatorname{Hom}_\mu(\operatorname{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho, S)$ is non-zero. Moreover, since $\operatorname{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho(H) = V$, we have $F_H = f_H = \mathrm{id} \neq 0$. Thus, the induced morphism $\tilde{F} : \tilde{S}_{H,V} \to S$ is non-zero, as required. □

Hereafter, we identify $\tilde{S}_{H,V}$ with $S_{H,V}^G$ and write $S_{H,V}$ when the group G is understood. We complete the proof of Theorem 6.3 by the following lemma.

Lemma 6.6. *The subfunctor $\mathcal{I} = \mathcal{I}_{\operatorname{coind}_\tau^\mu \mathcal{S}_{H,V}^\tau(H)}$ generated by $\mathcal{I}(H) = V$ is the unique minimal subfunctor. Moreover, the subfunctor $\mathcal{I}$ is isomorphic to $S_{H,V}$.*

Proof. Let T be a non-zero subfunctor of $\operatorname{coind}_\tau^\mu \mathcal{S}_{H,V}^\tau$. We must show that $\mathcal{I} \leqslant T$. It suffices to show that $T(H) \neq 0$. Indeed, in that case, since $\operatorname{coind}_\tau^\mu \mathcal{S}_{H,V}^\tau(H) = V$ is simple, $T(H) = V$ and hence $\mathcal{I} \leqslant T$. But $\mathcal{K}_{\operatorname{coind}_\tau^\mu \mathcal{S}_{H,V}^\tau(H)} = 0$ by the definition of the map r_H^K and the diagram

$$\begin{array}{ccc} T(H) & \xrightarrow{\ i\ } & \operatorname{coind}_\tau^\mu \mathcal{S}_{H,V}^\tau(H) \\ \big\downarrow{\scriptstyle r_H^K} & & \big\downarrow{\scriptstyle r_H^K} \\ T(K) & \xrightarrow[\ i\ ]{} & \operatorname{coind}_\tau^\mu \mathcal{S}_{H,V}^\tau(K) \end{array}$$

commutes for all $g \in G$. That is to say, $r_H^K T(K) \neq 0$, or $T(H) \neq 0$. The last claim follows from the classification Theorem 6.5. □

To find the modules $S_{H,V}(K)$ for $K \leqslant G$, we need to know the subfunctor $\mathcal{K}$ of $\operatorname{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho$. In particular, when $\mathcal{K}$ is zero, we get a more explicit description. The following is a characterization of the subfunctor $\mathcal{K}$.

Lemma 6.7. *The subfunctor $\mathcal{K}$ of $\operatorname{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho(K)$ coincides with the kernel of the mark homomorphism $\beta:\operatorname{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho \to \operatorname{coind}_\tau^\mu \mathcal{S}_{H,V}^\tau$. Moreover, the subfunctor $\mathcal{I}$ of $\operatorname{coind}_\tau^\mu \mathcal{S}_{H,V}^\tau$ is the image of β.*

Proof. As $\mathcal{K}$ is the unique maximal subfunctor of $\operatorname{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho(K)$, we have $\ker\beta \subset \mathcal{K}$. So, it suffices to show the inverse inclusion. Given $K \leqslant G$ and $x \in \mathcal{K}(K)$, then

$$\beta_K(x) = \big(\eta_L\big(r_L^K x\big)\big)_{L\leqslant K,\, L=_G H} = 0$$

since $r_L^K x = 0$ by definition of $\mathcal{K}(K)$. Therefore, $\mathcal{K} \subset \ker\beta$. The second claim is easy since β_H is identical. □

Now, using the next proposition from [7], and the above identification of the subfunctor $\mathcal{K}$, we describe $\mathcal{K}$, in some cases.

Proposition 6.8. *(Cf. [7, Proposition 1.3.2], [10, Section 3].) The mark homomorphism β_K is injective if $\operatorname{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho(K)$ has trivial $|K|$-torsion. It is an isomorphism if $|K|$ is invertible in R.*

Corollary 6.9.

(i) *If $\operatorname{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho(K)$ has trivial $|K|$-torsion, then $\mathcal{K}(K) = 0$.*
(ii) *If $\operatorname{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho(K)$ has trivial $|K|$-torsion for all $K \leqslant G$, then $\operatorname{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho$ is simple.*
(iii) *If $|G|$ is invertible in R, then $\operatorname{ind}_\rho^\mu \mathcal{S}_{H,V}^\rho \cong \operatorname{coind}_\tau^\mu \mathcal{S}_{H,V}^\tau$ is simple for any simple pair (H,V).*

Remark 6.10. In the case that $|G|$ is invertible in R, we get two different descriptions of $S_{H,V}(K)$ for $K \leqslant G$. By Corollary 6.9 and proof of Lemma 6.7, we have

$$S_{H,V}(K) = \Big(\bigoplus_{L\leqslant K:\ L={}^gH} {}^gV\Big)_K$$

with the maps t_K^N and r_K^N given explicitly in Section 2. Also, by Corollary 6.9, we have

$$S_{H,V}(K) = \Big(\prod_{L\leqslant K:\ L={}^gH} {}^gV\Big)^K$$

with the maps t_K^N and r_K^N given explicitly in Section 2.

7. Semisimplicity

Throughout this section, suppose R is a field in which $|G|$ is a unit. It is well known that the Mackey algebra over R is semisimple (see [7,11,12]). The first proof by Thévenaz and Webb [11] is constructive and uses the semisimplicity of the twin functor. In this section we reprove this result by giving a shorter proof of the fact that, in this case, the twin functor of a Mackey functor is isomorphic to itself.

Definition 7.1. (See Thévenaz [10].) Let M be a Mackey functor. A subgroup $H \leqslant G$ is called a *primordial subgroup for M* if $\mathrm{def}^{\tau}_{\gamma}\, \mathrm{res}^{\mu}_{\tau}\, M(H) \neq 0$.

Recall, without proof, the following lemma.

Lemma 7.2. *(See [11, Lemma 9.4].) Let M be a Mackey functor and χ be a subconjugacy closed family of subgroups of G. Then,*

$$M = \operatorname{Ker} r_{\chi} \oplus \operatorname{Im} t_{\chi}$$

where

$$\operatorname{Ker} r_{\chi}(K) = \bigcap_{L \leqslant K,\, L \in \chi} \operatorname{Ker} r^{K}_{L} \quad \text{and} \quad \operatorname{Im} t_{\chi}(K) = \sum_{L \leqslant K,\, L \in \chi} \operatorname{Im} t^{K}_{L}$$

are Mackey subfunctors.

As a consequence of this lemma, we obtain the following decomposition.

Lemma 7.3. *Let $\mathcal{P} = \{H_0, H_1, \ldots, H_n\}$ be the set of all primordial subgroups of a Mackey functor M taken up to conjugacy and indexed such that for $i < j$, no G-conjugate of H_j is contained in H_i. Let T_i denotes the subfunctor of M generated by* $\mathrm{def}^{\tau}_{\gamma}\, \mathrm{res}^{\mu}_{\tau}\, M(H_i)$*. Then*

$$M \cong \bigoplus_{H_i \in \mathcal{P}} T_i$$

as Mackey functors.

Proof. By Lemma 7.2, we have

$$M = T_0 \oplus \operatorname{Ker} r_{[H_0]}$$

where

$$\operatorname{Ker} r_{[H_0]} = \operatorname{Ker} r_{\chi} \quad \text{and} \quad T_0 = \operatorname{Im} t_{\chi}.$$

Here $[H_0]$ is the set of all G-conjugates of H_0 and χ is the subconjugacy closure of $[H_0]$. Indeed, we have the equalities since H_0 is a minimal subgroup for M. We denote $\operatorname{Ker} r_{[H_0]}$ by N_0. Then, clearly, $N_0(H_0) = 0$ and $N_0(H_1) = (\mathrm{def}^{\tau}_{\gamma}\, \mathrm{res}^{\mu}_{\tau}\, M)(H_1)$. Therefore, by Lemma 7.2 we obtain

$$N_0 = T_1 \oplus N_1$$

where $N_1 = \operatorname{Ker} r_{[H_1]}$. Note that H_1 is a minimal subgroup for N_0. Applying the same procedure, we obtain

$$M = \bigoplus_{H_i \in \mathcal{P}} T_i$$

as required. □

Let M_i denote the conjugation functor generated by $M_i(H_i) = \mathrm{def}^\tau_\gamma \, \mathrm{res}^\mu_\tau \, M(H_i)$.

Lemma 7.4. *There is an isomorphism of Mackey functors*

$$T_i \cong \mathrm{coind}^\mu_\tau \, \mathrm{inf}^\tau_\gamma \, M_i.$$

Proof. Decomposing M_i into simple summands and applying Corollary 6.9 to each summand, we obtain

$$\mathrm{ind}^\mu_\rho \, \mathrm{inf}^\rho_\gamma \, M_i \cong \mathrm{coind}^\mu_\tau \, \mathrm{inf}^\tau_\gamma \, M_i$$

where the isomorphism is given by the mark homomorphism. Note that we can decompose M_i into simple summands since it is clear that the conjugation algebra for G over R is semisimple when $|G|$ is invertible in R.

Now consider the following triangle.

$$\begin{array}{ccc} & T_i & \\ {}^{\psi}\nearrow & & \searrow^{\phi} \\ \mathrm{ind}^\mu_\rho \, \mathrm{inf}^\rho_\gamma \, M_i & \xrightarrow{\quad \beta \quad} & \mathrm{coind}^\mu_\tau \, \mathrm{inf}^\tau_\gamma \, M_i \end{array}$$

where β is the mark homomorphism and ψ is the induction morphism defined by $\psi(t^K_L \otimes v) = t^K_L v$ for $H =_G L \leqslant K \leqslant G$ and $v \in M_i(L)$. The map ψ is a morphism of Mackey functors since M_i is a minimal subgroup both for T_i and for $\mathrm{ind}^\mu_\rho \, \mathrm{inf}^\rho_\gamma \, M_i$ (thus ψ commutes with restriction). The map ϕ is given by

$$\phi_K(w) = \left(r^K_L w\right)_{L \leqslant K,\, L =_G H}$$

where $K \leqslant G$ and $w \in T_i(K)$. Note that ψ is surjective since T_i is generated by its value on the conjugacy class of H_i. Also, since t^K_L acts as the zero map on $\mathrm{inf}^\tau_\gamma \, M_i$ for $L \neq K$, the composition $\phi \circ \psi$ is the mark homomorphism, that is, the triangle commutes. Moreover since β is injective, the map ψ is also injective. Hence it is an isomorphism. Now it follows that $\phi = \beta\psi^{-1}$ is also an isomorphism, as required. □

Finally we are ready to prove the semisimplicity theorem.

Theorem 7.5. *(See [11, Theorem 9.1].) The Mackey algebra $\mu_R(G)$ is semisimple if R is a field of characteristic coprime to $|G|$.*

Proof. Assume the notation of the section. By Lemma 7.3 and Lemma 7.4, we have

$$M \cong \bigoplus_{H_i \in \mathcal{P}} \mathrm{coind}^\mu_\tau \, \mathrm{inf}^\tau_\gamma \, M_i.$$

Inflation and coinduction functors are additive. So decomposing $M_i(H_i)$ into simple $R\overline{N}_G(H_i)$-modules, we obtain a decomposition of the Mackey functor T_i. But, by Corollary 6.9, the Mackey functor $\mathrm{coind}^\mu_\tau \, \mathrm{inf}^\tau_\gamma \, M_i$ is simple if $M_i(H_i)$ is a simple $R\overline{N}_G(H_i)$-module. □

Corollary 7.6. *Let M and N be Mackey functors such that*

$$\mathrm{def}^{\tau}_{\gamma}\,\mathrm{res}^{\mu}_{\tau}\,M \cong \mathrm{def}^{\tau}_{\gamma}\,\mathrm{res}^{\mu}_{\tau}\,N$$

as conjugation functors. Then $M \cong N$ as Mackey functors. In particular,

$$M \cong \mathrm{coind}^{\mu}_{\tau}\,\mathrm{inf}^{\tau}_{\gamma}\,\mathrm{def}^{\tau}_{\gamma}\,\mathrm{res}^{\mu}_{\tau}\,M.$$

Proof. This follows from Theorem 7.5 since the simple summands of a Mackey functor M are determined by the γ-module $\mathrm{def}^{\tau}_{\gamma}\,\mathrm{res}^{\mu}_{\tau}\,M$. Note that the second statement is Corollary 4.4 in [10] and it holds for Mackey functors by Theorem 12.3 of that paper. □

Acknowledgment

I wish to thank my supervisor Laurence J. Barker and Ergün Yalçın for their guidance. I am also grateful to the referee for some important corrections and many detailed suggestions.

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