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# On lifting of operators to Hilbert spaces induced by positive selfadjoint operators

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## Abstract

We introduce the notion of induced Hilbert spaces for positive unbounded operators and show that the energy spaces associated to several classical boundary value problems for partial differential operators are relevant examples of this type. The main result is a generalization of the Krein–Reid lifting theorem to this unbounded case and we indicate how it provides estimates of the spectra of operators with respect to energy spaces.

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## 1. Introduction

One of the central problem in spectral theory refers to the estimation of the spectra of linear operators associated to different partial differential equations. Depending on the specific problem that is considered, we have to choose a certain space of functions, among

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which until now the most tractable have been proved to be the Banach spaces and the Hilbert spaces. The construction of the underlying Banach/Hilbert spaces associated to these linear operators is usually made by introduction of a norm, respectively an inner product. In this circle of ideas, the most general construction requires a factorization and a completion that introduces some ideal elements that are difficult to control. We mention here the pioneering work of K. Friedrichs [3].

On the other hand, since operators can be considered on different spaces, one of the problems of interest is to provide fairly general assumptions under which information on the spectrum of operators with respect to the spaces on which they are considered can be obtained from the preliminary information on the original spaces. Among the results on the invariance of the spectrum we recall, for instance, the Wiener's theorem stating that the convolution operator generated by a summable function has the same spectrum on each spaces of the type  $L_p$  for  $(1 \leq p \leq \infty)$ .

In addition, we mention that there are many other successful methods of investigation of this problem, among which we note those based on the theory of embedding spaces, interpolation theorems for operators [1], and on the maximum modulus principle for analytic functions, cf. N. Levinson [8]. There is a large number of articles on this topic, e.g., [4,12–17], to cite only a few.

In this paper we pursue a way opened by the works of M.G. Krein [6], W.T. Reid [11], P. Lax [7], and J. Dieudonné [2], where a general theory with applications to spectral properties of operators on different spaces is obtained. The core of this theory is a lifting theorem stating that under a certain intertwining relation, the operator can be lifted, with control on the norm. To our knowledge, the known results and applications of this theory have been considered only with respect to bounded operators. It is our aim to show how this can be extended to unbounded positive selfadjoint operators and to which extent the preservation of the spectra can be obtained in this case. We employed a rather general abstract scheme for induced spaces, having the advantage that it contains as special cases the energy spaces of K. Friedrichs (see also the further investigations of W.V. Petryshyn [10] and S.G. Mikhlin [9]).

The paper is organized as follows: in Section 2 we present the abstract definition and make some simple remarks, after which some motivation for this construction are presented, namely we show that this can be applied to some classical boundary value problems on fairly general domains. Section 3 is devoted to the main result of this paper, Theorem 3.1 on lifting of bounded operators, and we conclude this paper by taking into account a few consequences on the preservation of the spectra.

## 2. Hilbert spaces induced by positive operators

Let  $\mathcal{H}$  be a Hilbert space and  $A$  a densely defined positive operator in  $\mathcal{H}$  (in this paper, the positivity of an operator  $A$  means  $\langle Ax, x \rangle_{\mathcal{H}} \geq 0$  for all  $x \in \text{Dom}(A)$ ). A pair  $(\mathcal{K}, \Pi)$  is called a *Hilbert space induced by  $A$*  if:

- (i)  $\mathcal{K}$  is a Hilbert space;
- (ii)  $\Pi$  is a linear operator with domain  $\text{Dom}(\Pi) \supseteq \text{Dom}(A)$  and range in  $\mathcal{K}$ ;

- (iii)  $\Pi \text{Dom}(A)$  is dense in  $\mathcal{K}$ ;
- (iv)  $\langle \Pi x, \Pi y \rangle_{\mathcal{K}} = \langle Ax, y \rangle_{\mathcal{H}}$  for all  $x, y \in \text{Dom}(A)$ .

We first note that such an object always exists by showing actually that the so-called *energy space* introduced by K.O. Friedrichs [3] is an example of a Hilbert space induced by a positive operator. In addition, they are essentially unique in the following sense: two Hilbert spaces  $(\mathcal{K}_i, \Pi_i)$ ,  $i = 1, 2$ , induced by the same operator  $A$ , are called *unitary equivalent* if there exists a unitary operator  $U \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$  such that  $U\Pi_1 = \Pi_2$ .

**Proposition 2.1.** *Given a densely defined positive operator  $A$  in the Hilbert space  $\mathcal{H}$ , there exists a Hilbert space induced by  $A$  and it is unique, modulo unitary equivalence.*

**Proof.** We consider the inner product space  $(\text{Dom}(A), \langle \cdot, \cdot \rangle_A)$  where  $\langle x, y \rangle_A = \langle Ax, y \rangle_{\mathcal{H}}$  for all  $x, y \in \text{Dom}(A)$  and let  $\mathcal{K}_A$  be its quotient completion to a Hilbert space, that is, we factor out  $\text{Ker}(A)$  and complete the pre-Hilbert space  $(\text{Dom}(A)/\text{Ker}(A); \langle \cdot, \cdot \rangle_A)$  to a Hilbert space. Then letting  $\Pi_A$  be the composition of the quotient mapping  $\text{Dom}(A) \rightarrow \text{Dom}(A)/\text{Ker}(A)$  with the embedding of  $\text{Dom}(A)/\text{Ker}(A)$  into  $\mathcal{K}_A$ , we note that  $(\mathcal{K}_A, \Pi_A)$  is a Hilbert space induced by  $A$ .

On the other hand, if  $(\mathcal{K}_i, \Pi_i)$ ,  $i = 1, 2$ , are two Hilbert spaces induced by  $A$ , then

$$\langle \Pi_1 x, \Pi_1 y \rangle_{\mathcal{K}_1} = \langle Ax, y \rangle_{\mathcal{H}} = \langle \Pi_2 x, \Pi_2 y \rangle_{\mathcal{K}_2}, \quad x, y \in \text{Dom}(A),$$

and hence the operator  $U$  is correctly defined by  $U\Pi_1 x = \Pi_2 x$ , for all  $x \in \text{Dom}(A)$ , and it is isometric. Due to the minimality assumption,  $\Pi_i \text{Dom}(A)$  is dense in  $\mathcal{K}_i$  for  $i = 1, 2$ , it follows that  $U$  can be uniquely extended to a unitary operator  $U \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ .  $\square$

In this paper we will be interested mainly in the case when the operator  $A$  is unbounded. For this reason it is necessary to make clear the connection between the boundedness of  $A$  and that of the inducing operator  $\Pi$ .

**Proposition 2.2.** *Let  $(\mathcal{K}, \Pi)$  be a Hilbert space induced by the positive and densely defined operator  $A$  in  $\mathcal{H}$ . Then  $A$  is bounded if and only if  $\Pi$  is bounded.*

**Proof.** Indeed, the axiom (iv) can be interpreted as  $\Pi^* \Pi \supseteq A$ . Note that the axiom (iii) implies that  $\Pi^*$  is densely defined, and hence  $\Pi$  is closable. Without restricting the generality, we thus can assume that  $\Pi$  is closed. Thus, if  $A$  is bounded it follows that  $\Pi^* \Pi$  is bounded and hence  $\Pi$  is bounded: this follows, e.g., by the polar decomposition.

Conversely, if  $\Pi$  is bounded, then from  $A \subseteq \Pi^* \Pi$  it follows that  $A$  has a bounded extension and hence it is bounded.  $\square$

**Remark 2.3.** (a) In the proof of the existence of a Hilbert space  $(\mathcal{K}, \Pi)$  induced by a positive densely defined operator  $A$  as in Proposition 2.1, the strong topology on the Hilbert space  $\mathcal{K}$  is not explicit. This is remedied if  $A$  is selfadjoint. Thus, if  $A$  is a positive selfadjoint operator in the Hilbert space  $\mathcal{H}$ , then  $A^{1/2}$  exists as a positive selfadjoint operator in  $\mathcal{H}$ ,  $\text{Dom}(A^{1/2}) \supseteq \text{Dom}(A)$  and  $\text{Dom}(A)$  is a core of  $A^{1/2}$ . In particular, we have

$$\langle Ax, y \rangle_{\mathcal{H}} = \langle A^{1/2} x, A^{1/2} y \rangle_{\mathcal{H}}, \quad x, y \in \text{Dom}(A),$$

which shows that we can consider the seminorm  $\|A^{1/2} \cdot\|$  on  $\text{Dom}(A)$  and make the quotient completion with respect to this seminorm in order to get a Hilbert space  $\mathcal{K}_A$ . We denote by  $\Pi_A$  the corresponding canonical operator. It is easy to see that  $(\mathcal{K}_A, \Pi_A)$  is a Hilbert space induced by  $A$ . Since, as observed in Proposition 2.1, all the Hilbert spaces induced by  $A$  are unitary equivalent, in this case we have a concrete representation of the strong topology of any Hilbert space induced by  $A$ . We call  $(\mathcal{K}_A, \Pi_A)$  the Hilbert space induced by  $A$  in the *energy space representation*.

(b) The construction in (a) can be made a bit more general. Let  $T \in \mathcal{C}(\mathcal{H}, \mathcal{H}_1)$ , that is,  $T$  is a closed linear operator with domain  $\text{Dom}(T)$  dense in the Hilbert space  $\mathcal{H}$  and range in the Hilbert space  $\mathcal{H}_1$ . Then  $A = T^*T$  is a positive selfadjoint operator in  $\mathcal{H}$  and  $\text{Dom}(T)$  is a core of  $A$ . We consider the linear manifold  $\text{Dom}(T)$  and the quadratic seminorm  $\text{Dom}(T) \ni x \mapsto \|Tx\|$  and let  $\mathcal{K}$  denote its quotient-completion to a Hilbert space. If  $\Pi$  denotes the composition of the canonical projection  $\text{Dom}(T) \rightarrow \text{Dom}(T)/\text{Ker}(T)$  with the canonical embedding of  $\text{Dom}(T)/\text{Ker}(T)$  into  $\mathcal{K}$ , then  $(\mathcal{K}, \Pi)$  is a Hilbert space induced by  $A$ . The construction in item (a) corresponds to  $T = |T| = (T^*T)^{1/2} = A^{1/2}$ .

(c) In the constructions made above, the Hilbert spaces  $\mathcal{K}$  induced by the positive operator  $A$  have strong topologies different from the original Hilbert space  $\mathcal{H}$ . In the following, we show another related construction for which the strong topologies of  $\mathcal{K}$  and  $\mathcal{H}$  coincide, but the cost is a more involved operator  $\Pi$ . Let  $T \in \mathcal{C}(\mathcal{H})$  and denote by  $\mathcal{K}$  the closure of  $\text{Ran}(T)$  in  $\mathcal{H}$ . Thus,  $\mathcal{K}$  is a subspace, that is, a closed linear submanifold, of  $\mathcal{H}$ . Then  $(\mathcal{K}, T)$  is a Hilbert space induced by  $A = T^*T$ . A special case is when  $T = |T| = (T^*T)^{1/2} = A^{1/2}$ .

(d) Finally, we illustrate a mixed situation: the completion is made within the underlying Hilbert space  $\mathcal{H}$  but the strong topologies are yet different. To see this, let  $T \in \mathcal{C}(\mathcal{H})$  and on the linear manifold  $\mathcal{K} = \text{Dom}(T)$  consider the quadratic norm  $\text{Dom}(T) \ni x \mapsto |x|_T = \|x\| + \|Tx\|$ , that is, the so-called *graph norm*. Then  $(\mathcal{K}; |\cdot|_T)$  is a Hilbert space. We let  $\Pi: \mathcal{H} \rightarrow \mathcal{K}$  be the canonical identification of  $\text{Dom}(T) = \text{Dom}(\Pi)$  with  $\mathcal{K}$  as sets. Note that the operator  $A = I + T^*T$  is positive selfadjoint in  $\mathcal{H}$  and, in addition, it is boundedly invertible, equivalently, bounded away from 0. Then  $(\mathcal{K}, \Pi)$  is a Hilbert space induced by  $A$ .

We now show how the energy spaces associated to several classical boundary value problems for partial differential equations can be put into the framework of Hilbert spaces induced by positive (selfadjoint) operators. We first fix some notation and recall some terminology and facts about some function Hilbert spaces, especially Sobolev spaces.

Let  $x = (x_1, \dots, x_n)$  denote the position vector in  $\mathbb{R}^n$ . We denote the differentiation operator  $D = (D_1, \dots, D_n)$ , where  $D_j = i \frac{\partial}{\partial x_j}$ ,  $j = 1, \dots, n$ , and  $i^2 = -1$ . For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$  denote its length by  $|\alpha| = \alpha_1 + \dots + \alpha_n$ , its factorial by  $\alpha! = \alpha_1! \cdots \alpha_n!$ , and let  $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ .

Let  $\Omega$  be an open set in  $\mathbb{R}^n$ . Here and in the following we assume that its boundary  $\partial\Omega$  is sufficiently smooth to allow surface measure and unit normal. The class  $\mathcal{C}_0^\infty(\Omega)$  of indefinitely differentiable complex valued functions with compact support in  $\Omega$  is dense in  $L_2(\Omega)$ , the Hilbert space of square integrable complex valued functions on  $\Omega$ , identified modulo Lebesgue negligible sets. Let  $u, v \in L_{1,\text{loc}}(\Omega)$  denote the space of locally inte-

grable functions on  $\Omega$ , that is, for any compact set  $K \subset \Omega$  we have  $\int_K |u| dx < +\infty$  and  $\int_K |v| dx < +\infty$ . If

$$\int_{\Omega} u \overline{D^{\alpha} \varphi} dx = \int_{\Omega} v \bar{\varphi} dx, \quad \varphi \in C_0^{\infty}(\Omega), \quad \alpha \in \mathbb{Z}_+^n,$$

then it is said that  $u$  is differentiable in the sense of distributions on  $\Omega$  and  $v = D^{\alpha} u$ . Recall that the case  $n = 1$  is special: if  $\Omega = (a, b)$ , then  $v = D^{\alpha} u$  in the sense of distributions if and only if  $u$  is  $(\alpha - 1)$ -times differentiable on the interval  $(a, b)$ ,  $w = u^{(\alpha-1)}$  is absolutely continuous on any compact interval in  $(a, b)$ , and  $w' = i^{\alpha} v$  a.e. on  $(a, b)$ .

More generally, let  $p \in \mathbb{C}[X_1, \dots, X_n]$  be a complex valued polynomial in  $n$  indeterminates. By  $p(D)$  we denote a partial differential expression and let  $p^*(D)$  denote the formal conjugate expression. If  $u, v \in L_{1,\text{loc}}(\Omega)$  are such that

$$\int_{\Omega} u \overline{p^*(D) \varphi} dx = \int_{\Omega} v \bar{\varphi} dx, \quad \varphi \in C_0^{\infty}(\Omega),$$

then it is said that  $v = p(D)u$  in the sense of distributions.

With the notation as above and  $l \in \mathbb{N}$ , we denote

$$W_2^l(\Omega) = \{u \in L_2(\Omega) \mid D^{\alpha} u \in L_2(\Omega), |\alpha| = l\}, \quad (2.1)$$

$$\|u\|_{l,\Omega}^2 = \sum_{|\alpha|=l} \frac{l!}{\alpha!} \|D^{\alpha} u\|_{L_2(\Omega)}^2, \quad u \in W_2^l(\Omega), \quad (2.2)$$

$$\|u\|_{W_2^l(\Omega)}^2 = \|u\|_{L_2(\Omega)}^2 + \|u\|_{l,\Omega}^2, \quad u \in W_2^l(\Omega). \quad (2.3)$$

Then  $(W_2^l(\Omega); \|\cdot\|_{W_2^l(\Omega)})$  is a Hilbert space, usually called a *Sobolev space*. In general,

$$\overline{C_0^{\infty}(\Omega)} W_2^l(\Omega) = \dot{W}_2^l(\Omega)$$

is a subspace (that is, a closed linear submanifold) of the Hilbert space  $W_2^l(\Omega)$ , but  $\dot{W}_2^l(\Omega) \neq W_2^l(\Omega)$ . However, if  $\Omega = \mathbb{R}^n$  we have  $\dot{W}_2^l(\Omega) = W_2^l(\Omega)$ , that is,  $C_0^{\infty}(\mathbb{R}^n)$  is dense in  $W_2^l(\mathbb{R}^n)$ .

On the other hand, if the open subset  $\Omega$  is bounded in  $\mathbb{R}^n$ , then the norm  $\|\cdot\|_{l,\Omega}$  on  $\dot{W}_2^l(\Omega)$  is equivalent with the norm  $\|\cdot\|_{W_2^l(\Omega)}$ . In addition, the class  $C^{\infty}(\overline{\Omega})$ , of complex valued functions on  $\Omega$  that admit an indefinitely differentiable prolongation to  $\mathbb{R}^n$ , is dense in  $W_2^l(\Omega)$ . Also, if  $u \in \dot{W}_2^l(\Omega)$ , then  $u|_{\partial\Omega} = 0$  a.e. with respect to the surface measure  $dS$  on the boundary  $S = \partial\Omega$ .

Again, the case  $l = 1$  is special. Letting

$$|u|_2^2 = \|u\|_{1,\Omega}^2 + \int_{\partial\Omega} |u|^2 dS, \quad u \in W_2^1(\Omega),$$

we have

$$\begin{aligned} \dot{W}_2^1(\Omega) &= \{u \in W_2^1(\Omega) \mid |u|_2 = \|u\|_{1,\Omega}\} \\ &= \{u \in W_2^1(\Omega) \mid u|_{\partial\Omega} = 0, dS\text{-a.e.}\}. \end{aligned} \quad (2.4)$$

### 2.1. The Neumann boundary value problem

Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  such that its boundary  $\partial\Omega$  is sufficiently smooth. We consider the selfadjoint operator  $A$  in  $L_2(\Omega)$  associated to the Neumann problem:

$$\begin{cases} u - \Delta u = f & \text{on } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.5)$$

where  $\partial/\partial\nu$  denotes the derivation with respect to the exterior normal.

More precisely, consider the Sobolev space  $W_2^1(\Omega)$  and note that, by its definition (2.1), we have  $W_2^1(\Omega) \hookrightarrow L_2(\Omega)$ , that is, it is continuously embedded in  $L_2(\Omega)$ . The hermitian form

$$a[u, v] = \int_{\Omega} u \bar{v} \, dx + \int_{\Omega} \langle \nabla u, \nabla v \rangle \, dx, \quad u, v \in D[a] = W_2^1(\Omega),$$

where  $D[a]$  denotes the form domain, is positive, bounded from below by 1, closed, and determines the positive selfadjoint operator  $A$ . By the Friedrichs theory we have that  $\text{Dom}(A^{1/2}) = D[a]$ , and hence  $A = A^*$  and  $A \gg 0$ , in particular  $\text{Ker}(A) = 0$ .

We let  $\mathcal{H} = L_2(\Omega)$ ,  $\mathcal{K} = W_2^1(\Omega)$ , as well as the linear operator  $\Pi : \mathcal{H} \rightarrow \mathcal{K}$  with  $\text{Dom}(\Pi) = W_2^1(\Omega)$ ,  $\Pi : u \mapsto u$  ( $u \in W_2^1(\Omega)$ ). We show that  $(\mathcal{K}, \Pi)$  is a Hilbert space induced by  $A$ :

- (i)  $\mathcal{K} = W_2^1(\Omega)$  is a Hilbert space with inner product

$$\begin{aligned} \langle u, v \rangle_{\mathcal{K}} &= \langle u, v \rangle_{L_2(\Omega)} + \langle u, v \rangle_{1, \Omega} \\ &= \int_{\Omega} u \bar{v} \, dx + \int_{\Omega} \langle \nabla u, \nabla v \rangle \, dx, \quad u, v \in W_2^1(\Omega). \end{aligned}$$

- (ii)  $\text{Dom}(\Pi) \supset \text{Dom}(A)$ . To see this, note that  $\text{Dom}(\Pi) = W_2^1(\Omega) = \text{Dom}(A^{1/2}) \supset \text{Dom}(A)$ .

- (iii)  $\Pi(\text{Dom}(A))$  is dense in  $\mathcal{K}$ . Since  $\Pi$  acts like identity this means that  $\text{Dom}(A)$  is dense in  $W_2^1(\Omega)$ .

- (iv)  $\langle \Pi u, \Pi v \rangle_{\mathcal{K}} = \langle Au, v \rangle_{\mathcal{H}}$ ,  $u, v \in \text{Dom}(A)$ . To see this, let  $u, v \in \text{Dom}(A)$ . Note that, by (ii), we have  $u, v \in \text{Dom}(\Pi)$ . By definition,

$$\langle \Pi u, \Pi v \rangle_{\mathcal{K}} = \langle u, v \rangle_{W_2^1(\Omega)} = \int_{\Omega} u \bar{v} \, dx + \int_{\Omega} \langle \nabla u, \nabla v \rangle \, dx.$$

Integrating by parts into the latter integral and taking into account that  $\partial\Omega$  is sufficiently smooth and that  $u, v \in \text{Dom}(A)$  implies that  $u$  and  $v$  are two times differentiable, it follows that

$$\begin{aligned} \langle \Pi u, \Pi v \rangle_{\mathcal{K}} &= \int_{\Omega} u \bar{v} \, dx - \int_{\Omega} u \overline{\Delta v} \, dx + \int_{\partial\Omega} u \frac{\partial \bar{v}}{\partial \nu} \, dS = \int_{\Omega} u \bar{v} \, dx - \int_{\Omega} u \overline{\Delta v} \, dx \\ &= \langle Au, v \rangle_{L_2(\Omega)}, \end{aligned}$$

where we took into account that  $\frac{\partial v}{\partial \nu} = 0$  on  $\partial\Omega$ .

This example is in the energy space representation of type  $(\mathcal{K}_A, \Pi_A)$  as in Remark 2.3(a). Equivalently, it can be treated in the representation  $A = I + T^*T$  as in Remark 2.3(d), where  $T = i\nabla$  is the operator in  $L_2(\Omega)$  with the domain

$$\text{Dom}(T) = \left\{ u \in W_2^1(\Omega) \mid \frac{\partial u}{\partial \nu} = 0 \right\}.$$

## 2.2. The Neumann boundary value problem for the Poisson equation

Let  $\Omega$  be an open and bounded subset in  $\mathbb{R}^n$  such that  $\partial\Omega$  is sufficiently smooth. Since  $\Omega$  is bounded it follows that  $1 \in L_2(\Omega)$ . Let

$$L_2^1(\Omega) = \{ f \in L_{1,\text{loc}}(\Omega) \mid D_j u \in L_2(\Omega), \ j = 1, \dots, n \}.$$

On  $L_2^1(\Omega)$  it is defined the nonnegative inner product

$$\langle u, v \rangle_{L_2^1(\Omega)} = \int_{\Omega} \langle \nabla u, \nabla v \rangle dx, \quad u, v \in L_2^1(\Omega).$$

This inner product is degenerate, in general, and hence it only yields a seminorm

$$\|u\|_{L_2^1(\Omega)} = \int_{\Omega} |\nabla u|^2 dx, \quad u \in L_2^1(\Omega).$$

In the following we assume that  $\Omega$  is chosen in such a way that the Poincaré inequality holds,

$$\int_{\Omega} |u|^2 dx \leq \left| \frac{1}{|\Omega|} \int_{\Omega} u dx \right|^2 + c \int_{\Omega} |\nabla u|^2 dx, \quad u \in L_2^1(\Omega), \quad (2.6)$$

where  $c > 0$  is a fixed constant, good for all  $u \in L_2^1(\Omega)$ . For example, this is true if  $\Omega$  is convex, star-shaped, etc. Under this assumption we have the continuous embedding  $L_2^1(\Omega) \hookrightarrow L_2(\Omega)$ . On  $L_2^1(\Omega)$  we introduce a new norm

$$\|u\|_1^2 = \left| \frac{1}{|\Omega|} \int_{\Omega} u dx \right|^2 + c \int_{\Omega} |\nabla u|^2 dx, \quad u \in L_2^1(\Omega).$$

Then  $(L_2^1(\Omega), \|\cdot\|_1)$  is unitarily equivalent with the Sobolev space  $W_2^1(\Omega)$ .

Let  $A$  be the positive selfadjoint operator generated by the Neumann problem for the Poisson equation:

$$\begin{cases} -\Delta u = f, & \text{on } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \quad (2.7)$$

More precisely, let

$$W_2^{1,0} = \left\{ u \in W_2^1(\Omega) \mid \int_{\Omega} u dx = 0 \right\}.$$

Due to the continuity of the integral in  $W_2^1(\Omega)$ , the linear manifold  $W_2^{1,0}$  is actually a subspace in  $W_2^1(\Omega)$ . On this subspace we consider the Dirichlet norm

$$\|u\|_{W_2^{1,0}(\Omega)}^2 = \int_{\Omega} |\nabla u|^2 dx, \quad u \in W_2^{1,0}(\Omega).$$

Then

$$W_2^{1,0}(\Omega) \subset L_2^0(\Omega) = \{u \in L_2(\Omega) \mid \langle u, 1 \rangle_{L_2(\Omega)} = 0\}.$$

Due to the Poincaré inequality it follows that  $W_2^{1,0}(\Omega)$  is continuously embedded and dense in  $L_2^0(\Omega)$ . The hermitian form

$$a[u, v] = \int_{\Omega} \langle \nabla u, \nabla v \rangle dx, \quad u, v \in D[a] = W_2^{1,0}(\Omega),$$

is closed and densely defined in  $L_2^0(\Omega)$  and

$$a[u, u] = \int_{\Omega} \langle \nabla u, \nabla u \rangle dx = \int_{\Omega} |\nabla u|^2 dx \geq c^{-1} \|u\|_{L_2^0(\Omega)}^2. \quad (2.8)$$

By the Friedrichs extension theory, it follows that there exists uniquely a positive selfadjoint operator  $A$  in  $L_2^0(\Omega)$  associated to the hermitian form  $a$ , that is,

$$a[u, v] = \langle Au, v \rangle_{L_2^0(\Omega)}, \quad u, v \in \text{Dom}(A). \quad (2.9)$$

In addition, by (2.8),  $A \gg 0$ , more precisely, the lower bound  $m(A) \geq c^{-1} > 0$ , where

$$m(A) = \inf \{ \langle Au, u \rangle_{L_2^0(\Omega)} \mid \|u\|_{L_2^0(\Omega)} = 1, u \in \text{Dom}(A) \}.$$

Let  $\mathcal{H} = L_2^0(\Omega)$ ,  $\mathcal{K} = W_2^{1,0}(\Omega)$  and the linear operator  $\Pi$  defined on  $\text{Dom}(\Pi) = W_2^{1,0}(\Omega) \subset L_2^0(\Omega)$  and valued in  $W_2^{1,0}(\Omega)$ ,  $\Pi u = u$  for all  $u \in W_2^{1,0}(\Omega)$ . We verify that  $(\mathcal{K}, \Pi)$  is a Hilbert space induced by  $A$ :

- (i)  $\mathcal{K} = W_2^{1,0}(\Omega)$  is a Hilbert space, as mentioned above.
- (ii)  $\text{Dom}(\Pi) \supset \text{Dom}(A)$ . Indeed, by (2.9) and Friedrichs construction, we have  $\text{Dom}(A) \subseteq \text{Dom}(A^{1/2}) = D[a] = W_2^{1,0}(\Omega) = \text{Dom}(\Pi)$ .
- (iii)  $\Pi(\text{Dom}(A))$  is dense in  $\mathcal{K}$ . Again, since  $\Pi$  acts like identity, this means that  $\text{Dom}(A)$  is dense in  $\mathcal{K}$ .
- (iv)  $\langle \Pi u, \Pi v \rangle_{\mathcal{K}} = \langle Au, v \rangle_{\mathcal{H}}$  for all  $u, v \in \text{Dom}(A)$ . Indeed, let  $u, v \in W_2^{1,0}(\Omega)$ . Then

$$\langle \Pi u, \Pi v \rangle_{\mathcal{K}} = \langle u, v \rangle_{W_2^{1,0}(\Omega)} = \int_{\Omega} \langle \nabla u, \nabla v \rangle dx,$$

and then, integrating by parts, we get

$$\langle \Pi u, \Pi v \rangle_{\mathcal{K}} = - \int_{\Omega} u \overline{\Delta v} dx + \int_{\partial \Omega} \frac{\partial v}{\partial \nu} dS = - \int_{\Omega} u \overline{\Delta v} dx = \langle Au, v \rangle_{L_2^0(\Omega)}.$$



Alternatively, we might take  $\mathcal{H} = L_2(\Omega)$  and let  $A$  be the positive selfadjoint operator on  $L_2(\Omega)$  generated by the Dirichlet problem. In this case  $A$  has a nontrivial kernel  $\mathcal{N}$ , of dimension 1,

$$\mathcal{N} = \{u \in L_2^1(\Omega) \mid \|u\|_{L_2^1(\Omega)} = 0\},$$

and the operator  $\Pi$  should be defined through the factorization

$$L_2(\Omega) \longrightarrow W_2^1(\Omega)/\mathcal{N} \simeq W_2^{1,0}(\Omega).$$

Moreover, we have  $A = T^*T$ , where  $T = i\nabla$  is the operator in  $L_2(\Omega)$  with the dense domain

$$\text{Dom}(T) = \left\{ u \in W_2^1(\Omega) \mid \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\},$$

and the boundary value condition should be understood in the sense of distributions.

### 2.3. The mixed boundary value problem of Zaremba for the Poisson equation

Again, let  $\Omega$  be a bounded and open subset of  $\mathbb{R}^n$ , with  $\partial\Omega$  sufficiently smooth, and  $\Gamma \subseteq \partial\Omega$  measurable with respect to the (hyper)surface measure  $dS$ , and such that  $|\Gamma| > 0$ . Denote  $\Gamma' = \partial\Omega \setminus \Gamma$ . We consider the space  $\mathring{W}_{2,\Gamma}^1(\Omega)$

$$\mathring{W}_{2,\Gamma}^1(\Omega) = \{u \in W_2^1(\Omega) \mid u|_{\Gamma} = 0\},$$

where the boundary condition should be understood in the sense of the restriction operator  $u \mapsto u|_{\partial\Omega}$ . We consider the restriction operator  $u \rightarrow u|_{\partial\Omega}$  with domain  $W_2^1(\Omega)$  and range in  $L_2(\partial\Omega)$ ; note that this operator is correctly defined on the dense set  $C^\infty(\bar{\Omega})$  in  $W_2^1(\Omega)$  and then it can be extended by continuity onto the whole space  $W_2^1(\Omega)$ . Clearly,  $u|_{\partial\Omega} = 0$  a.e. on  $\Gamma$  for all  $u \in \mathring{W}_{2,\Gamma}^1(\Omega)$ .

On the space  $\mathring{W}_{2,\Gamma}^1(\Omega)$  we consider the Dirichlet norm

$$\|u\|_{\mathring{W}_{2,\Gamma}^1(\Omega)}^2 = \int_{\Omega} |\nabla u|^2 dx, \quad u \in \mathring{W}_{2,\Gamma}^1(\Omega).$$

Due to the assumption  $|\Gamma| > 0$ , it follows that the norm  $\|\cdot\|_{\mathring{W}_{2,\Gamma}^1(\Omega)}$  is equivalent with the norm  $\|\cdot\|_{W_2^1(\Omega)}$ .

Recall that we have assumed  $\partial\Omega$  sufficiently smooth to admit surface measure and unit normal. Let  $A$  be the positive selfadjoint operator associated to the mixed (Zaremba) boundary value problem:

$$\begin{cases} -\Delta u = f, & \text{on } \Omega, \\ u = 0, & \text{on } \Gamma, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma'. \end{cases} \quad (2.10)$$

To describe the operator  $A$ , we proceed analogously as in the previous subsection. We consider, in the space  $L_2(\Omega)$ , the hermitian form

$$a[u, v] = \int_{\Omega} \langle \nabla u, \nabla v \rangle dx, \quad u, v \in D[a] = \mathring{W}_{2,\Gamma}^1(\Omega).$$

The form  $a$  is closed and densely defined in  $L_2(\Omega)$  and again, using the Friedrichs extension theory, we get a positive selfadjoint operator  $A$  in  $L_2(\Omega)$  for which

$$\langle Au, v \rangle_{L_2(\Omega)} = \int_{\Omega} \langle \nabla u, \nabla v \rangle dx, \quad u, v \in \text{Dom}(A).$$

Now, let  $\mathcal{H} = L_2(\Omega)$ ,  $\mathcal{K} = \dot{W}_{2,\Gamma}^1(\Omega)$  with the inner product

$$\langle u, v \rangle_{\mathcal{K}} = \int_{\Omega} \langle \nabla u, \nabla v \rangle dx, \quad u, v \in \mathcal{K},$$

and the linear operator  $\Pi$  with domain  $\text{Dom}(\Pi) = \dot{W}_{2,\Gamma}^1(\Omega) \subset L_2(\Omega)$  and valued in  $\mathcal{K}$ ,  $\Pi u = u$  for all  $u \in \text{Dom}(\Pi)$ . We verify that  $(\mathcal{K}, \Pi)$  is a Hilbert space induced by  $A$ :

- (i)  $\mathcal{K}$  is a Hilbert space. This is true because  $\dot{W}_{2,\Gamma}^1(\Omega)$  is a subspace of  $W_2^1(\Omega)$  and the equivalence of the norms  $\|\cdot\|_{\dot{W}_{2,\Gamma}^1(\Omega)}$  and  $\|\cdot\|_{W_2^1(\Omega)}$  on  $\dot{W}_{2,\Gamma}^1(\Omega)$ .
- (ii)  $\text{Dom}(\Pi) \supset \text{Dom}(A)$ . Indeed, this follows as in the previous examples, by observing that  $\text{Dom}(A) \subset \dot{W}_{2,\Gamma}^1(\Omega)$ .
- (iii)  $\Pi(\text{Dom}(A))$  is dense in  $\mathcal{K}$ . This is equivalent with saying that  $\text{Dom}(A)$  is dense in  $\dot{W}_{2,\Gamma}^1(\Omega)$ .
- (iv)  $\langle \Pi u, \Pi v \rangle_{\mathcal{K}} = \langle Au, v \rangle_{\mathcal{H}}$  for all  $u, v \in \text{Dom}(A)$ . Indeed, for  $u, v \in \text{Dom}(A)$  we integrate by parts, as in the previous examples, and use the boundary conditions to obtain

$$\begin{aligned} \langle \Pi u, \Pi v \rangle_{\mathcal{K}} &= \int_{\Omega} \langle \nabla u, \nabla v \rangle dx = - \int_{\Omega} u \overline{\Delta v} dx + \int_{\partial\Omega} u \frac{\partial \overline{v}}{\partial \nu} dS \\ &= - \int_{\Omega} u \overline{\Delta v} dx + \int_{\Gamma} u \frac{\partial \overline{v}}{\partial \nu} dS + \int_{\Gamma'} u \frac{\partial \overline{v}}{\partial \nu} dS \\ &= - \int_{\Omega} u \overline{\Delta v} dx = \langle Au, v \rangle_{L_2(\Omega)}. \end{aligned}$$

### 3. Lifting of bounded operators

The main result of this paper is the following lifting theorem for bounded operators with respect to Hilbert spaces induced by positive selfadjoint unbounded operators.

**Theorem 3.1.** *Let  $A$  and  $B$  be positive selfadjoint operators in the Hilbert spaces  $\mathcal{H}_1$  and respectively  $\mathcal{H}_2$ , and let  $(\mathcal{K}_A, \Pi_A)$  and  $(\mathcal{K}_B, \Pi_B)$  be the Hilbert spaces induced by  $A$  and respectively  $B$ . For any operators  $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$  and  $S \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$  such that*

$$\langle Bx, Ty \rangle_{\mathcal{H}_2} = \langle Sx, Ay \rangle_{\mathcal{H}_1}, \quad x \in \text{Dom}(B), \quad y \in \text{Dom}(A), \quad (3.1)$$

*there exist uniquely determined operators  $\tilde{T} \in \mathcal{B}(\mathcal{K}_A, \mathcal{K}_B)$  and  $\tilde{S} \in \mathcal{B}(\mathcal{K}_B, \mathcal{K}_A)$  such that  $\tilde{T}\Pi_A x = \Pi_B T x$  for all  $x \in \text{Dom}(A)$ ,  $\tilde{S}\Pi_B y = \Pi_A S y$  for all  $y \in \text{Dom}(B)$ , and*

$$\langle \tilde{S}h, k \rangle_{\mathcal{K}_A} = \langle h, \tilde{T}k \rangle_{\mathcal{K}_B}, \quad h \in \mathcal{K}_B, \quad k \in \mathcal{K}_A. \quad (3.2)$$

We divide the proof of Theorem 3.1 in three lemmas. The main technical ingredient is the inequality that makes the subject of the following Lemma 3.4. Basically, we employ the same idea as in [6,11], (see also [2,7]), to iterate the Schwarz inequality, but technically much more precautions should be taken: these are illustrated in the next two lemmas.

**Lemma 3.2.** *Under the notation and assumptions of Theorem 3.1 we have*

$$BTx = S^*Ax, \quad x \in \text{Dom}(A), \quad (3.3)$$

*in the sense that for any  $x \in \text{Dom}(A)$  we have  $Tx \in \text{Dom}(B)$  and (3.3) holds.*

**Proof.** Indeed, if  $x \in \text{Dom}(A)$ , then by (3.1) we have

$$\langle Tx, By \rangle_{\mathcal{H}_2} = \langle S^*Ax, y \rangle_{\mathcal{H}_1}, \quad y \in \text{Dom}(B),$$

and hence  $Tx \in \text{Dom}(B^*) = \text{Dom}(B)$  and  $BTx = S^*Ax$ .  $\square$

**Lemma 3.3.** *Under the notation and assumptions of Theorem 3.1, for any integer  $n \geq 0$  we have*

$$A(ST)^nh = (T^*S^*)^nAh, \quad h \in \text{Dom}(A), \quad (3.4)$$

*in the sense that for any  $h \in \text{Dom}(A)$  we have  $(ST)^nh \in \text{Dom}(A)$  and (3.4) holds.*

**Proof.** To prove this, we use induction. The case  $n = 0$  is trivial, so let  $n = 1$ , and  $h \in \text{Dom}(A)$  be arbitrary. By (3.3) we have  $Th \in \text{Dom}(B)$  and using (3.1) it follows that for any  $x \in \text{Dom}(A)$  we have

$$\langle Ax, STh \rangle_{\mathcal{H}_1} = \langle Tx, BTh \rangle_{\mathcal{H}_2} = \langle x, T^*S^*Ah \rangle_{\mathcal{H}_1},$$

and hence  $STh \in \text{Dom}(A^*) = \text{Dom}(A)$  and  $A(ST)h = (T^*S^*)Ah$ . To check the general induction step, let us assume that for an arbitrary, but fixed,  $n \geq 0$  and any  $h \in \text{Dom}(A)$  we have  $(ST)^nh \in \text{Dom}(A)$  and  $A(ST)^nh = (T^*S^*)^nAh$ . Fix  $h \in \text{Dom}(A)$ . Then  $(ST)^nh \in \text{Dom}(A)$  and, by (3.1) we have  $T(ST)^nh \in \text{Dom}(B)$  and

$$BT(ST)^nh = S^*A(ST)^nh = S^*(T^*S^*)^nAh.$$

Therefore, for arbitrary  $x \in \text{Dom}(A)$  we have

$$\begin{aligned} \langle Ax, (ST)^{n+1}h \rangle_{\mathcal{H}_1} &= \langle Ax, ST(ST)^nh \rangle_{\mathcal{H}_1} = \langle Tx, BT(ST)^nh \rangle_{\mathcal{H}_2} \\ &= \langle x, (T^*S^*)^{n+1}Ah \rangle_{\mathcal{H}_1}, \end{aligned}$$

and hence  $(ST)^{n+1}h \in \text{Dom}(A^*) = \text{Dom}(A)$  and  $(T^*S^*)^{n+1}Ah = A(ST)^{n+1}h$ . Thus, (3.4) is completely proved.  $\square$

**Lemma 3.4.** *Under the notation and assumptions of Theorem 3.1, we have*

$$\langle BTx, Tx \rangle_{\mathcal{H}_2}^2 \leq r(ST) \langle Ax, x \rangle_{\mathcal{H}_1}^2, \quad x \in \text{Dom}(A), \quad (3.5)$$

*in the sense that for any  $x \in \text{Dom}(A)$  we have  $Tx \in \text{Dom}(B)$  and the inequality (3.5) holds. Here,  $r(ST)$  denotes the spectral radius of the bounded operator  $ST$ .*

**Proof.** To this end, we repeatedly use (3.4) and the Schwarz inequality for the positive inner product  $\langle A \cdot, \cdot \rangle_{\mathcal{H}_1}$ , to get

$$\begin{aligned}
 \|B^{1/2}Tx\|_{\mathcal{H}_2}^2 &= \langle BTx, Tx \rangle_{\mathcal{H}_2} = \langle S^*Ax, Tx \rangle_{\mathcal{H}_2} = \langle Ax, STx \rangle_{\mathcal{H}_1} \\
 &\leq \langle Ax, x \rangle_{\mathcal{H}_1}^{\frac{1}{2}} \langle A(ST)x, (ST)x \rangle_{\mathcal{H}_1}^{\frac{1}{2}} \\
 &= \langle Ax, x \rangle_{\mathcal{H}_1}^{\frac{1}{2}} \langle (T^*S^*)Ax, (ST)x \rangle_{\mathcal{H}_1}^{\frac{1}{2}} \\
 &= \langle Ax, x \rangle_{\mathcal{H}_1}^{\frac{1}{2}} \langle Ax, (ST)^2x \rangle_{\mathcal{H}_1}^{\frac{1}{2}} \\
 &\leq \langle Ax, x \rangle_{\mathcal{H}_1}^{\frac{1}{2} + \frac{1}{4}} \langle A(ST)^2x, (ST)^2x \rangle_{\mathcal{H}_1}^{\frac{1}{4}} \\
 &\vdots \\
 &\leq \langle Ax, x \rangle_{\mathcal{H}_1}^{\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}} \langle A(ST)^{2^{n-1}}x, (ST)^{2^{n-1}}x \rangle_{\mathcal{H}_1}^{\frac{1}{2^n}} \\
 &\leq \langle Ax, x \rangle_{\mathcal{H}_1}^{\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}} \langle (T^*S^*)^{2^{n-1}}Ax, (ST)^{2^{n-1}}x \rangle_{\mathcal{H}_1}^{\frac{1}{2^n}} \\
 &= \langle Ax, x \rangle_{\mathcal{H}_1}^{\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}} \langle Ax, (ST)^{2^n}x \rangle_{\mathcal{H}_1}^{\frac{1}{2^n}} \\
 &\leq \langle Ax, x \rangle_{\mathcal{H}_1}^{\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}} \| (ST)^{2^n}x \|_{\mathcal{H}_1}^{\frac{1}{2^n}} \\
 &\leq \| (ST)^{2^n}x \|_{\mathcal{H}_1}^{\frac{1}{2^n}} \langle Ax, x \rangle_{\mathcal{H}_1}^{1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}}} \| Ax \|_{\mathcal{H}_1}^{\frac{1}{2^n}},
 \end{aligned}$$

and hence

$$\langle BTx, Tx \rangle_{\mathcal{H}_2}^2 \leq \| (ST)^{2^n} \|_{\mathcal{H}_1}^{\frac{1}{2^{n+1}}} \langle Ax, x \rangle_{\mathcal{H}_1}^{1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}}} \| Ax \|_{\mathcal{H}_1}^{\frac{1}{2^n}} \| x \|_{\mathcal{H}_1}^{\frac{1}{2^n}}. \quad (3.6)$$

Further, let us note that, if  $Ax = 0$  then by (3.3) the inequality (3.5) is trivial. Thus, assuming  $Ax \neq 0$ , hence  $x \neq 0$ , we can pass to the limit in (3.6) and, taking into account that

$$\lim_{n \rightarrow \infty} \| (ST)^{2^n} \|_{\mathcal{H}_1}^{\frac{1}{2^n}} = r(ST), \quad \lim_{n \rightarrow \infty} \| Ax \|_{\mathcal{H}_1}^{\frac{1}{2^n}} = \lim_{n \rightarrow \infty} \| x \|_{\mathcal{H}_1}^{\frac{1}{2^n}} = 1,$$

we get the inequality (3.5).  $\square$

We are now in a position to finish off the proof of Theorem 3.1.

**Proof of Theorem 3.1.** By Proposition 2.1 and Remark 2.3, it is sufficient to prove the result for the energy space representations  $(\mathcal{K}_A, \Pi_A)$  and  $(\mathcal{K}_B, \Pi_B)$ , when the strong topologies are explicitly defined in terms of the seminorms  $\|A^{1/2} \cdot\|_{\mathcal{H}_1}$  and, respectively,  $\|B^{1/2} \cdot\|_{\mathcal{H}_2}$ . Let us note that the inequality (3.5) can be reformulated as

$$\|B^{1/2}Tx\|_{\mathcal{H}_2} \leq \sqrt{r(ST)} \|A^{1/2}x\|_{\mathcal{H}_1}, \quad x \in \text{Dom}(A). \quad (3.7)$$

On the ground of (3.7) it follows that the operator  $T$  factors to a linear operator  $\text{Dom}(A)/\text{Ker}(A) \rightarrow \text{Dom}(B)/\text{Ker}(B)$  and is continuous with respect to the strong topologies of the induced Hilbert space  $\mathcal{K}_A$  and  $\mathcal{K}_B$ , and hence it is uniquely extended to

a bounded operator  $\tilde{T} : \mathcal{K}_A \rightarrow \mathcal{K}_B$ . Clearly, we then have  $\tilde{T}\Pi_A x = \Pi_B T x$  for all  $x \in \text{Dom}(A)$ . In a similar way,  $S$  can be lifted to an operator  $\tilde{S} \in \mathcal{B}(\mathcal{K}_B, \mathcal{K}_A)$  such that  $\tilde{S}\Pi_B = \Pi_A S$ , for all  $y \in \text{Dom}(B)$ . Finally, once the existence of  $\tilde{T}$  and  $\tilde{S}$  is established, (3.2) is a simple consequence of (3.1) and a continuity argument.  $\square$

We finally present some properties of preservation for spectra of operators lifted to induced Hilbert spaces. The conclusions will be obtained as applications of Theorem 3.1 and will generalize most of the known properties previously obtained in case the selfadjoint operator  $A$  is bounded.

Let  $A$  be a positive self-adjoint operator in the Hilbert space  $\mathcal{H}$  and let us consider a linear bounded operator  $T$  on  $\mathcal{H}$ . In the sequel, it is assumed that the operator  $T$  commutes with  $A$  in the sense that the following relation

$$\langle Ax, Ty \rangle = \langle Tx, Ay \rangle, \quad x \in \text{Dom}(A), \quad (3.8)$$

holds. This means that all conditions with  $T = S$  in Theorem 3.1 are satisfied. Therefore, by virtue of the mentioned theorem applied to the operator  $T$ , there corresponds a uniquely determined operator  $\tilde{T}$  on the space  $\mathcal{K}_A$  induced by  $A$ . Thus  $\tilde{T} \in \mathcal{B}(\mathcal{K}_A)$  and  $\tilde{T}\Pi_A x = \Pi_A T x$  for each  $x \in \text{Dom}(A)$ . In the following we indicate how some spectral properties of the operator  $T$  remain valid for the corresponding lifted operator  $\tilde{T}$ . We start with the following result which turns out to be an easy consequence of Theorem 3.1.

**Theorem 3.5.** *The spectrum of  $\tilde{T}$ , as a linear operator on the Hilbert space  $\mathcal{K}_A$ , is a subset of the spectrum of  $T$ , as a linear operator on the Hilbert space  $\mathcal{H}$ .*

**Proof.** Let  $z$  be a complex number which is a regular point of the operator  $T$ , i.e.,  $T - zI$  has a bounded inverse in  $\mathcal{H}$ . Denote it by  $R(z; T) = (T - zI)^{-1}$ . We remark that condition (3.7) means that the operator  $T$  commutes with  $A$ , in the sense that for each  $u \in \text{Dom}(A)$  it follows  $Tu \in \text{Dom}(A)$  and  $TAu = ATu$ , i.e.,  $TA \subseteq AT$ . This is equivalent with the fact that the operator  $T$  commutes with the spectral measure  $E$  of  $A$ , i.e.,  $E(\alpha)T = TE(\alpha)$ , where  $\alpha$  denotes any Borel set of the real line  $\mathbb{R}$ . But then, the same is true for the resolvent operator  $R(z; T)$ . Consequently, the resolvent operator  $R(z; T)$  commutes with  $A$ , and hence we can apply Theorem 3.1 to obtain the corresponding lifted operator  $\tilde{R}$  which becomes bounded with respect to the norm of  $\mathcal{K}_A$ .

Further, we note that the transformation  $T \rightarrow \tilde{T}$ , which by Theorem 5.1 is well-defined, is an algebraic homomorphism, from the set of all bounded operators on  $\mathcal{H}$  that commute with  $A$ , into  $\mathcal{B}(\mathcal{K}_A)$ . Then, we can conclude that the operator  $\tilde{R}$  is the inverse operator of  $\tilde{T} - zI_A$  on  $\mathcal{K}_A$  ( $I_A$  means the identity operator on  $\mathcal{K}_A$ ).  $\square$

Theorem 3.5 can be extended for some classes of unbounded operators. As an example, let  $T$  be a selfadjoint (not necessarily bounded) operator in  $\mathcal{H}$ . Suppose that the operator  $T$  commutes with  $A$  in the sense of commutativity of their spectral measures. Then the resolvent operator  $R(z; T) = (T - zI)^{-1}$  ( $\Im z \neq 0$ ) satisfies all the assumptions that makes it a resolvent for some densely defined operator  $\tilde{T}$  in  $\mathcal{K}_A$  (see, for instance, [5]). Therefore we have

**Corollary 3.6.** *If  $T$  is a self-adjoint operator in  $\mathcal{H}$  such that it commutes with  $A$ , then there exists a uniquely determined operator  $\tilde{T}$  on the space  $\mathcal{K}_A$ ,  $\tilde{T}$  is self-adjoint on  $\mathcal{K}_A$ , and its spectrum is a subset of the spectrum of  $T$  on  $\mathcal{H}$ .*

Again, let  $T$  be a bounded operator on  $\mathcal{H}$  and let  $\lambda$  be a point in the discrete spectrum of  $T$ . This means that  $\lambda$  is an isolated point in the spectrum of  $T$  and the null-space of  $T - \lambda I$  is finite-dimensional. The point  $\lambda$  is an eigenvalue of the operator  $T$  with finite multiplicity. In other words, consider the projection

$$P_\lambda = -\frac{1}{2\pi i} \int_{\gamma} R(z; T) dz,$$

where  $\gamma$  is a circumference centered in  $\lambda$  of sufficiently small radius such that the disk  $|z - \lambda| \leq r$  does not contain other singularities except  $z = \lambda$ . Then  $P_\lambda$  is a finite-rank operator in  $\mathcal{H}$ . The subspace  $P_\lambda \mathcal{H}$  is the root subspace of  $A$  which corresponds to the eigenvalue  $\lambda$ . Let  $\tilde{T}$  be the corresponding lifted operator of  $T$ . In view of the previous remarks, it follows that the projection

$$\tilde{P}_\lambda = -\frac{1}{2\pi i} \int_{\gamma} R(z, \tilde{T}) dz$$

is the lifted operator of  $P_\lambda$ , respectively. Since  $\text{Dom}(A)$  is dense in  $\mathcal{H}$  and  $\dim P_\lambda \mathcal{H} < \infty$ , it follows that  $\tilde{P}_\lambda \mathcal{K}_A = P_\lambda \mathcal{H}$ . Therefore, we obtain the following result.

**Theorem 3.7.** *Under the assumptions from above, if  $\lambda$  belongs to the discrete spectrum of  $T$ , then  $\lambda$  belongs to the discrete spectrum of  $\tilde{T}$  and their corresponding root subspaces are the same.*

Finally, an immediate consequence of Theorems 3.5 and 3.7 is the following result often useful for concrete applications.

**Corollary 3.8.** *Let  $A$  and  $T$  be as in Theorem 3.5 and suppose that the operator  $T$  has only discrete spectrum, i.e., each point of the spectrum  $\sigma(T)$  except  $\lambda = 0$  is an isolated eigenvalue of finite multiplicity. Then the spectrum of  $\tilde{T}$  is discrete as well,  $\sigma(T) = \sigma(\tilde{T})$ , and the root subspaces corresponding to the same nonzero eigenvalues of  $T$  and  $\tilde{T}$  coincide, respectively. In particular, if  $T$  is completely continuous on  $\mathcal{H}$ , then  $\tilde{T}$  is completely continuous on  $\mathcal{K}_A$ .*

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