Journal of Pure and Applied Algebra 213 (2009) 2238–2243

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Essential cohomology for elementary abelian p-groups

F. Altunbulak Aksu [a], D.J. Green [b,*]

[a] *Department of Mathematics, Bilkent University, Bilkent, 06800, Ankara, Turkey*

[b] *Mathematical Institute, Friedrich-Schiller-Universität Jena, 07737 Jena, Germany*

A R T I C L E   I N F O

Article history:
Received 25 September 2008
Received in revised form 19 February 2009
Available online 17 May 2009
Communicated by M. Broué

MSC:
Primary: 20J06
13A50
55S10

A B S T R A C T

For an odd prime p the cohomology ring of an elementary abelian p-group is polynomial tensor exterior. We show that the ideal of essential classes is the Steenrod closure of the class generating the top exterior power. As a module over the polynomial algebra, the essential ideal is free on the set of Mùi invariants.

1. Introduction

Let G be a finite group and k a field whose characteristic p divides the order of G. A cohomology class $x \in H^n(G, k)$ is called *essential* if its restriction $\mathrm{Res}_H(x)$ is zero for every proper subgroup H of G. The essential classes form an ideal, called the essential ideal and denoted by $\mathrm{Ess}(G)$. It is standard that restriction to a Sylow p-subgroup of G is a split injection (see for example Theorem XII,10.1 of [1]), and so the essential ideal can only be non-zero if G is a p-group. Many p-groups have non-zero essential ideal, for instance the quaternion group of order eight. The essential ideal plays an important role and has therefore been the subject of many studies: two such being Carlson's work on the depth of a cohomology ring [2], and the cohomological characterization due to Adem and Karagueuzian of those p-groups whose order p elements are all central [3].

The nature of the essential ideal depends crucially on whether or not the p-group G is elementary abelian. If G is not elementary abelian, then a celebrated result of Quillen (Theorem 7.1 of [4]) implies that $\mathrm{Ess}(G)$ is a nilpotent ideal. By contrast, the essential ideal of an elementary abelian p-group contains non-nilpotent classes. Work to date on the essential ideal has concentrated on the non-elementary abelian case. In this paper we give a complete treatment of the outstanding elementary abelian case. As we shall recall in the next section, the case $p = 2$ is straightforward and well known. So we shall concentrate on the case of an odd prime p.

So let p be an odd prime and V a rank n elementary abelian p-group. We may equally well view V as an n-dimensional $\mathbb{F}_p$-vector space. Recall that the cohomology ring has the form

$$H^*(V, \mathbb{F}_p) \cong S(V^*) \otimes_{\mathbb{F}_p} \Lambda(V^*), \tag{1}$$

where the exterior copy of the dual space V^* is $H^1(V, \mathbb{F}_p)$, and the polynomial copy lies in $H^2(V, \mathbb{F}_p)$: specifically, the polynomial copy is the image of the exterior copy under the Bockstein boundary map β. Our first result is as follows:

Theorem 1.1. *Let p be an odd prime and V a rank n elementary abelian p-group. Then the essential ideal $\mathrm{Ess}(V)$ is the Steenrod closure of $\Lambda^n(V^*)$. That is, $\mathrm{Ess}(V)$ is the smallest ideal in $H^*(V, \mathbb{F}_p)$ which contains the one-dimensional space $\Lambda^n(V^*) \subseteq H^n(V, \mathbb{F}_p)$ and is closed under the action of the Steenrod algebra.*

* Corresponding author.
E-mail addresses: fatma@fen.bilkent.edu.tr (F. Altunbulak Aksu), david.green@uni-jena.de (D.J. Green).

0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2009.04.016

Our second result concerns the structure of $\operatorname{Ess}(V)$ as a module over the polynomial subalgebra $S(V^*)$ of $H^*(V, \mathbb{F}_p)$. It was conjectured by Carlson (Question 5.4 in [5]) – and earlier in a less precise form by Mùi [6] – that the essential ideal of an arbitrary p-group is free and finitely generated as a module over a certain polynomial subalgebra of the cohomology ring. In [7], the second author demonstrated finite generation, and for most p-groups of a given order was able to prove freeness as well: specifically the method works provided the group is not a direct product in which one factor is elementary abelian of rank at least two. Our second result states that Carlson's conjecture holds for elementary abelian p-groups too, and gives explicit free generators.

Theorem 1.2. *Let p be an odd prime and V a rank n elementary abelian p-group. Then as a module over the polynomial part $S(V^*)$ of the cohomology ring $H^*(V, \mathbb{F}_p)$, the essential ideal* $\operatorname{Ess}(V)$ *is free on the set of Mùi invariants, as defined in Definition* 3.3*.*

Structure of the paper. In Section 2 we briefly cover the well-known case $p = 2$. We introduce the Mùi invariants in Section 3. After proving Theorem 1.2 in Section 4 we consider the action of the Steenrod algebra on the Mùi invariants in order to prove Theorem 1.1 in Section 5.

2. Elementary abelian p-groups and the case $p = 2$

The cohomology group $H^1(G, \mathbb{F}_p)$ may be identified with the set of group homomorphisms $\operatorname{Hom}(G, \mathbb{F}_p)$. This set is an $\mathbb{F}_p$-vector space, and – assuming that G is a p-group – the maximal subgroups of G are in bijective correspondence with the one-dimensional subspaces: the maximal subgroup corresponding to $\alpha: G \to \mathbb{F}_p$ being $\ker(\alpha)$. Of course, the cohomology class $\alpha \in H^1(G, \mathbb{F}_p)$ has zero restriction to the maximal subgroup $\ker(\alpha)$. Note that in order to determine $\operatorname{Ess}(G)$ it suffices to consider restrictions to maximal subgroups.

Definition. Denote by L_n the polynomial

$$L_n(X_1, \ldots, X_n) = \det \begin{vmatrix} X_1 & X_2 & \cdots & X_n \\ X_1^p & X_2^p & \cdots & X_n^p \\ \vdots & \vdots & \ddots & \vdots \\ X_1^{p^{n-1}} & X_2^{p^{n-1}} & \cdots & X_n^{p^{n-1}} \end{vmatrix} \in \mathbb{F}_p[X_1, \ldots, X_n].$$

There is a well-known alternative description of L_n.

Lemma 2.1. *L_n is the product of all monic linear forms in $X_1, \ldots, X_n$. So for an n-dimensional $\mathbb{F}_p$-vector space V we may define $L_n(V) \in S(V^*)$ up to a non-zero scalar multiple by*

$$L_n(V) = \prod_{[x] \in \mathbb{P}V^*} x. \tag{2}$$

Proof. First part: Here we call a linear form monic if the first non-zero coefficient is one. The right hand side divides the left. Both sides have the same total degree. And the coefficient of $X_1 X_2^p X_3^{p^2} \cdots X_n^{p^{n-1}}$ is $+1$ in both cases. The second part follows. $\square$

Let V be an elementary abelian 2-group. Then $H^*(V, \mathbb{F}_2) \cong S(V^*)$, where the dual space V^* is identified with $H^1(V, \mathbb{F}_2)$. Pick $x_1, \ldots, x_n$ to be a basis for $H^1(V, \mathbb{F}_2)$. The following is well-known:

Lemma 2.2. *For an elementary abelian 2-group V, the essential ideal is the principal ideal in $H^*(V, \mathbb{F}_2)$ generated by $L_n(x_1, \ldots, x_n)$.*

Moreover, $\operatorname{Ess}(V)$ *is the free $S(V^*)$-module on $L_n(V)$, and the Steenrod closure of this one generator.*

Proof. $L_n(V)$ is essential, because every non-zero linear form is a factor and every maximal subgroup is the kernel of a non-zero linear form. Now suppose that y is essential, and let $x \in V^*$ be a non-zero linear form. Let $U \subseteq V^*$ be a complement of the subspace spanned by x. So $y = y'x + y''$ with $y' \in S(V^*)$ and $y'' \in S(U)$. Hence $\operatorname{Res}_H(y'') = 0$ for $H = \ker(x)$, as y is essential and $\operatorname{Res}_H(x) = 0$. But the map $\operatorname{Res}_H: V^* \to H^*$ satisfies $\ker(\operatorname{Res}_H) \cap U = 0$, and so Res_H is injective on $S(U)$. Hence $y'' = 0$, and x divides y. By unique factorization in $S(V^*)$ it follows that $L_n(V)$ divides y. So $\operatorname{Ess}(V)$ is the principal ideal generated by $L_n(V)$, and the free module on this one generator. Finally, the definition of the essential ideal means that it is closed under the action of the Steenrod algebra. $\square$

We finish off this section by recalling the action of the Steenrod algebra on the cohomology of an elementary abelian p-group in the case of an odd prime. So let p be an odd prime and V an elementary abelian p-group. Recall that the mod-p-cohomology ring is the free graded commutative algebra

$$H^*(V, \mathbb{F}_p) \cong \mathbb{F}_p[x_1, \ldots, x_n] \otimes_{\mathbb{F}_p} \Lambda(a_1, \ldots, a_n),$$

where $a_i \in H^1(V, \mathbb{F}_p)$, $x_i \in H^2(V, \mathbb{F}_p)$, and n is the rank of V. That is, $a_1, \dots, a_n$ is a basis of the exterior copy of V^*, and $x_1, \dots, x_n$ is a basis of the polynomial copy. The product $a_1 a_2 \cdots a_n \in H^n(V, \mathbb{F}_p)$ is a basis of the top exterior power $\Lambda^n(V^*)$. The Steenrod algebra $\mathcal{A}$ acts on the cohomology ring, making it an unstable $\mathcal{A}$-algebra with $\beta(a_i) = x_i$ and $\mathcal{P}^1(x_i) = x_i^p$. Observe that $L_n(x_1, \dots, x_n)$ is essential, for the same reason as in the case $p = 2$.

3. The Mùi invariants

Let k be a finite field and V a finite dimensional k-vector space. Consider the natural action of $GL(V)$ on V^*. The Dickson invariants generate the invariants for the induced action of $GL(V)$ on the polynomial algebra $S(V^*)$. But there is also an induced action on the polynomial tensor exterior algebra $S(V^*) \otimes_k \Lambda(V^*)$, and the Mùi invariants are $SL(V)$-invariants of this action: see Mùi's original paper [8] as well as Crabb's modern treatment [9].

We shall need several properties of the Mùi invariants. For the convenience of the reader, we rederive these from scratch: but see Mùi's papers [8,10] and Sum's work [11].

Notation. Often we shall work with the direct sum decomposition

$$H^*(V, \mathbb{F}_p) = \bigoplus_{r=0}^{n} N_r(V)\,,$$

where n is the rank of V and we set

$$N_r(V) = S(V^*) \otimes_{\mathbb{F}_p} \Lambda^r(V^*)\,.$$

Observe that restriction to each subgroup respects this decomposition. This means that the essential ideal is well-behaved with respect to this decomposition:

$$\mathrm{Ess}(V) = \bigoplus_{r=0}^{n} N_r(V) \cap \mathrm{Ess}(V)\,. \tag{3}$$

Definition. Recall that $L_n(x_1, \dots, x_n)$ is the determinant of the $n \times n$-matrix

$$C = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{p^{n-1}} & x_2^{p^{n-1}} & \cdots & x_n^{p^{n-1}} \end{pmatrix},$$

where $C_{s,i} = x_i^{p^{s-1}}$ for $1 \leq s \leq n$. For each such s, define $E(s)$ to be the matrix obtained from C by deleting row s and then prefixing $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$ as new first row: so

$$\det E(s) = \sum_{i=1}^{n} (-1)^{i+1} \gamma_{s,i} a_i,$$

where $\gamma_{s,i}$ is the determinant of the minor of C obtained by removing row s and column i.

Now define the Mùi invariant $M_{n,s} \in H^*(V, \mathbb{F}_p)$ by $M_{n,s} = \det E(s)$. Note that our indexing differs from Mùi's: our $M_{n,s}$ is his $M_{n,s-1}$.

Example. So $M_{4,3} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^p & x_2^p & x_3^p & x_4^p \\ x_1^{p^3} & x_2^{p^3} & x_3^{p^3} & x_4^{p^3} \end{vmatrix}$ and $\gamma_{2,3} = \begin{vmatrix} x_1 & x_2 & x_4 \\ x_1^{p^2} & x_2^{p^2} & x_4^{p^2} \\ x_1^{p^3} & x_2^{p^3} & x_4^{p^3} \end{vmatrix}$.

Lemma 3.1. $M_{n,s} \in N_1(V) \cap \mathrm{Ess}(V)$.

Proof. By construction $M_{n,s} \in N_1(V)$. Restricting to a maximal subgroup of V involves killing a non-zero linear form on V^*: That is, one imposes a linear dependence on the a_i and consequently the same linear dependency on the x_i. So one obtains a linear dependency between the columns of $E(s)$, meaning that restriction kills $M_{n,s} = \det E(s)$. $\square$

Lemma 3.2. $\mathrm{Ess}(V)^2 = L_n(V) \cdot \mathrm{Ess}(V)$.

Proof. As $L_n(V)$ is essential, the left hand side contains the right. Now let H be a maximal subgroup of V. Then $H = \ker(a)$ for some non-zero $a \in H^1(V, \mathbb{F}_p)$. Let $x = \beta(a) \in H^2$. Observe that the kernel of restriction to H is generated by a, x. Suppose that f, g both lie in this kernel: then we may write $f = f'a + f''x$, $g = g'a + g''x$, and so $fg = (f''g' \pm f'g'')ax + f''g''x^2$, that is $fg = xh$ for $h = (f''g' \pm f'g'')a + f''g''x \in \ker \mathrm{Res}_H$.

Since $H^*(V, \mathbb{F}_p)$ is a free module over the unique factorization ring $S(V^*)$, this means that $fg = L_n(V) \cdot y$ for some $y \in H^*(V, \mathbb{F}_p)$. So $h = \frac{L_n(V)}{x} \cdot y$. As $\mathrm{Res}_H(h) = 0$ and $\mathrm{Res}_H\left(\frac{L_n(V)}{x}\right)$ is a non-zero divisor, we deduce that $\mathrm{Res}_H(y) = 0$. So $y \in \mathrm{Ess}(V)$. $\square$

Definition 3.3. Let $S = \{s_1, \dots, s_r\} \subseteq \{1, \dots, n\}$ be a subset with $s_1 < s_2 < \cdots < s_r$. In view of Lemmas 3.1 and 3.2 we may define the Mùi invariant $M_{n,S} \in N_r(V) \cap \mathrm{Ess}(V)$ by

$$M_{n,S} = \frac{1}{L_n(V)^{r-1}} M_{n,s_1} M_{n,s_2} \cdots M_{n,s_r}.$$

Note in particular that $M_{n,\emptyset} = L_n(V)$.

Remark. Observe that

$$M_{n,S} M_{n,T} = \begin{cases} \pm L_n(V) M_{n,S\cup T} & \text{if } S \cap T = \emptyset; \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

4. Joint annihilators

In this section we study the joint annihilators of the $M_{n,S}$ with $|S| = r$ as a means to prove Theorem 1.2.

Lemma 4.1. *The joint annihilator of* $M_{n,1}, \dots, M_{n,n}$ *is* $N_n(V)$.

Proof. The element $a_1 \dots a_n$ is a basis for $\Lambda^n(V)$ and is clearly annihilated by each $M_{n,s}$. Conversely, suppose that $y \neq 0$ is annihilated by every $M_{n,s}$. As $M_{n,s} N_r(V) \subseteq N_{r+1}(V)$ we may assume without loss of generality that $y \in N_r(V)$ for some r. If $r \leq n-1$, then for some i we have $0 \neq a_i y \in N_{r+1}(V)$. So as $a_i y$ also lies in the joint annihilator, it will suffice by iteration to eliminate the case $y \in N_{n-1}(V)$.

Suppose therefore that $0 \neq y \in N_{n-1}(V)$ lies in the joint annihilator. Denote by K the field of fractions of $S(V^*)$, and let $W = K \otimes_k \Lambda^{n-1}(V^*)$. Each $M_{n,s}$ induces a linear form $\phi_s \colon W \to K$ given by $\phi_s(w) a_1 \cdots a_n = M_{n,s} w$. By assumption, $y \neq 0$ lies in the kernel of every ϕ_s. A basis for W consists of the elements $a_1 \cdots \widehat{a_r} \cdots a_n$ for $1 \leq r \leq n$, where the hat denotes omission. Now,

$$M_{n,s} \cdot a_1 \cdots \widehat{a_r} \cdots a_n = (-1)^{r+1} \gamma_{s,r} a_r \cdot a_1 \cdots \widehat{a_r} \cdots a_n,$$

and so

$$\phi_s(a_1 \cdots \widehat{a_r} \cdots a_n) = \gamma_{s,r}.$$

Now consider the matrix $\Gamma \in M_n(K)$ given by $\Gamma_{s,r} = \gamma_{s,r}$. If one transposes and then multiplies the ith row by $(-1)^i$ and the jth column by $(-1)^j$, then one obtains the adjugate matrix of C. As the determinant of C is $L_n(V)$ and in particular non-zero, it follows that $\det \Gamma \neq 0$.

So by construction of Γ, the ϕ_s form a basis of W^*. So their common kernel is zero, contradicting our assumption on y. $\square$

Corollary 4.2. *The joint annihilator of* $\{M_{n,S} \colon |S| = r\}$ *is* $\bigoplus_{s \geq n-r+1} N_s(V)$.

Proof. By induction on r, Lemma 4.1 being the case $r = 1$. As $M_{n,S} \in N_{|S|}(V)$ and $N_r(V) N_s(V) \subseteq N_{r+s}(V)$, the annihilator is at least as large as claimed. Now suppose that $y \in H^*(V, \mathbb{F}_p)$ does not lie in $\bigoplus_{s \geq n-r+1} N_s(V)$. We may therefore write

$$y = \sum_{s=0}^{n} y_s$$

with $y_s \in N_s(V)$, and we know that $s_0 \leq n - r$ for $s_0 = \min\{s \mid y_s \neq 0\}$. As $y_{s_0} \neq 0$ and $y_{s_0} \notin N_n(V)$, Lemma 4.1 tells us that $y_{s_0} M_{n,t} \neq 0$ for some $1 \leq t \leq n$. As $y_{s_0} M_{n,t} \in N_{s_0+1}(V)$, we conclude that $y M_{n,t}$ lies outside $\bigoplus_{s \geq n-r+2} N_s(V)$. So the inductive hypothesis means that there is some T with $|T| = r - 1$ and $y M_{n,t} M_{n,T} \neq 0$. So $y M_{n,S} \neq 0$ for $S = T \cup \{t\}$ and $|S| = r$: Note that $t \in T$ is impossible. $\square$

Corollary 4.3. *Every* $M_{n,S}$ *is non-zero. For* $S = \underline{n} = \{1, \dots, n\}$ *we have*

$M_{n,\underline{n}}$ *is a non-zero scalar multiple of* $a_1 a_2 \cdots a_n$.

Proof. Observe that $M_{n,\underline{n}}$ is a scalar multiple of $a_1 \cdots a_n$ for degree reasons. The case $r = n$ of Corollary 4.2 says that $1 \in N_0(V)$ does not annihilate $M_{n,\underline{n}}$ and therefore $M_{n,\underline{n}} \neq 0$. But from Eq. (4) we see that every $M_{n,S}$ divides $L_n(V)M_{n,\underline{n}} \neq 0$. □

Proof of Theorem 1.2. In view of Eq. (3) it suffices to show that for each r the Mùi invariants $M_{n,S}$ with $|S| = r$ are a basis of the $S(V^*)$-module $N_r(V) \cap \mathrm{Ess}(V)$. We observed in Definition 3.3 that these $M_{n,S}$ lie in this module.

So suppose that $y \in N_r(V) \cap \mathrm{Ess}(V)$. We should like there to be $f_S \in S(V^*)$ such that

$$y = \sum_{|S|=r} f_S M_{n,S}. \tag{5}$$

Note that for $T = \underline{n} - S$ we have $M_{n,S}M_{n,T} = \pm L_n(V)M_{n,\underline{n}}$ by Eq. (4). Define $\varepsilon_S \in \{+1, -1\}$ by $M_{n,S}M_{n,T} = \varepsilon_S L_n(V)M_{n,\underline{n}}$. So Eq. (5) implies that we should define f_S by

$$f_S M_{n,\underline{n}} = \frac{1}{L_n(V)} \varepsilon_S y M_{n,T},$$

since $T \cap S' \neq \emptyset$ and therefore $M_{n,S'}M_{n,T} = 0$ for all $S' \neq S$ with $|S| = r$. Note that this definition of f_S makes sense, as $yM_{n,T}$ lies in both $N_r(V)N_{n-r}(V) = N_n(V)$ and $L_n(V)\,\mathrm{Ess}(V)$, the latter inclusion coming from Lemma 3.2.

With this definition of f_S we have

$$\left(y - \sum_{|S|=r} f_S M_{n,S}\right) M_{n,T} = 0$$

for every $|T| = n - r$. As $y - \sum_{|S|=r} f_S M_{n,S}$ lies in $N_r(V)$, this means that $y = \sum_{|S|=r} f_S M_{n,S}$ by Corollary 4.2.

Finally we show linear independence. Suppose that $g_S \in S(V^*)$ are such that $\sum_{|S|=r} g_S M_{n,S} = 0$. Pick one S and set $T = \underline{n} - S$. Multiplying by $M_{n,T}$, we deduce that $g_S = 0$. □

5. The action of the Steenrod algebra

To prepare for the proof of Theorem 1.1 we shall study the operation of the Steenrod algebra on the Mùi invariants.

Lemma 5.1.

$$\beta(M_{n,s}) = \begin{cases} L_n(V) & s = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \beta(L_n(V)) = 0. \tag{6}$$

For $0 \leq s \leq n - 2$ we have:

$$\mathcal{P}^{p^s}(M_{n,r}) = \begin{cases} M_{n,r-1} & r = s + 2 \\ 0 & \text{otherwise} \end{cases} \qquad \mathcal{P}^{p^s}(L_n(V)) = 0. \tag{7}$$

Proof. One sees Eq. (6) by inspecting the determinants in the definition of $M_{n,s}$ and $L_n(V)$. The proof of Eq. (7) is also based on an inspection of these determinants. Recall that $\mathcal{P}^m(a_i) = 0$ for every $m > 0$, and that $\mathcal{P}^m(x_i^{p^s})$ is zero too except for $\mathcal{P}^{p^s}(x_i^{p^s}) = x_i^{p^{s+1}}$. We may use the Cartan formula

$$\mathcal{P}^m(xy) = \sum_{a+b=m} \mathcal{P}^a(x)\mathcal{P}^b(y)$$

to distribute $\mathcal{P}^{p^s}$ over the rows of the determinant. As p^s cannot be expressed as a sum of distinct smaller powers of p, we only have to consider summands where all of $\mathcal{P}^{p^s}$ is applied to one row and the other rows are unchanged. This will result in two rows being equal unless it is the row consisting of the $x_i^{p^{s+1}}$ that is missing. □

Lemma 5.2. *Let $S = \{s_1, \ldots, s_r\}$ with $1 \leq s_1 < s_2 < \cdots < s_r \leq n$.*

1. *Suppose that $1 \notin S$. Then $M_{n,S} = \beta(M_{n,S\cup\{1\}})$.*
2. *$L_n(V)^{r-1}\mathcal{P}^m(M_{n,S}) = \mathcal{P}^m(M_{n,s_1} \cdots M_{n,s_r})$ for each $m < p^{n-1}$.*
3. *For $2 \leq u \leq n$ set $X = \{s \in S \mid s \leq u\}$ and $Y = \{s \in S \mid s > u\}$. Then*
$$L_n(V)\mathcal{P}^{p^{u-2}}(M_{n,S}) = \mathcal{P}^{p^{u-2}}(M_{n,X}) \cdot M_{n,Y}.$$
4. *For $1 \leq r \leq n$ and $0 < m < p^{n-1}$ one has $\mathcal{P}^m(M_{n,\{1,\ldots,r\}}) = 0$.*
5. *For $2 \leq u \leq n$ one has $\mathcal{P}^{p^{u-2}}(M_{n,\{1,\ldots,u-2,u\}}) = M_{n,\{1,\ldots,u-1\}}$.*

Proof. Recall that

$$L_n(V)^r M_{n,S} = L_n(V) M_{n,s_1} \cdots M_{n,s_r} \,. \tag{8}$$

The first two parts follow by applying Eqs. (6) and (7).

Recall that by the Adem relations each $\mathcal{P}^m$ may be expressed in terms of the $\mathcal{P}^{p^s}$ with $p^s \le m$. So the third part follows from the second, since we deduce from Eq. (7) that $\mathcal{P}^m(M_{n,s}) = 0$ if $0 < m \le p^{u-2}$ and $s > u$.

Fourth part: By induction on r. Follows for $r = 1$ from the Adem relations and Eq. (7). Inductive step: Enough to consider $\mathcal{P}^{p^s}$ for $0 \le s \le n-2$. By the inductive hypothesis and a similar argument to the third part, deduce that

$$L_n(V)\mathcal{P}^{p^s}(M_{n,\{1,\ldots,r\}}) = M_{n,\{1,\ldots,r-1\}}\mathcal{P}^{p^s}(M_{n,r}).$$

But this is zero by Eq. (7), since $M_{n,\{1,\ldots,r-1\}}M_{n,r-1} = 0$.

Fifth part: Using the fourth part and an argument similar to the third, deduce that

$$L_n(V)\mathcal{P}^{p^{u-2}}(M_{n,\{1,\ldots,u-2,u\}}) = M_{n,\{1,\ldots,u-2\}}\mathcal{P}^{p^{u-2}}(M_{n,u}) = M_{n,\{1,\ldots,u-2\}}M_{n,u-1}:$$

but this is $L_n(V)M_{n,\{1,\ldots,u-1\}}$. □

Proof of Theorem 1.1. We shall show that for every $M_{n,S}$ there is an element θ of the Steenrod algebra with $M_{n,S} = \theta(M_{n,\underline{n}})$. We do this by decreasing induction on $r = |S|$. It is trivially true for $r = n$, so assume now that $r < n$. Amongst the S with $|S| = r$ we shall proceed by induction over u, the smallest element of $\underline{n} - S$. So

$$S = \{1, \ldots, u-1\} \cup Y \quad \text{with } s > u \text{ for every } s \in Y.$$

Part 1 of Lemma 5.2 covers the case $u = 1$, so assume that $u \ge 2$. Set $T = \{1, \ldots, u-2, u\}$. We complete the induction by showing that $M_{n,S} = \mathcal{P}^{p^{u-2}}(M_{n,T\cup Y})$. Part 3 of Lemma 5.2 gives us

$$L_n(V)\mathcal{P}^{p^{u-2}}(M_{n,T\cup Y}) = \mathcal{P}^{p^{u-2}}(M_{n,T})M_{n,Y}.$$

But $\mathcal{P}^{p^{u-2}}(M_{n,T}) = M_{n,\{1,\ldots,u-1\}}$, by Part 5 of that lemma. So $\mathcal{P}^{p^{u-2}}(M_{n,T\cup Y}) = M_{n,S}$, as claimed. □

Remark. Theorem 1.2 shows that the $S(V^*)$-module generated by the Mùi invariants $M_{n,S}$ is the essential ideal and therefore closed under the action of the Steenrod algebra. One may however see more directly that this $S(V^*)$-module is Steenrod closed. This is observed for example in [11]. In view of Lemma 5.2 and Eqs. (6) and (7) it only remains to show that $\mathcal{P}^{p^{n-1}}(M_{n,s})$ lies in our $S(V^*)$-module. Now $\mathcal{P}^{p^{n-1}}(M_{n,n}) = 0$ by the unstable condition, so suppose $s < n$. Recall that $M_{n,s}$ is a determinant, the last row of the matrix having entries $x_i^{p^{n-1}}$. So applying $\mathcal{P}^{p^{n-1}}$ replaces these entries by $x_i^{p^n}$. But it is well known that $x_i^{p^n}$ is an $S(V^*)$-linear combination of the $x_i^{p^r}$ for $r \le n-1$, and that the coefficients are independent of i: this is the "fundamental equation" in the sense of [12], and the coefficients are the Dickson invariants $c_{n,r}$ in $S(V^*)$. Applying $S(V^*)$-linearity of the determinant in the bottom row of the matrix, one deduces that $\mathcal{P}^{p^{n-1}}(M_{n,s})$ is an $S(V^*)$-linear combination of the $M_{n,r}$.

Acknowledgements

The first author was supported by a Ph.D. research scholarship from the Scientific and Technical Research Council of Turkey (TÜBİTAK-BAYG). The first author wishes to express her gratitude to her research supervisor Prof. Ergün Yalçın for advice and guidance. Both authors thank him for suggesting this problem.

References

[1] H. Cartan, S. Eilenberg, Homological algebra, Princeton University Press, Princeton, NJ, 1956.
[2] J.F. Carlson, Depth and transfer maps in the cohomology of groups, Math. Z. 218 (3) (1995) 461–468.
[3] A. Adem, D. Karagueuzian, Essential cohomology of finite groups, Comment. Math. Helv. 72 (1) (1997) 101–109.
[4] D. Quillen, The spectrum of an equivariant cohomology ring: I, Ann. Math. 94 (2) (1971) 549–572.
[5] J.F. Carlson, Problems in the calculation of group cohomology, in: P. Dräxler, G.O. Michler, C.M. Ringel (Eds.), Computational Methods for Representations of Groups and Algebras (Essen, 1997), Birkhäuser, Basel, 1999, pp. 107–120.
[6] H. Mùi, The mod p cohomology algebra of the extra-special group $E(p^3)$, unpublished essay (1982).
[7] D.J. Green, The essential ideal is a Cohen–Macaulay module, Proc. Amer. Math. Soc. 133 (11) (2005) 3191–3197.
[8] H. Mùi, Modular invariant theory and cohomology algebras of symmetric groups, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 22 (3) (1975) 319–369.
[9] M.C. Crabb, Dickson–Mui invariants, Bull. London Math. Soc. 37 (6) (2005) 846–856.
[10] H. Mùi, Cohomology operations derived from modular invariants, Math. Z. 193 (1) (1986) 151–163.
[11] N. Sum, Steenrod operations on the modular invariants, Kodai Math. J. 17 (3) (1994) 585–595. workshop on Geometry and Topology (Hanoi, 1993).
[12] C. Wilkerson, A primer on the Dickson invariants, in: H.R. Miller, S.B. Priddy (Eds.), Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982), in: Contemporary Math., vol. 19, Amer. Math. Soc., Providence, RI, 1983, pp. 421–434. Revised version: URL http://hopf.math.purdue.edu//Wilkerson-80s/dickson.pdf.