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Two remarks on monomial Gotzmann sets

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ABSTRACT

A homogeneous set of monomials in a quotient of the polynomial ring $S := F[x_1, \dots, x_n]$ is called Gotzmann if the size of this set grows minimally when multiplied with the variables. We note that Gotzmann sets in the quotient $R := F[x_1, \dots, x_n]/(x_1^a)$ arise from certain Gotzmann sets in S. Secondly, we prove a combinatorial result about the deletion of a variable in a Gotzmann set in S.

1. Introduction

Let $S = F[x_1, \dots, x_n]$ be a polynomial ring over a field F with $\deg(x_i) = 1$ for $1 \le i \le n$. We use the lexicographic order on S with $x_1 > \cdots > x_n$. For a homogeneous ideal I in S, the *Hilbert function* $H(I, -) : \boldsymbol{Z}_{\ge 0} \to \boldsymbol{Z}_{\ge 0}$ of I is the numerical function defined by $H(I, t) = \dim_F I_t$, where I_t is the homogeneous component of degree t of I. A set M of monomials in S is called *lexsegment* if for monomials $m \in M$ and $v \in S$ we have: if $\deg m = \deg v$ and $v > m$, then $v \in M$. A monomial ideal I is called *lexsegment* if the set of monomials in I is lexsegment. For a set of monomials M in the homogeneous component S_t of degree t in S, let $\operatorname{lex}_S(M)$ denote the lexsegment set of $|M|$ monomials in S_t. Also for a set of monomials M, $S_1 \cdot M$ denotes the set of monomials of the form um, where u is a variable and $m \in M$. By a classical theorem of Macaulay [6, C4] we have

$$|(S_1 \cdot \operatorname{lex}_S(M))| \le |(S_1 \cdot M)|. \tag{1}$$

Since the Hilbert function of a homogeneous ideal is the same as the Hilbert function of its lead term ideal this inequality implies that for each homogeneous ideal in S there is a lexsegment ideal with the same Hilbert function. One course of research inspired by Macaulay's theorem is the study of the homogeneous ideals I such that every Hilbert function in S/I is obtained by a lexsegment ideal in S/I. Such quotients are called *Macaulay-lex* rings. These rings have important applications in combinatorics and algebraic geometry and for more background on them we direct the reader to Mermin and Peeva [8,9]. Some recently discovered classes of Macaulay-lex rings can be found in Mermin and Murai [7].

Monomial sets in S whose sizes grow minimally in the sense of Macaulay's inequality have also attracted attention: a homogeneous set M of monomials is called *Gotzmann* if $|(S_1 \cdot \operatorname{lex}_S(M))| = |(S_1 \cdot M)|$ and a monomial ideal I is *Gotzmann* if the set of monomials in I_t is a Gotzmann set for all t. In [12], Gotzmann ideals in S that are generated by at most n homogeneous polynomials are classified in terms of their Hilbert functions. In [10] Murai finds all integers j such that every Gotzmann set of size j in S is lexsegment up to a permutation. He also classifies all Gotzmann sets for $n \le 3$. The Gotzmann persistence theorem states that if M is a Gotzmann set in S, then $S_1 \cdot M$ is also a Gotzmann set; see [2]. In [11] Murai gives a combinatorial proof of this theorem using binomial representations. More recently, Hoefel and Mermin classified Gotzmann square-free ideals; [5] see also [4]. Also some results on the generation of lexsegment and Gotzmann ideals by invariant monomials can be found in [13].

In this paper we first consider Gotzmann sets in the Macaulay-lex quotient $R := F[x_1, \dots, x_n]/(x_1^a)$, where a is a positive integer. A set M of monomials in R can also be considered as a set of monomials in S and by $R_1 \cdot M$ we mean the set

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of monomials in $S_1 \cdot M$ that are not zero in R, i.e., $R_1 \cdot M = (S_1 \cdot M) \cap R$. A set M of monomials in R_t is Gotzmann if $|(R_1 \cdot \mathrm{lex}_R(M))| = |(R_1 \cdot M)|$, where R_t is the homogeneous component of degree t of R and $\mathrm{lex}_R(M)$ denotes the lexsegment set of monomials in R_t that has the same size as M. We show that Gotzmann sets in R arise from certain Gotzmann sets in S: when a Gotzmann set in R_t with $t \geq a$ is added to the set of monomials in S_t that are divisible by x_1^a, one gets a Gotzmann set in S_t. Secondly, we partition the monomials in a Gotzmann set in S with respect to the multiplicity of x_i and show that if the growth of the size of a component is larger than the size of the following component, then this component is a multiple of a Gotzmann set in $F[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. Otherwise, we obtain restrictions on the size of the component in terms of sizes of neighboring components.

For a general reference for Hilbert functions we recommend [1,3].

2. Monomial Gotzmann sets

We continue with the notation and the convention of the previous section. For a homogeneous lexsegment set L in S with $|L| = d$, the size of $S_1 \cdot L$ was computed by Macaulay. This number is very closely related to the nth binomial representation of d and is denoted by $d^{\langle n-1\rangle}$. We refer the reader to [1, Section 4] for more information on this number. In contrast to the situation in S, for the homogeneous lexsegment set $L \subseteq R_t$ of size d, the size of the set $R_1 \cdot L$ depends also on t. We let $d_{n,t}$ denote this size. Notice that we have $d_{n,t} = d^{\langle n-1\rangle}$ for $t < a-1$. For a non-negative integer i, let S_t^i and R_t^i denote the set of monomials in S_t and R_t respectively that are divisible by x_1^i but not by x_1^{i+1}. For a set of monomials M in R_t, let M^i denote the set $R_t^i \cap M$. Similarly, if M is in S_t, then M^i denotes $S_t^i \cap M$. Also let $\min(M)$ denote the smallest integer such that $M^{\min(M)} \neq \emptyset$. Set $S' = F[x_2, \ldots, x_n]$ and let $S_1' \cdot M$ denote the set of monomials of the form $x_i m$, where $2 \leq i \leq n$ and $m \in M$. For a monomial $u \in S$ (resp. R) and a monomial set M in S (resp. R) we let $u \cdot M$ denote the set of monomials in S (resp. R) that are of the form um with $m \in M$. We also let M^i/x_1 denote the set of monomials m in S' such that $mx_1^i \in M^i$. We start by noting down a result of Murai [11, 1.5] that is very useful for our purposes.

Lemma 1. *Let b_1, b_2, n be positive integers. Then*

$$b_1^{\langle n\rangle} + b_2^{\langle n\rangle} > (b_1 + b_2)^{\langle n\rangle}.$$

The following lemmas squeeze $d_{n,t}$ between $d^{\langle n-2\rangle}$ and $d^{\langle n-1\rangle}$.

Lemma 2. *Let $t \geq a-1$. Then $d_{n,t} \geq d^{\langle n-2\rangle}$.*

Proof. Let L be the lexsegment set of size d in R_t with $t \geq a-1$ and let j denote $\min(L)$. Since L is lexsegment, we have $L^i = R_t^i$ for $j < i \leq a-1$ giving $x_1 \cdot L^i \subseteq S_1' \cdot L^{i+1}$ for $j \leq i < a-1$. Moreover $x_1 \cdot L^{a-1}$ is empty and so we get $R_1 \cdot L = \bigsqcup_{j \leq i \leq a-1} S_1' \cdot L^i$. Note that L^i/x_1 is a lexsegment set in S' and so $|S_1' \cdot L^i| = |L^i|^{\langle n-2\rangle}$ because $|S_1' \cdot (L^i/x_1)| = |S_1' \cdot L^i|$. It follows that

$$d_{n,t} = |R_1 \cdot L| = \sum_{j \leq i \leq a-1} |L^i|^{\langle n-2\rangle}.$$

From this identity and Lemma 1 we obtain $d_{n,t} \geq d^{\langle n-2\rangle}$, as desired. □

Lemma 3. *Let M be a set of monomials in R_t with $t \geq a$. Let B denote the set of monomials in S_t that are divisible by x_1^a. We have the disjoint union*

$$S_1 \cdot (B \sqcup M) = (S_1 \cdot B) \sqcup ((S_1 \cdot M) \cap R).$$

Therefore $d_{n,t} = (d + |B|)^{\langle n-1\rangle} - |B|^{\langle n-1\rangle}$. In particular, $d_{n,t} < d^{\langle n-1\rangle}$.

Proof. Since $t \geq a$, B is non-empty. Note also that B is a lexsegment set in S because x_1 is the highest ranked variable. Since no monomial in R is divisible by x_1^a, the sets $S_1 \cdot B$ and $(S_1 \cdot M) \cap R$ are disjoint and we clearly have $S_1 \cdot (B \sqcup M) \supseteq (S_1 \cdot B) \sqcup ((S_1 \cdot M) \cap R)$. Conversely, let m be a monomial in $S_1 \cdot (B \sqcup M)$. We may take $m \in (S_1 \cdot M) \setminus R$. Then $m = x_1^a m'$ for some monomial m' that is not divisible by x_1. Since the degree of m is at least $a+1$, m' is divisible by one of the variables, say x_i for some $2 \leq i \leq n$. Then $m = x_i(x_1^a m'/x_i) \in S_1 \cdot B$. Secondly, putting a lexsegment set L for M in this formula yields $d_{n,t} = (d+|B|)^{\langle n-1\rangle} - |B|^{\langle n-1\rangle}$ because $R_1 \cdot L = (S_1 \cdot L) \cap R$ and $L \sqcup B$ is lexsegment in S_t. It also follows that $d_{n,t} < d^{\langle n-1\rangle}$ by Lemma 1. □

Since $R_1 \cdot M = (S_1 \cdot M) \cap R$, the previous lemma yields the following theorem.

Theorem 4. *Let M be a set of monomials in R_t. Then we have:*

(1) *If $t \geq a$, then M is Gotzmann in R_t if and only if $B \sqcup M$ is Gotzmann in S_t.*
(2) *If $t = a-1$, then M is Gotzmann in R_t if and only if M is Gotzmann in S_t and $x_1^{a-1} \in M$.*
(3) *If $t < a-1$, then M is Gotzmann in R_t if and only if M is Gotzmann in S_t.*

Proof. Let L denote the lexsegment set in R_t of the same size as M with $t \geq a$. Then Lemma 3 implies that $|R_1 \cdot L| = |R_1 \cdot M|$ if and only if $|S_1 \cdot (B \sqcup M)| = |S_1 \cdot (B \sqcup L)|$. Hence the first statement of the theorem follows because $B \sqcup L$ is lexsegment in S_t.

For $t = a - 1$, we have $d_{n,a-1} = d^{\langle n-1\rangle} - 1$. Let $M \in S_{a-1}$ be a set of monomials with $x_1^{a-1} \notin M$. Then $R_1 \cdot M = S_1 \cdot M$ and so $|R_1 \cdot M| \geq d^{\langle n-1\rangle} > d_{n,a-1}$. Conversely, if $x_1^{a-1} \in M$, then $|R_1 \cdot M| = |S_1 \cdot M| - 1$. Hence the second assertion of the theorem follows.

Finally, the last statement follows easily because for $t < a - 1$ we have $R_1 \cdot M = S_1 \cdot M$ and lexsegment sets in R_t and S_t are the same. □

Remark 5. This theorem does not generalize to all Macaulay-lex quotients. Consider the set of monomials $A := \{x_1^3x_2, x_1^3x_3, x_1x_2^3, x_2^3x_3\}$ whose size grows minimally in $F[x_1, x_2, x_3]/(x_1^4, x_2^4)$. But $A \sqcup \{x_2^4\}$ is not Gotzmann in $F[x_1, x_2, x_3]/(x_1^4)$. Furthermore, $A \sqcup \{x_1^4, x_2^4\}$ is not Gotzmann in $F[x_1, x_2, x_3]$.

Our second result concerns a Gotzmann set M in S. We show that M^i is a product of x_1^i with a Gotzmann set in S' if $|M^i|^{\langle n-2\rangle} \geq |M^{i-1}|$. Otherwise we provide lower bounds on $|M^i|$ depending on the sizes of neighboring components.

Lemma 6. *Let M be a Gotzmann set of monomials in S_t with $t \geq 0$. For $0 \leq i \leq t$ set $d_i = |M^i|$. For $0 \leq i \leq t + 1$ we have*

$$|(S_1 \cdot M)^i| = \max\{d_i^{\langle n-2\rangle}, d_{i-1}\}.$$

Proof. Note that we have $S_1' \cdot K^i = x_1^i \cdot (S_1' \cdot (K^i/x_1))$ and $(S_1 \cdot K)^i = S_1' \cdot K^i \cup x_1 \cdot K^{i-1}$ for any set K of monomials in S. Therefore $|S_1' \cdot M^i| = |S_1' \cdot (M^i/x_1)|$ which is at least $d_i^{\langle n-2\rangle}$. Meanwhile $|x_1 \cdot M_{i-1}| = d_{i-1}$. So

$$|(S_1 \cdot M)^i| \geq \max\{d_i^{\langle n-2\rangle}, d_{i-1}\} \quad \text{for } 0 \leq i \leq t + 1. \tag{2}$$

Let $\text{lex}_{S'}(M^i/x_1)$ denote the lexsegment set of $|M^i/x_1|$ monomials in S'_{t-i}. Define

$$T = \bigsqcup_{0\leq i\leq t} x_1^i \cdot (\text{lex}_{S'}(M^i/x_1)).$$

Notice that we have $|T^i| = d_i$ for $0 \leq i \leq t$. We compute $|(S_1 \cdot T)^i|$ for $0 \leq i \leq t+1$ as follows. Firstly, T^i/x_1 is a homogeneous lexsegment set by construction and so $|S_1' \cdot T^i| = |S_1' \cdot (T^i/x_1)| = d_i^{\langle n-2\rangle}$. On the other hand $|x_1 \cdot T^{i-1}| = d_{i-1}$. Moreover, since T^{i-1}/x_1 is a lexsegment set in S'_{t-i+1}, the identity $x_1 \cdot T^{i-1} = x_1^i \cdot (T^{i-1}/x_1)$ gives that $x_1 \cdot T^{i-1}$ is obtained by multiplying each element in a homogeneous lexsegment set in S' with x_1^i. Since $S_1' \cdot T^i$ is also obtained by multiplying the lexsegment set $S_1' \cdot (T^i/x_1)$ with x_1^i we have either $S_1' \cdot T^i \subseteq x_1 \cdot T^{i-1}$ or $S_1' \cdot T^i \supseteq x_1 \cdot T^{i-1}$. Hence $(S_1 \cdot T)^i = S_1' \cdot T^i$ if $d_i^{\langle n-2\rangle} \geq d_{i-1}$ and $(S_1 \cdot T)^i = x_1 \cdot T^{i-1}$ otherwise. It also follows that $|(S_1 \cdot T)^i| = \max\{d_i^{\langle n-2\rangle}, d_{i-1}\}$. Since the size of M has minimal growth, from Inequality (2) we get $|(S_1 \cdot M)^i| = \max\{d_i^{\langle n-2\rangle}, d_{i-1}\}$ as desired. □

We remark that the statement of the following theorem (and the previous lemma) stays true if we permute the variables and write the assertion with respect to another variable. It is also instructive to compare this with [10, 2.1].

Theorem 7. *Assume the notation of the previous lemma. If $d_i^{\langle n-2\rangle} \geq d_{i-1}$, then M^i/x_1 is Gotzmann in S'. Moreover, if $d_i^{\langle n-2\rangle} < d_{i-1}$, then we have either $(d_i + 1)^{\langle n-2\rangle} > d_{i-1} - 1$ or $d_i + 1 > d_{i+1}^{\langle n-2\rangle}$.*

Proof. Assume that $d_i^{\langle n-2\rangle} \geq d_{i-1}$ for some $0 \leq i \leq t$. Then from the previous lemma we have $|(S_1 \cdot M)^i| = d_i^{\langle n-2\rangle}$. But $S_1' \cdot M^i$ is a subset of $(S_1 \cdot M)^i$ and $|S_1' \cdot M^i| = |x_1^i \cdot (S_1' \cdot (M^i/x_1))| = |S_1' \cdot (M^i/x_1)| \geq d_i^{\langle n-2\rangle}$. It follows that $|S_1' \cdot (M^i/x_1)| = d_i^{\langle n-2\rangle}$ and so M^i/x_1 is Gotzmann.

We now prove the second assertion of the theorem. Assume that $d_q^{\langle n-2\rangle} < d_{q-1}$ for some $1 \leq q \leq t$. By way of contradiction assume further that $(d_q + 1)^{\langle n-2\rangle} \leq d_{q-1} - 1$ and $d_q + 1 \leq d_{q+1}^{\langle n-2\rangle}$. We obtain a contradiction by constructing a set W of size $|M|$ in S_t whose size grows strictly less than the size of M. Let w_{q-1} be the minimal monomial in T^{q-1}, where as before, $T = \sqcup_{0\leq i\leq t} x_1^i \cdot (\text{lex}_{S'}(M^i/x_1))$. Notice also that $d_q^{\langle n-2\rangle} < d_{q-1}$ implies that $S_t^q \setminus T^q \neq \emptyset$ and let w_q be the monomial that is maximal among the monomials in $S_t^q \setminus T^q$. Define

$$W = \left(\bigsqcup_{0\leq i\leq t, i\neq q-1,q} T^i\right) \sqcup (T^{q-1} \setminus \{w_{q-1}\}) \sqcup (T^q \cup \{w_q\}).$$

It suffices to show $|S_1 \cdot W| < |S_1 \cdot T|$ because $|S_1 \cdot T| = |S_1 \cdot M|$ by the (proof of the) previous lemma. Notice that by construction W^i/x_1 is a lexsegment set in S' for all $0 \leq i \leq t$. Therefore, just as we saw for T, we have $|(S_1 \cdot W)^i| = \max\{|W^i|^{\langle n-2\rangle}, |W^{i-1}|\}$. Again by construction, we also have $|W^i| = d_i$ for $i \neq q - 1, q$, and $|W^{q-1}| = d_{q-1} - 1$ and $|W^q| = d_q + 1$. It follows that $|(S_1 \cdot T)^i| = |(S_1 \cdot W)^i|$ for all $i \neq q - 1, q, q + 1$. We finish the proof by showing that

$$\sum_{q-1\leq i\leq q+1} |(S_1 \cdot W)^i| < \sum_{q-1\leq i\leq q+1} |(S_1 \cdot T)^i|.$$

We have $|(S_1 \cdot W)^{q-1}| = \max\{(d_{q-1} - 1)^{\langle n-2\rangle}, d_{q-2}\} \leq \max\{(d_{q-1})^{\langle n-2\rangle}, d_{q-2}\} = |(S_1 \cdot T)^{q-1}|$. Notice also that $|(S_1 \cdot W)^q| = \max\{(d_q + 1)^{\langle n-2\rangle}, d_{q-1} - 1\} = d_{q-1} - 1 < d_{q-1} = \max\{d_q^{\langle n-2\rangle}, d_{q-1}\} = |(S_1 \cdot T)^q|$. Finally, $|(S_1 \cdot W)^{q+1}| = \max\{d_{q+1}^{\langle n-2\rangle}, d_q + 1\} = d_{q+1}^{\langle n-2\rangle} = |(S_1 \cdot T)^{q+1}|$. □

Remark 8. Let M be a Gotzmann set in S. Since $d_{\min(M)-1} = 0$, the above theorem implies that $M^{\min(M)}/x_1$ is Gotzmann in S'. On the other hand M^i/x_1 does not need to be Gotzmann for all i. An example already exists in [10, 2.2] where

Murai lists all Gotzmann sets of size 11 in degree 4 in $S = F[x_1, x_2, x_3]$ up to a permutation. The set $M = \{x_1^2x_2^2, x_1^2x_3^2, x_1x_2^3, x_1x_2^2x_3, x_1x_2x_3^2, x_1x_3^3, x_2^4, x_2^3x_3, x_2^2x_3^2, x_2x_3^3, x_3^4\}$ is a Gotzmann set in S_4 but $M^2/x_1 = \{x_2^2, x_3^2\}$ is not Gotzmann in $S' = F[x_2, x_3]$. We have $d_2 = 2$ and $d_1 = 4$ and so $d_2^{\langle 1\rangle} = 3 < 4 = d_1$ as projected by the previous theorem. We remark that M is the only one among the ten Gotzmann sets in S_4 such that there exists an integer i with M^i/x_1 that is not Gotzmann in S'.

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