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On robust solutions to linear least squares problems affected by data uncertainty and implementation errors with application to stochastic signal modeling

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Abstract

Engineering design problems, especially in signal and image processing, give rise to linear least squares problems arising from discretization of some inverse problem. The associated data are typically subject to error in these applications while the computed solution may only be implemented up to limited accuracy digits, i.e., quantized. In the present paper, we advocate the use of the robust counterpart approach of Ben-Tal and Nemirovski to address these issues simultaneously. Approximate robust counterpart problems are derived, which leads to semidefinite programming problems yielding stable solutions to overdetermined systems of linear equations affected by both data uncertainty and implementation errors, as evidenced by numerical examples from stochastic signal modeling.

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1. Introduction

In this paper we consider the problem of computing robust solutions to the linear least squares problem

$$\min_x \|Ax - b\|_2, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$ with $m > n$ and $b \in \mathbb{R}^m$ may be uncertain and the computed solution can only be implemented up to a certain number of accuracy digits. By uncertainty in (A, b) we mean that we are not facing a single pair (A, b) but a family of matrices $(A + \Delta A, b + \Delta b)$ where $(\Delta A, \Delta b)$ are unknown but bounded (in norm) perturbations. Furthermore, there is an implementation uncertainty on x of the form $x + \Delta x$ where Δx represents a certain quantization error inherent in many engineering applications.

There exist many references on the least squares problem. To avoid trying to list all at the expense of omitting some we adopt the excellent book by Björck [5] as our desktop reference. The problem of uncertainty in (A, b) is addressed by using several remedies, such as total least squares and variants thereof, truncated SVD, Tikhonov regularization, iterated regularization, L-curve analysis and so on. The interested reader is directed to Section 7 of Chapter 2 and Section 6 of Chapter 4 in [5]. An important reference on total least squares and its applications in engineering is by Van Huffel and Vandevelle [17]. Ample information on regularization methods can be found in the book by Hansen [10]. For recent related articles on least squares problems under uncertainty, the reader is directed to [6,7,18] as well. Another important line of research dealing with uncertainty in linear systems of equations summarized by Kreinovich et al. [11] is the subject of interval computations, with an emphasis on complexity issues. However, none of the above references nor any other reference that we know of treat the problem of perturbations in the data elements and implementation inaccuracies together.

The purpose and contribution of the present paper are to present a methodology to address the problem of uncertainty in (A, b) and the solution vector x simultaneously. Although the method to be discussed below originates in the contributions by Ben-Tal et al. [1–4], and also from [8] where the authors treat the problem of uncertainty in (A, b) with bounded but otherwise unknown perturbation matrices, the present paper is the first, to the best of the authors' knowledge, to blend the ingredients of these references to solve the problem of uncertainty in both data and implementation levels. Our main results are given in Theorems 2 and 3, and Propositions 2 and 3, respectively. Given the pervasive nature of least squares problem in signal and image processing applications where typically some discretization of an inverse problem gives rise to these problems, we believe the present undertaking to be worth the attention of the signal and image processing communities. Therefore, the contribution of the paper is not only methodological but also application oriented. We substantiate this claim in Section 3 where we offer a successful and, to the best of our knowledge, novel application of the proposed methodology to the stochastic signal modeling problem.

The rest of the paper is organized as follows. In Section 2, we describe the robust counterpart methodology as applied to least squares problems under both data uncertainty and implementation errors. In Section 3, we give numerical application examples from stochastic signal modeling illustrating the effectiveness of the proposed method.

Concerning notation, we use $\mathscr{S}^m$ to denote the space of $m \times m$ symmetric matrices, and $\mathscr{S}^m_+$ to denote the space of $m \times m$ symmetric, positive semidefinite matrices, while $X \succeq 0$ means the symmetric matrix X is positive semidefinite. When we write $X \succeq Y$ for two symmetric matrices X and Y, we mean that $X - Y \succeq 0$. The Frobenius norm of a matrix $H \in \mathbb{R}^{m\times n}$ with entries H_{ij}, denoted $\|H\|_{\mathrm{F}}$, is equal to $\sqrt{\sum_{i=1}^m \sum_{j=1}^n H_{ij}^2}$.

2. The robust counterpart methodology

In this section we derive approximate robust counterparts for linear least squares problems in three cases: (1) (A, b) is subject to unknown but bounded perturbation measured in Frobenius norm while x is subject to implementation errors measured in the Euclidean norm, (2) (A, b) is subject to interval uncertainty (equivalent to imposing ∞-norm uncertainty bound on (A, b)) while x is still subject to errors measured in the Euclidean norm, and (3) each row of A is subject to independent uncertainty measured in the Euclidean norm, b is uncertain, x is subject to implementation errors in the Euclidean norm.

These different models of representing uncertainty are appropriate under different modeling environments. Interval form of uncertainty is relevant in cases where the modeler is able to quantify an upper bound on the maximum error in each data element separately (or, in each computed variable). Euclidean norm or Frobenius norm uncertainty representation may be more appropriate when measurement or implementation errors concerning different data elements and/or variables are interrelated, i.e., interdependent, and when only aggregate error information is available on uncertain quantities. In such cases some probabilistic information about the errors may alternatively be given or inferred from observed data, which may be incorporated as a joint norm bound into the uncertainty representation.

Since in all three cases we consider below the first step is identical, we present it here. Assume that the problem data (A, b) are uncertain, subject to unknown but bounded perturbation of the form

$$\|(\Delta A, \Delta b)\|_p \leqslant \gamma, \tag{2}$$

where $\|\cdot\|_p$ denotes some suitable norm and, γ is a positive scalar. We also assume that the vector x is subject to implementation errors of a similar form:

$$\|\Delta x\|_2 \leqslant \epsilon,$$

where $\|\cdot\|_2$ denotes the usual Euclidean vector norm, and ϵ is a positive scalar. To guarantee optimal performance in a worst-case scenario we adopt the min–max view of the least squares problem as follows: compute an x that minimizes the function

$$\max_{\|\Delta x\|_2 \leqslant \epsilon, \|(\Delta A, \Delta b)\|_p \leqslant \gamma} \|(A + \Delta A)(x + \Delta x) - (b + \Delta b)\|_2. \tag{3}$$

We refer to the problem of minimizing function (3) over x as the "robust counterpart" problem to the uncertain least squares problem, or robust least squares problem for short. We can equally treat the problem in epigraph form:

$$\min_{x,t} t \tag{4}$$

subject to

$$\max_{\|\Delta x\|_2 \leqslant \epsilon, \|(\Delta A, \Delta b)\|_p \leqslant \gamma} \|(A + \Delta A)(x + \Delta x) - (b + \Delta b)\|_2 \leqslant t, \tag{5}$$

which is more convenient for our purposes. Now let us deal with the perturbations in x and (A, b) one at a time. Keeping $(\Delta A, \Delta b)$ fixed, we concentrate on the inequality

$$\max_{\|\Delta x\|_2 \leqslant \epsilon} \|(A + \Delta A)(x + \Delta x) - (b + \Delta b)\|_2 \leqslant t. \tag{6}$$

In a recent paper [12], using Proposition 4.5.60 of [3] Lewis studied a slightly more general version of problem (6). Based on Theorem 2.1 of [12] we can immediately state that inequality (6) is equivalent to the following: there exists $\mu \in \mathbb{R}$ such that

$$\begin{bmatrix} tI_m & (A + \Delta A)x - (b + \Delta b) & \epsilon(A + \Delta A) \\ x^{\mathrm{T}}(A + \Delta A)^{\mathrm{T}} - (b + \Delta b)^{\mathrm{T}} & t - \mu & 0 \\ \epsilon(A + \Delta A)^{\mathrm{T}} & 0 & \mu I_n \end{bmatrix} \succeq 0. \tag{7}$$

Now, according to the specification of the norm in (2) we obtain different formulations as detailed below.

2.1. *Bounded perturbations in (A, b) measured in the Frobenius norm*

Now, we deal with the maximization problem over $(\Delta A, \Delta b)$. Assume that the problem data (A, b) are subject to unknown but bounded perturbation of the form

$$\|(\Delta A, \Delta b)\|_{\mathrm{F}} \leqslant \gamma. \tag{8}$$

Now we want the above inequality (7) to hold for all realizations of $(\Delta A, \Delta b)$ such that $\|(\Delta A, \Delta b)\|_{\mathrm{F}} \leqslant \gamma$, i.e., we want the linear matrix inequality

$$\begin{bmatrix} tI_m & (A + \Delta A)x - (b + \Delta b) & \epsilon(A + \Delta A) \\ x^{\mathrm{T}}(A + \Delta A)^{\mathrm{T}} - (b + \Delta b)^{\mathrm{T}} & t - \mu & 0 \\ \epsilon(A + \Delta A)^{\mathrm{T}} & 0 & \mu I_n \end{bmatrix} \succeq 0 \tag{9}$$

to be satisfied for all $(\Delta A, \Delta b)$ such that $\|(\Delta A, \Delta b)\|_{\mathrm{F}} \leqslant \gamma$. This is a system of semiinfinite linear matrix inequalities (LMI). Now, we pose the above semidefiniteness condition in the following form:

$$\mathscr{A}^0(x, t, \mu) + \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta A_{ij}\mathscr{A}^{ij}(x) + \sum_{i=1}^{m} \Delta b_i \mathscr{B}^i \succeq 0 \quad \forall (\Delta A, \Delta b),$$
$$\text{with } \|(\Delta A, \Delta b)\|_{\mathrm{F}} \leqslant \gamma,$$

where ΔA_{ij} denotes the (i, j)-entry of ΔA, Δb_i denotes the ith component of Δb, each symmetric $(m + n + 1) \times (m + n + 1)$ matrix $\mathscr{A}^{ij}$ has at most four non-zero entries, x_j in column $m + 1$ and row j, ϵ in column $m + 1 + j$ and row i, and the two symmetric entries below the diagonal. Similarly, each symmetric $(m + n + 1) \times (m + n + 1)$ matrix $\mathscr{B}^i$ has two non-zero entries, -1 in column $m + 1$ and row i, and the symmetric entry with respect to the diagonal, and finally $\mathscr{A}^0$ is given as

$$\mathscr{A}^0(x, t, \mu) = \begin{bmatrix} tI_m & Ax - b & \epsilon A \\ (Ax - b)^{\mathrm{T}} & t - \mu & 0 \\ \epsilon A^{\mathrm{T}} & 0 & \mu I_n \end{bmatrix}. \tag{10}$$

Unfortunately, the inequality above leads to a computationally intractable (NP-hard) robust counterpart (see [1, pp. 791–792]). Related complexity results on uncertain linear systems are also contained in the book [11] especially in Chapter 23 where ellipsoidal uncertainty in linear systems is discussed. On the other hand, it is possible to derive an approximate but computationally tractable (e.g., solvable in polynomial time) robust counterpart to the problem

$$\min_{x,t,\mu} t$$

subject to the semiinfinite system of linear matrix inequalities (9). To do this we need some background material from [1]. We note here that another important reference on robust linear matrix inequalities is El Ghaoui et al. [9].

Consider an "uncertain" convex programming problem in the form

$$\text{(UNCCP)} \quad \min_x c^{\mathrm{T}}x\colon \mathscr{A}^i(x) \in \mathscr{S}_+^{l_i}, \quad i = 1, \ldots, m,$$

where the term "uncertain" refers to the specification of the data that are allowed to take a continuum of values within a specific uncertainty set, namely, $\mathscr{A}^i(\cdot) \in \mathscr{A}^{i0}(\cdot) + \mathscr{V}_i$, $i = 1, \ldots, K$, $\mathscr{A}^{i0}$ being an affine mapping from $\mathbb{R}^n$ to $\mathscr{S}^{l_i}$, and $\mathscr{V}_i$, $i = 1, \ldots, K$, with $0 \in \mathscr{V}_i$, are convex perturbations sets in the space of mappings of this type. Furthermore, we assume that each set $\mathscr{V}_i$, $i = 1, \ldots, K$, can be approximated by an ellipsoid "up to factor γ_i", i.e., we can find k_i affine mappings $\mathscr{A}^{ij}(\cdot)$: $\mathbb{R}^n \mapsto \mathscr{S}^{l_i}$, $j = 1, \ldots, k_i$ in such a way that $\mathscr{V}_i^- \subseteq \mathscr{V}_i \subseteq \gamma_i \mathscr{V}_i^-$, where

$$\mathscr{V}_i^- = \left\{ \sum_{j=1}^{k_i} u_j \mathscr{A}^{ij}(\cdot)\colon u^{\mathrm{T}}u \leqslant 1 \right\}.$$

The robust counterpart problem to (UNCCP) is a "certain" (i.e., it does not involve uncertainty) optimization problem of the form

$$\min_x c^{\mathrm{T}}x\colon \mathscr{A}^i(x) \in \mathscr{S}_+^{l_i} \quad \forall \mathscr{A}^i(\cdot) \in \mathscr{A}^{i0}(\cdot) + \mathscr{V}_i, \;\; i = 1, \ldots, K.$$

I.e., we want to compute an optimal x among all those points that satisfy the constraints of the problem for all possible realizations of the data according to our model of uncertainty. The robust counterpart problem has an infinite number of constraints. Unfortunately, computing an optimal solution to the above robust counterpart problem is NP-hard in general, [1], hence the need to look for an "approximate robust counterpart".

Definition 1. A certain optimization problem Π is an approximate robust counterpart of an uncertain optimization problem P if the objective functions of both problems are identical and the feasible set of Π (or the projection of the feasible set of Π on the plane of x-variables) is contained in the feasible set of the robust counterpart P^* of P, i.e., Π is more conservative than P^*.

Let $G_+(\Pi)$ denote the feasible set of the approximate robust counterpart problem Π and $G_*(\gamma)$ (where γ is our estimate of uncertainty in the data, e.g., as in (2)) denote the feasible set of the robust counterpart P^* (in our case, the problem of minimizing function (3) over x, or alternatively, problems (4) and (5)). A measure of conservatism of Π as an approximation to P^* is as follows:

$$\mathrm{cons}(\Pi) = \inf\{\rho \geqslant 1\colon G_*(\rho\gamma) \subset G_+(\Pi)\}.$$

Definition 2. Π is an α-conservative approximation of P^* if $\mathrm{cons}(\Pi) \leqslant \alpha$, i.e., if

$$\begin{aligned} x \in G_+(\Pi) \quad &\Longrightarrow \quad x \in G_*(\gamma),\\ x \notin G_+(\Pi) \quad &\Longrightarrow \quad x \notin G_*(\rho\gamma) \quad \forall \rho > \alpha. \end{aligned}$$

Put in other words, we are looking for the smallest ρ-enlargement of the uncertainty level initially specified as γ in (2) such that the feasible solutions of the approximate robust counterpart would be contained in the set of feasible solutions of the true robust counterpart (which we do not compute) corresponding to uncertainty specification γ. On the other hand, if a point is not in the set of feasible solutions of the approximate robust counterpart, then it is not a feasible point for the robust counterpart corresponding to a ρ-enlargement of the uncertainty level, i.e., an uncertainty specification of the form

$$\|(\Delta A, \Delta b)\|_{\mathrm{F}} \leqslant \rho\gamma$$

with $\rho \geqslant 1$. Notice that a 1-conservative approximate robust counterpart coincides exactly with the robust counterpart.

We can now cite the following theorem [1, Theorem 3.4].

Theorem 1. *The convex programming problem*

$$\min_x c^{\mathrm{T}} x$$

subject to

$$\begin{bmatrix} \mathscr{A}^{i0}(x) & \gamma_i \mathscr{A}^{i1}(x) & \gamma_i \mathscr{A}^{i2}(x) & \cdots & \gamma_i \mathscr{A}^{ik_i}(x) \\ \gamma_i \mathscr{A}^{i1}(x) & \mathscr{A}^{i0}(x) & & & \\ \gamma_i \mathscr{A}^{i2}(x) & & \mathscr{A}^{i0}(x) & & \\ \vdots & & & \ddots & \\ \gamma_i \mathscr{A}^{ik_i}(x) & & & & \mathscr{A}^{i0}(x) \end{bmatrix} \succeq 0, \quad i = 1, \ldots, K \tag{11}$$

is an α*-conservative approximate robust counterpart of the problem* (UNCCP) *with*

$$\alpha = \max_{i=1,\ldots,K} \gamma_i \min\left[\sqrt{k_i}, \sqrt{l_i}\right].$$

Now, using the above result an approximate robust counterpart of the uncertain least squares problem is the following semidefinite programming problem (see [16] for a review of semidefinite programming) that we refer to as [RLS]:

$$\min_{x,t,\mu} t$$

subject to

$$\begin{bmatrix} \mathscr{A}_0(x,t,\mu) & \gamma \mathscr{A}_{11}(x) & \gamma \mathscr{A}_{12}(x) & \cdots & \gamma \mathscr{A}_{mn}(x) & \gamma \mathscr{B}_1 & \cdots & \gamma \mathscr{B}_m \\ \gamma \mathscr{A}_{11}(x) & \mathscr{A}_0(x) & & & & & & \\ \gamma \mathscr{A}_{12}(x) & & \mathscr{A}_0(x) & & & & & \\ \vdots & & & \ddots & & & & \\ \gamma \mathscr{A}_{mn}(x) & & & & \mathscr{A}_0(x) & & & \\ \gamma \mathscr{B}_1 & & & & & \mathscr{A}_0(x) & & \\ \vdots & & & & & & \ddots & \\ \gamma \mathscr{B}_m & & & & & & & \mathscr{A}_0(x) \end{bmatrix} \succeq 0. \tag{12}$$

Furthermore, the conservatism level of [RLS] is given by

$$\alpha = \gamma \min\left(\sqrt{m(n+1)}, \sqrt{m+n+1}\right).$$

To see this, it suffices to observe that the problem of finding a robust counterpart to uncertain least squares problem, which we posed as

$$\min_{x,t,\mu} t$$

subject to the semiinfinite system of linear matrix inequalities (9), is exactly in the form of problem (UNCCP) with $K = 1$, $k_1 = m(n + 1)$ and $l_1 = m + n + 1$.

Now, we can summarize our findings in the result below.

Theorem 2. *The semidefinite programming problem* [RLS] *over the variables* (x, t, μ) *is an* α*-conservative approximation to the robust least squares problem*

$$\min_{x} \max_{\|\Delta x\|_2 \leqslant \epsilon, \|(\Delta A, \Delta b)\|_{\mathrm{F}} \leqslant \gamma} \|(A + \Delta A)(x + \Delta x) - (b + \Delta b)\|_2$$

with

$$\alpha = \gamma \min\left(\sqrt{m(n+1)}, \sqrt{m+n+1}\right).$$

Problem [RLS] is a special convex optimization problem, known as a semidefinite programming problem (SDP) which is efficiently solvable with polynomial interior point methods [16,19]. Many efficient and reliable software systems already exist for the solution of SDPs, among them the package SeDuMi of Sturm [14] that we use in the present paper, cf. Section 3. On the other hand, the SDP problem of Theorem 2 above has data matrices of dimension $(mn + m + 1)(m + n + 1) \times (mn + m + 1)(m + n + 1)$ which, given the current computational state-of-the-art, limits its use to problems of relatively small sizes. One such engineering problem, of genuine interest in spite of its small size, is treated successfully using the above results in Section 3, namely, stochastic signal modeling in digital signal processing.

2.2. (A, b) *subject to interval uncertainty*

Consider now the uncertain least squares problem where (A, b) is subject to *interval uncertainty* of the form $\|(A, b)\|_\infty \leqslant \rho$ (notice that $\|H\|_\infty = \max_{i,j} |H_{ij}|$) and x is still subject to errors measured in the Euclidean norm. Complexity issues regarding interval computations on linear systems of equations are discussed in detail in Chapter 11 of [11] where the authors prove several NP-hardness results.

The robust counterpart problem we are interested in is now formulated as follows:

$$\min_{x} \max_{\|\Delta x\|_2 \leqslant \epsilon, \|(A,b)\|_\infty \leqslant \rho} \|A(x + \Delta x) - b\|_2.$$

Using our transformation at the beginning of this section we transform the problem into

$$\min_{x,t,\mu} t$$

subject to

$$\begin{bmatrix} tI_m & Ax - b & \epsilon A \\ x^{\mathrm{T}}A^{\mathrm{T}} - b^{\mathrm{T}} & t - \mu & 0 \\ \epsilon A^{\mathrm{T}} & 0 & \mu I_n \end{bmatrix} \succeq 0 \quad \forall (A, b) \text{ s.t. } \|(A, b)\|_\infty \leqslant \rho. \tag{13}$$

At this point, it is useful to recollect the work of Ben-Tal and Nemirovski [2] in a form suitable for our purposes.

Ben-Tal and Nemirovski study the following problem that they term the "matrix cube" problem.

MatrCube: Given $x \in \mathbb{R}^n$ and an affine mapping $u \mapsto \mathscr{B}(u) = B^0[x] + \sum_{\ell=1}^{L} u_\ell B^\ell[x]$ from $\mathbb{R}^L$ to the space $\mathscr{S}^m$ of $m \times m$ symmetric matrices ($B^0[x]$, $B^1[x], \ldots, B^L[x]$ are affine functions of x) and $\rho > 0$ check whether the image

$$\mathscr{C}[\rho] = \{A \mid \exists (u, \|u\|_\infty \leqslant \rho) \colon A = \mathscr{B}(u)\}$$

of the box $\{u \colon \|u\|_\infty \leqslant \rho\}$ under this mapping is contained in the cone $\mathscr{S}_+^m$ of positive semidefinite symmetric matrices. That is, check whether $B^0[x] + \sum_{\ell=1}^{L} u_\ell B^\ell[x]$ is positive semidefinite for all u such that $\|u\|_\infty \leqslant \rho$.

A simple sufficient condition for the inclusion $\mathscr{C}[\rho] \subset \mathscr{S}_+^m$ is given in [2] as

(S) Assume there exist matrices $X_1, \ldots, X_L$ satisfying the system of linear matrix inequalities in variables $x, X_1, \ldots, X_L$

$$X_\ell \succeq \pm \rho B^\ell[x], \quad \ell = 1, \ldots, L, \tag{14}$$

$$\sum_{\ell=1}^{L} X_\ell \preceq B^0[x]. \tag{15}$$

Then $\mathscr{C}[\rho] \subset \mathscr{S}_+^m$. I.e., if x can be extended to a solution of (14) and (15), then $B^0[x] + \sum_{\ell=1}^{L} u_\ell B^\ell[x]$ is positive semidefinite for all u such that $\|u\|_\infty \leqslant \rho$.

Furthermore, Ben-Tal and Nemirovski [2] prove the following result [2, Theorem 2.1].

Theorem 3. *Consider the problem* MatrCube *along with system of LMIs* (14) *and* (15) *in variables* $x, X_1, \ldots, X_L$, *and let*

$$\mu = \max_{x, 1 \leqslant \ell \leqslant L} \operatorname{rank}(B^\ell[x]).$$

Then

1. *If the system of LMIs* (14) *and* (15) *is solvable, the matrix box* $\mathscr{C}[\rho]$ *is contained in the positive semidefinite cone* $\mathscr{S}_+^m$.
2. *If the system of LMIs* (14) *and* (15) *is not solvable, the* $\theta(\mu)$*-enlargement* $\mathscr{C}[\theta(\mu)\rho]$ *of the matrix box* $\mathscr{C}[\rho]$ *is not contained in the positive semidefinite cone* $\mathscr{S}_+^m$, *where the function* $\theta(\cdot)$ *is given by*

$$\frac{1}{\theta(k)} = \min_{\alpha} \left\{ \int_{\mathbb{R}^k} \left| \sum_{i=1}^{k} \alpha_i u_i^2 \right| (2\pi)^{-k/2} \exp\left(-\frac{u^{\mathrm{T}} u}{2}\right) \mathrm{d}u \colon \sum_{i=1}^{k} |\alpha_i| = 1 \right\}.$$

Furthermore, $\theta(\cdot)$ *satisfies the relations*

$$\theta(k) \leqslant \frac{\pi\sqrt{k}}{2} \quad \forall k$$

and $\theta(2) = \frac{\pi}{2}$.

Now, we can pose the semidefiniteness condition (13) that we refer to as $\Xi(\rho)$ in the following form:

$$\mathscr{A}^0(t,\mu) + \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\mathscr{A}^{ij}(x) + \sum_{i=1}^{m} b_i\mathscr{B}^i \succeq 0 \quad \forall (A,b),$$
$$\text{with } \|(A,b)\|_\infty \leqslant \rho,$$

where a_{ij} denotes the (i,j)-entry of A, b_i denotes the ith component of b, each symmetric $(m+n+1)\times(m+n+1)$ matrix $\mathscr{A}^{ij}$ has at most four non-zero entries, x_j in column $m+1$ and row j, ϵ in column $m+1+j$ and row i, and the two symmetric entries below the diagonal. Similarly, each symmetric $(m+n+1)\times(m+n+1)$ matrix $\mathscr{B}^i$ has two non-zero entries, -1 in column $m+1$ and row i, and the symmetric entry with respect to the diagonal, and finally $\mathscr{A}^0$ is given as

$$\mathscr{A}^0(t,\mu) = \begin{bmatrix} tI_m & 0 & 0 \\ 0 & t-\mu & 0 \\ 0 & 0 & \mu I_n \end{bmatrix}. \tag{16}$$

Interestingly, this structure fits perfectly into the uncertain LMI structure treated in [2] that we summarized above. Therefore, we can immediately make the following statement.

The positive semidefiniteness condition (13) holds for a given x if there exist matrices X_{ij}, $i=1,\dots,m$, $j=1,\dots,n$, matrices Y_i, $i=1,\dots,n$ satisfying the system of linear matrix inequalities:

$$X_{ij} \succeq \pm\rho\mathscr{A}^{ij}(x), \quad i=1,\dots,n, \quad j=1,\dots,m, \tag{17}$$

$$Y_i \succeq \pm\rho\mathscr{B}^i, \quad i=1,\dots,n, \tag{18}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij} + \sum_{i=1}^{n} Y^i \preceq \mathscr{A}^0(t,\mu). \tag{19}$$

Although conditions (17)–(19) provide only a sufficient condition for the positive semidefiniteness condition (13) to hold, they constitute a $\frac{\pi}{2}$-conservative approximation (in the sense of Definitions 1 and 2) to the robust counterpart problem

$$\min_{x,t,\mu} t$$

subject to (13). More precisely, if the system consisting of (17)–(19) is solvable, i.e., a given x can be extended to a solution of (17)–(19), then $\Xi(\rho)$ is solvable. If the system consisting of (17)–(19) is not solvable, i.e., a given x cannot be extended to a solution of (17)–(19), then a $\frac{\pi}{2}$-enlargement $\Xi(\frac{\pi}{2}\rho)$ of $\Xi(\rho)$ is not solvable. We state this result below. We refer to the optimization problem

$$\min_{x,t,\mu,X_{ij},Y_i,i=1,\ldots,m,j=1,\ldots,n} t$$

subject to (17)–(19) as problem (RLSint).

Theorem 4. *The semidefinite programming problem* [RLSint] *over* (X_{ij}, Y_i, x, t, μ), $i = 1, \ldots, m$ *and* $j = 1, \ldots, n$ *is a* $\frac{\pi}{2}$*-conservative approximation to the robust least squares problem*

$$\min_x \max_{\|\Delta x\|_2 \leqslant \epsilon, \|(A,b)\|_\infty \leqslant \rho} \|A(x + \Delta x) - b\|_2.$$

Proof. The result follows directly from Theorem 3 above after observing that the matrices $\mathscr{A}^{ij}$ and $\mathscr{B}^i$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ are rank-2 matrices. To see this observe that each matrix $\mathscr{A}^{ij}(\cdot)$ is given by

$$\mathscr{A}^{ij}(x) = c^i(d^j)^{\mathrm{T}}[x] + d^j[x](c^i)^{\mathrm{T}},$$

where $c^i \in \mathbb{R}^{m+n+1}$ with $c^i_i = 1$ and all remaining entries zero, and $d^j[x] \in \mathbb{R}^{m+n+1}$ with x_j as the $(m+1)$th component and ϵ as the $(m+1+j)$th component with all remaining components zero. Similarly, each matrix $\mathscr{B}^i$ is given by

$$\mathscr{B}^i = e^i f^{\mathrm{T}} + f(e^i)^{\mathrm{T}},$$

where $e^i \in \mathbb{R}^{m+n+1}$ with $e_i = 1$ and all remaining entries zero, and $f \in \mathbb{R}^{m+n+1}$ with with -1 as the $(m+1)$th component with all remaining components zero. □

Notice that although (RLSint) is polynomially solvable by efficient interior point methods, the dimensions of the linear matrix inequality and the number of matrix variables involved make the problem too large for numerical processing. To be precise, the approximate robust counterpart problem [RLSint] has $mn + m$ matrix variables, each matrix variable being a symmetric $(m + n + 1) \times (m + n + 1)$ matrix, in addition to the original x variables, and scalars t and μ. Furthermore, the system involves $mn + m + 1$ linear matrix inequalities. However, Ben-Tal and Nemirovski [2] exhibit a transformation result that applies in the presence of rank-2 matrices $\mathscr{A}^{ij}$ and $\mathscr{B}^i$ leading to a reduction in the design dimension of the problem (the size of variables), for an entirely different problem: quadratic Lyapunov stability analysis and synthesis. Their result is the following. When the symmetric $m \times m$ matrices $B^0[x]$, $B^1[x], \ldots, B^L[x]$ (affinely dependent on x) are of the form

$$B^\ell[x] = a_\ell b_\ell^{\mathrm{T}}[x] + b_\ell[x]a_\ell^{\mathrm{T}}, \quad \ell = 1, \ldots, L,$$

where $a_\ell \neq 0$ and $b_\ell[x]$ is not equal to the null vector for all x and affinely dependent on x, Ben-Tal and Nemirovski prove the following [2, Proposition 3.3].

Proposition 1. *The LMI system* (14) *and* (15) *is equivalent to the following system of LMIs in variables x and additional variables $Y \in \mathscr{S}^m$ and $\lambda \in \mathbb{R}^L$:*

$$\begin{bmatrix} Y - \sum_{\ell=1}^{L} \lambda_\ell a_\ell (a_\ell)^{\mathrm{T}} & b_1[x] & b_2[x] & \cdots & b_L[x] \\ b_1^{\mathrm{T}}[x]T & \lambda_1 & & & \\ b_2^{\mathrm{T}}[x] & & \lambda_2 & & \\ \vdots & & & \ddots & \\ b_L^{\mathrm{T}}[x] & & & & \lambda_L \end{bmatrix} \succeq 0, \tag{20}$$

$$\rho Y \preceq B^0[x]. \tag{21}$$

Our application fits exactly into their framework, which leads to the following result which follows by direct application of Proposition 1 above.

Proposition 2. *The LMI system composed of* (17)–(19) *is equivalent to the following system of LMIs in variables x and additional variables $Y \in \mathscr{S}^{m+n+1}$, $\lambda \in \mathbb{R}^{mn}$ and $\beta \in \mathbb{R}^m$:*

$$\begin{bmatrix} Y - \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij} c^i (c^i)^{\mathrm{T}} + \sum_{i=1}^{n} \beta_i e^i (e^i)^{\mathrm{T}} & d^{11}[x] & d^{12}[x] & \cdots & d^{mn}[x] & f & \cdots & f \\ (d^{11}[x])^{\mathrm{T}} & \lambda_{11} & & & & & & \\ (d^{12}[x])^{\mathrm{T}} & & \lambda_{12} & & & & & \\ \vdots & & & \ddots & & & & \\ (d^{mn}[x])^{\mathrm{T}} & & & & \lambda_{mn} & & & \\ f^{\mathrm{T}} & & & & & \beta_1 & & \\ \vdots & & & & & & \ddots & \\ f^{\mathrm{T}} & & & & & & & \beta_m \end{bmatrix} \succeq 0, \tag{22}$$

$$\rho Y \preceq \mathscr{A}^0(t, \mu). \tag{23}$$

In Proposition 2 above, for convenience in the exposition we have used $d^{ij}[x]$ to mean d^i, i.e., $d^{ij}[x] = d_i$ for all $i = 1, \ldots, m,\ j = 1, \ldots, n$. Notice that in the equivalent approximate robust counterpart problem

$$\min_{x,t,\mu,Y,\lambda_{ij},\beta_j,i=1,\ldots,m,j=1,\ldots,n} t$$

subject to (22) and (23), we are dealing with a single linear matrix inequality of row and column dimension equal to $(mn + m + 1)(m + n + 1)$ and another linear matrix inequality of dimension $(m+n+1) \times (m+n+1)$ while we deal with a single symmetric matrix variable $Y \in \mathscr{S}^{m+n+1}$, a vector $\lambda \in \mathbb{R}^{mn}$ and a vector $\beta \in \mathbb{R}^m$ in addition to the original variables x, and scalars t and μ.

2.3. Each row of A subject to independent uncertainty measured in the Euclidean norm

In the case where each row of A is subject to independent uncertainty measured in the Euclidean norm (as well as b) we can give another approximate robust counterpart. Consider the least squares problem (1) where each row A_i of A is subject to uncertainty of the form $A_i + \Delta A_i$ with $\Delta A_i = (\Delta A_{i1}, \ldots, \Delta A_{in})$ such that $\|\Delta A_i\|_2 \leqslant \gamma_i$, for all $i = 1, \ldots, m$. The vector b is also assumed to be uncertain of the form $b + \Delta b$ where $\Delta b \in \mathbb{R}^m$ with $\|\Delta b\|_2 \leqslant \epsilon_b$. We assume as usual implementation errors in x of the form $x + \Delta x$, with $\|\Delta x\|_2 \leqslant \epsilon$. The robust problem we are interested in is now formulated as follows:

$$\min_x \max_{\substack{\|\Delta x\|_2 \leqslant \epsilon, \\ \|\Delta A_i\|_2 \leqslant \gamma_i,\, i=1,\ldots,m, \\ \|\Delta b\|_2 \leqslant \epsilon_b}} \sqrt{\sum_{i=1}^m \left[(A_i + \Delta A_i)^{\mathrm{T}}(x + \Delta x) - (b_i + \Delta b_i)\right]^2}.$$

The transformed problem after treating the uncertainty in x first as in the beginning of this section is the following problem referred to as [RLSIND]:

$$\min_{x,t,\mu} t$$

subject to

$$\begin{bmatrix} tI_m & (A+\Delta A)x - (b+\Delta b) & \epsilon(A+\Delta A) \\ x^{\mathrm{T}}(A+\Delta A)^{\mathrm{T}} - (b+\Delta b)^{\mathrm{T}} & t-\mu & 0 \\ \epsilon(A+\Delta A)^{\mathrm{T}} & 0 & \mu I_n \end{bmatrix} \succeq 0. \tag{24}$$

Now we want the above inequality (24) to hold for all realizations of $(\Delta A, \Delta b)$ such that $\|\Delta A_i\|_2 \leqslant \gamma_i$, for all $i = 1, \ldots, m$ and Δb with $\|\Delta b\|_2 \leqslant \epsilon_b$. Now, the above semidefiniteness condition is equivalent to the following:

$$\mathscr{A}^0(x,t,\mu) + \sum_{i=1}^m \sum_{j=1}^n \Delta A_{ij} \mathscr{A}^{ij}(x) + \sum_{i=1}^m \Delta b_i \mathscr{B}^i \succeq 0$$
$$\forall \|\Delta A_i\|_2 \leqslant \gamma_i, \quad \|\Delta b\|_2 \leqslant \epsilon_b, \tag{25}$$

where ΔA_{ij} denotes the (i, j)-entry of ΔA, Δb_i denotes the ith component of Δb, as in Section 2.1 each symmetric $(m+n+1) \times (m+n+1)$ matrix $\mathscr{A}^{ij}$ has at most four non-zero entries, x_j in column $m+1$ and row j, ϵ in column $m+1+j$ and row i, and the two symmetric entries below the diagonal. Similarly, each symmetric $(m+n+1) \times (m+n+1)$ matrix $\mathscr{B}^i$ has two non-zero entries, -1 in column

$m+1$ and row i, and the symmetric entry with respect to the diagonal, and finally $\mathscr{A}^0$ is given as

$$\mathscr{A}^0(x,t,\mu)=\begin{bmatrix} tI_m & Ax-b & \epsilon A\\ (Ax-b)^{\mathrm{T}} & t-\mu & 0\\ \epsilon A^{\mathrm{T}} & 0 & \mu I_n\end{bmatrix}. \tag{26}$$

Notice that as in the previous section each matrix $\mathscr{A}^{ij}(\cdot)$ is of rank two, given by the formula

$$\mathscr{A}^{ij}(x)=c^i(d^j)^{\mathrm{T}}[x]+d^j[x](c^i)^{\mathrm{T}},$$

where $c^i\in\mathbb{R}^{m+n+1}$ with $c^i_i=1$ and all remaining entries zero, and $d^j[x]\in\mathbb{R}^{m+n+1}$ with x_j as the $(m+1)$th component and ϵ as the $(m+1+j)$th component with all remaining components zero. Similarly, each matrix $\mathscr{B}^i$ is of rank two, given by

$$\mathscr{B}^i=e^if^{\mathrm{T}}+f(e^i)^{\mathrm{T}},$$

where $e^i\in\mathbb{R}^{m+n+1}$ with $e_i=1$ and all remaining entries zero, and $f\in\mathbb{R}^{m+n+1}$ with -1 as the $(m+1)$th component with all remaining components zero. Although we do not know how to find an exact equivalent to this problem, an approximation is obtained, inspired from Proposition 3.1 of [1].

Proposition 3. *The minimization problem over the variables x,t,μ,λ with $\lambda\in\mathbb{R}^{m+1}_+$*

$$\min_{x,t,\mu,\lambda} t$$

subject to

$$\begin{bmatrix} \mathscr{A}_0(x,t,\mu)-\sum_{i=1}^m\lambda_i\gamma_i^2c_ic_i^{\mathrm{T}}-\epsilon_b^2\lambda_{m+1}ff^{\mathrm{T}} & \beta[x] & \cdots & \beta[x] & E\\ \beta[x]^{\mathrm{T}} & \lambda_1I_n & & & \\ \vdots & & \ddots & & \\ \beta[x]^{\mathrm{T}} & & & \lambda_mI_n & \\ E^{\mathrm{T}} & & & & \lambda_{m+1}I_m\end{bmatrix}\succeq 0, \tag{27}$$

where $\beta[x]=[d^1[x];\ldots;d^n[x]]$ and $E=[e^1;\ldots;e^m]$, is an approximation (approximate robust counterpart) of the problem [RLSIND] *in the sense of Definition* 1.

Proof. A vector x,t,μ satisfies (25) iff for all $\xi\in\mathbb{R}^{m+n+1}$ and ΔA_i, Δb with $\|\Delta A_i\|_2\leqslant\gamma_i$ and $\|\Delta b\|_2\leqslant\epsilon_b$, $i=1,\ldots,m$ one has

$$\xi^{\mathrm{T}}\mathscr{A}_0(x,t,\mu)\xi+\sum_{i=1}^m2((c^i)^{\mathrm{T}}\xi)(\Delta A_i^{\mathrm{T}}\beta^{\mathrm{T}}[x]\xi)+2(f^{\mathrm{T}}\xi)(\Delta b^{\mathrm{T}}E^{\mathrm{T}}\xi)\geqslant 0.$$

This is equivalent to saying that for all $\xi \in \mathbb{R}^{m+n+1}$

$$\xi^{\mathrm{T}}\mathscr{A}_0(x,t,\mu)\xi - 2\sum_{i=1}^{m}\gamma_i|(c^i)^{\mathrm{T}}\xi|\,\|\beta^{\mathrm{T}}[x]\xi\|_2 - 2\epsilon_b|f^{\mathrm{T}}\xi|\,\|E^{\mathrm{T}}\xi\|_2 \geqslant 0,$$

which is, in turn, equivalent to: for all $\xi \in \mathbb{R}^{m+n+1}$ and $\eta_i \in \mathbb{R}^n$, $i = 1,\ldots,m$, and $\eta_{m+1} \in \mathbb{R}^m$

$$\begin{aligned}
&P_i(\xi,\eta_i) = \gamma_i^2[(c^i)^{\mathrm{T}}\xi]^2 - \eta_i^{\mathrm{T}}\eta_i \geqslant 0 \quad \forall\, i = 1,\ldots,m,\\
&P_{m+1}(\xi,\eta_{m+1}) = \epsilon_b^2[f^{\mathrm{T}}\xi]^2 - \eta_{m+1}^{\mathrm{T}}\eta_{m+1} \geqslant 0,\\
&\quad\Downarrow\\
&Q(\xi,\eta_1,\ldots,\eta_{m+1}) = \xi^{\mathrm{T}}\mathscr{A}_0(x,t,\mu)\xi + 2\sum_{i=1}^{m}\eta_i\beta^{\mathrm{T}}[x]\xi + 2\eta_{m+1}^{\mathrm{T}}E^{\mathrm{T}}\xi \geqslant 0.
\end{aligned}$$

Now, invoking the S-procedure (cf. [8, Lemma 2.1]), the above implication holds if there exist non-negative λ_i, $i = 1,\ldots,m+1$ such that the quadratic form

$$Q(\xi,\eta_1,\ldots,\eta_{m+1}) - \sum_{i=1}^{m}\lambda_i P_i(\xi,\eta_i) - \lambda_{m+1}P_{m+1}(\xi,\eta_{m+1})$$

in ξ, η_i, $i = 1,\ldots,m+1$ is positive semidefinite. □

The level of conservativeness of the above approximation is unknown, to the best of the authors' knowledge. Notice, however, that the dimensions of the matrices involved in the semidefinite program are much smaller in comparison to those of the previous two sections. In particular, each data matrix is only $(mn + 2m + n + 1) \times (mn + 2m + n + 1)$. Hence, in practice the associated SDPs are solved significantly faster than, e.g., those of Theorem 2, an expectation which is confirmed in our own experimentation.

3. An application to stochastic signal modeling

The robust least squares solution technique has wide application areas in digital signal processing. In this section we will consider one of the important applications: autoregressive moving average (ARMA) modeling of stochastic signals or sequences. ARMA models have been used in modeling of wide variety of discrete signals including biomedical, economical, seismic and speech.

In stochastic ARMA(p, q) modeling the signal is assumed to be generated by a linear time invariant filter whose input is a white noise sequence with unit variance. The filter transfer function is

$$H(z) = \frac{\sum_{k=0}^{q} b_q(k)z^{-k}}{1 + \sum_{k=1}^{p} a_p(k)z^{-k}}, \tag{28}$$

where p, q, $b_q(k)$'s and $a_p(k)$'s are the model parameters to be determined from the available signal data. There are many approaches in obtaining the model parameters [15]. Here, we consider the simpler case of ARMA modeling with known model orders p and q. The model coefficients $b_q(k)$'s and $a_p(k)$'s are typically estimated by using modified Yule–Walker equations (MYWE) in two stages: estimate of $a_p(k)$'s and then use the estimated $a_p(k)$'s in estimating $b_q(k)$'s [15]. The second stage of processing is commonly performed by using causal power spectrum factorization requiring roots of a polynomial whose coefficients are determined by the estimated $a_p(k)$'s [15]. Since roots of a polynomial are highly sensitive to the coefficient accuracies, the overall performance of the modeling is heavily dependent on the accuracy of the first stage: estimation of $a_p(k)$'s. In the remaining part of this section, we focus on this critical stage of modeling, and compare the accuracies and stability with respect to perturbations of commonly used least squares and the proposed robust least squares solution techniques in solving the following MYWE:

$$Aa = b, \tag{29}$$

where a is the unknown coefficient vector $[a_p(1) \;\; a_p(2) \;\; \cdots \;\; a_p(p)]^{\mathrm{T}}$ and the entries of A and b are related to the autocorrelation sequence r of the modeled signal as

$$\begin{aligned} &A(i, j) = r_x(q + i - j), \quad b(i) = -r_x(q + i), \quad p \leqslant m, \\ &\quad 1 \leqslant i \leqslant m \;\text{ and }\; 1 \leqslant j \leqslant p. \end{aligned} \tag{30}$$

Since the autocorrelation sequence of the modeled signal is typically unknown, the entries of A and b have to be estimated from the available signal samples $x[1], \ldots, x[N]$. Unbiased estimates for the autocorrelation sequence can be obtained by

$$\hat{r}_x(n) = \frac{1}{N-n} \sum_{k=1}^{N-n} x[k]x^*[k+n], \quad 0 \leqslant n \leqslant N-1. \tag{31}$$

Since the true entries of A and b, i.e., $A(i, j) = r_x(q + i - j)$ and $b(i) = -r_x(q + i)$, where $m \geqslant p$, $i = 1, \ldots, m$ and $j = 1, \ldots, p$ are not known but estimated from (31), both A and b in (29) have uncertainties in their entries. Furthermore, in practice entries of a must be quantized prior to the implementation of the estimated signal model in a fixed point processor. Therefore, we are dealing with an uncertain least squares problem of the form $\min_a \|Aa - b\|_2$, where the entries of A and b are subject to estimation error, and the computed solution can be implemented only to a fixed number of digits. More precisely, one of the many ways that we can model uncertainty in this problem is as follows: assume we are dealing exactly as in Section 2.1 with a collection of matrices $(A + \Delta A, b + \Delta b)$ instead of a single matrix (A, b) such that $\|(\Delta A, \Delta b)\|_{\mathrm{F}} \leqslant \gamma$, where the entries of the matrix ΔA and the vector Δb represent perturbations of the corresponding entries in A and b, and γ is computed as the Frobenius norm of the matrix formed using the computed standard deviations of the corresponding entries of A and b. We compute the standard deviation values while forming A and b using (30) and (31). We also assume that the vector a is subject to implementation errors of a similar form:

$$\|\Delta a\|_2 \leqslant \epsilon,$$

where ϵ is a positive scalar, e.g., equal to $1/2^8$ corresponding to 8 digits of accuracy in fixed point implementation. Note that it is equally interesting and valid from a modeling viewpoint to adopt the uncertainty models of Sections 2.2 and 2.3 in solving the stochastic signal modeling problem as we can make use of the computed standard deviation values to impose a different model of uncertainty on A and b, e.g., the interval uncertainty model of Section 2.2. Since the robust least squares solution technique can account for such inaccuracies, we investigate its performance and compare it with the commonly used least squares solution (obtained by ignoring uncertainty and solving the problem $\min_a \|Aa - b\|_2$ using A and b computed from (30) and (31)) over some synthetic test examples where the actual model parameter vector a (i.e., a_p) is known.

3.1. Experimental results

In this section we report the results of our experimentation with the ARMA modeling of stochastic signals. The simulation is conducted by using an ARMA(p, q) model with $p = 3$ and $q = 1$ (we take $m = 3$ in (30)), and the true model coefficient vectors $a_p = [-1.4141 \;\; 0.5781 \;\; 0.1250]^{\mathrm{T}}$ and $b_q = [1 \;\; -0.3359]^{\mathrm{T}}$ (the $a_p(k)$'s and the $b_q(k)$'s of (28)). Note that in (28), the index k runs from 0 to q, hence we have $q + 1$ entries for b_q. The chosen model has a real pole at -0.1529 and a pair of complex conjugate poles with magnitude 0.9043 and respective angles of $\pm 30°$. The resulting overdetermined systems of linear equations have six equations and three variables. Consequently, the semidefinite program of Theorem 2 have data matrices of dimension 225×225 with five scalar variables. To keep the paper at a reasonable length, we only report results obtained with the SDP approximation of Section 2.1 (cf. Theorem 2). To solve the resulting SDP problems, we use the SeDuMi 1.05 software package in the MATLAB™ environment with the YALMIP LMI (linear matrix inequality) interpreter [13]. The SDP problems are solved to ten to eleven digits of accuracy within at most 2 min of CPU time. For the value of γ we use the Frobenius norm of the matrix formed using the computed standard deviations of the corresponding entries of A and b. We used 8-bit uniform quantization. The corresponding quantization error is $1/2^8$, hence we take ϵ equal to this value.

Our first experiment consists in evaluating the predictive power of the robust solution reported by the SDP solver SeDuMi 1.05 (the "robust" solution) in comparison to the usual least squares solution. We conducted the following experiment. Based on the chosen ARMA model, we generated 100 random digital signals of length 64 and formed the corresponding (A, b) matrices using (30) and (31). For these 100 instances of (A, b) we computed the least squares solution by solving $\min_a \|Aa - b\|_2$ for each instance of (A, b), and the robust least squares solution. We compared

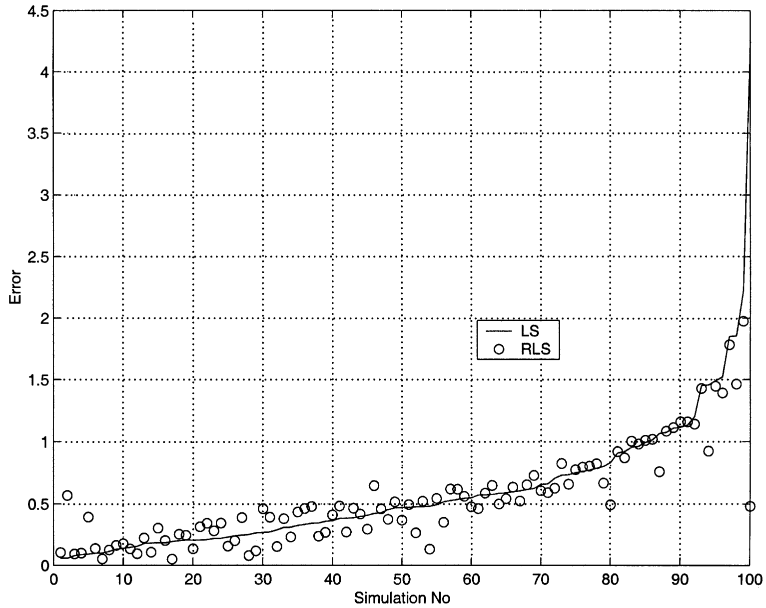

Fig. 1. Estimation error.

the least squares and robust least squares solutions to the known coefficient vectors. In Fig. 1, we present the absolute estimation errors (measured in the norm of the difference between the computed solution and known coefficient values), plotted in increasing order of the error incurred by the least squares solution, of both the least squares and robust least squares approaches showing that both error curves follow each other closely in the region where the least squares solution performs well. However, to the right of the plot where the least squares solution incurs a large estimation error (up to 4.25 in absolute terms) the robust solution deviates from it to make much smaller estimation errors. This result suggests that the robust solution should always be the solution of choice, since it does not deteriorate performance appreciably in instances where no large estimation error occurs, and results in significantly better estimation performance where the least squares solution is far from the true solution.

To test the stability of the solution (with respect to both data perturbations and implementation inaccuracies) reported by SeDuMi using the SDP approximation of Theorem 2, we conducted the following experiments the results of which are presented in Tables 1 and 2. We generated random perturbation matrices ΔA and vectors Δb of appropriate dimension in such a way that the joint norm $\|(\Delta A, \Delta b)\|$ does not exceed the norm bound γ which is derived from the easily computable estimation error bounds (standard deviation of each individual entry in A and b) in (30) and (31). Then, we observed the effect of these perturbations on the quality of the objective function. More precisely, let x_{rob} denote the robust least squares solution we computed by solving our SDP, and x_{ls} denote the "nominal" least squares solution (the

Table 1
Stability with respect to data perturbations

	a-rob$_d$ (robust solution)	a-ls$_d$ (least squares solution)
Mean	2.2×10^{-4}	5.45×10^{-4}
Std	1.24×10^{-4}	4.14×10^{-4}

Table 2
Stability with respect to quantization error

	a-rob$_x$ (robust solution)	a-ls$_x$ (least squares solution)
Mean	6.27×10^{-4}	1.2×10^{-3}
Std	3.72×10^{-4}	7.05×10^{-4}

"nominal" least squares solution is the least squares solution obtained by ignoring uncertainty, i.e., by solving $\min_a \|Aa - b\|_2$, where A and b were computed using (30) and (31)). We compute the quantity $|\|(A + \Delta A)x_{\text{rob}} - (b + \Delta b)\| - \|Ax_{\text{rob}} - b\||$ referred to as a-rob$_d$, and the quantity $|\|(A + \Delta A)x_{\text{ls}} - (b + \Delta b)\| - \|Ax_{\text{ls}} - b\||$ referred to as a-ls$_d$, where we denote the random perturbations we generated as ΔA, Δb. We compute these quantities over a sample of 100 random perturbations, and report average and standard deviation figures in Table 1. The "mean" and "std" stand for the average and standard deviation of the above quantities over 100 trials.

In order to observe the quantization effects on the estimated coefficients due to fixed point implementation, we perturbed both the computed robust solution and the computed "nominal" least squares solution by the same random perturbation vector bounded in norm by ϵ. We compute the difference $|\|(A(x_{\text{rob}} + \Delta x) - b\| - \|Ax_{\text{rob}} - b\||$ referred to as a-rob$_x$, and the quantity $|\|(A(x_{\text{ls}} + \Delta x) - b\| - \|Ax_{\text{ls}} - b\||$ referred to as a-ls$_x$, where we denote the random perturbations we generated as Δx. The results are reported in Table 2 exactly in the same format as in Table 1.

We observe that in both tables, the robust solution achieves a greater stability with respect to the least squares solution. More precisely, from Table 1, we see that when the data (A, b) is perturbed randomly, the objective function value corresponding to the robust solution $\|(A + \Delta A)x_{\text{rob}} - (b + \Delta b)\|$, under the perturbed data $(A + \Delta A, b + \Delta b)$ deviates from the objective function value corresponding to nominal data A and b, $\|Ax_{\text{rob}} - b\|$, on the average by 2.2×10^{-4}. The same deviation when the robust solution is replaced by the "nominal" least squares solution is on the average 5.45×10^{-4}, that is, more than twice the average deviation corresponding to the robust solution. Furthermore, the standard deviation figures in the second line in Table 1 also show that the standard deviation of the average value corresponding to the nominal least squares solution is more than three times the standard deviation value corresponding to the robust solution. Similar observations can be made for Table 2 where the average figures show an even more pronounced difference in favor of the robust solution.

The observed stability of the robust solution to the coefficient quantization will be very helpful in the implementation of the estimated ARMA model by using commonly used processing boards with fixed point arithmetic.

4. Conclusions

In this paper, we investigated the robust solutions to least squares problems with data uncertainties and coefficient quantizations. Our investigation resulted in approximate robust problems inspired by the important contributions of Ben-Tal and Nemirovski. The approximate robust counterpart problems are semidefinite programming problems efficiently solved by polynomial interior point algorithms. To illustrate the performance of the proposed framework, the well known digital signal processing application of ARMA modeling was used. Our experimental results demonstrated that the robust least squares solution approach provides significantly more stable solutions even in the presence of coefficient quantization effects. Thus, the robust least squares approach addresses the right issues in the implementation of ARMA signal modeling, and provides reliable models.

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