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Economic design of EWMA control charts based on loss function

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ABSTRACT

For monitoring the stability of a process, various control charts based on exponentially weighted moving average (EWMA) statistics have been proposed in the literature. We study the economic design of EWMA-based mean and dispersion charts when a linear, quadratic, or exponential loss function is used for computing the costs arising from poor quality. The chart parameters (sample size, sampling interval, control limits and smoothing constant) minimizing the overall cost of the control scheme are determined via computational methods. Using numerical examples, we compare the performances of the EWMA charts with Shewhart $\overline{X}$ and S charts, and investigate the sensitivity of the chart parameters to changes in process parameters and loss functions. Numerical results imply that rather than sample size or control limits, the users need to adjust the sampling interval in response to changes in the cost of poor quality.

1. Introduction

Statistical process control charts are commonly used for monitoring production processes where standardized products are produced in a repetitive manner. From time to time, the assignable causes such as worker errors, machine wear, or changes in raw material quality cause a negative effect on the process, and consequently, the process output quality deteriorates. The primary goal of using control charts is to detect assignable (special) causes as quickly as possible when they occur. Control charts function as a filter to separate the variation due to an assignable cause from the natural variation inherent in a process. The variation attributed to an assignable cause is temporary, and eliminated by proper corrective action. The search for an assignable cause is conducted only when there is a high likelihood that the process has gone out of control. In a typical implementation, the sample statistics computed based on random samples taken periodically from the process are compared against predetermined control limits, and a decision is made regarding whether the process is currently in control or out of control. False alarms occur when an in-control process is erroneously classified as out of control (type 1 error) and an assignable cause is searched. There also exists the risk of concluding that the process is in control based on the sample test statistic although the process is actually out of control (type 2 error).

Various types of control charts have been suggested to monitor the two important process characteristics: process mean, and process variability (range or standard deviation). The Shewhart $\overline{X}$ chart is the most common chart used in practice for monitoring the process mean. However, the previous research has shown that exponentially weighted moving average (EWMA) charts can perform better than $\overline{X}$ charts in detecting the shifts in process mean when the size of the shift is small [1]. The main difference between an $\overline{X}$ chart and an EWMA chart is that, as opposed to an $\overline{X}$ chart, an EWMA chart takes into account the information from not only the current sample, but also from the samples taken previously.

Economic design of process control charts has been investigated by many researchers in the literature [2,3]. In order to use control charts for monitoring the process, the control limits, sample size, and time between samples must be specified. The

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economic design approach to control charts involves the determination of these parameters based on a cost-minimization model that takes into account costs due to sampling, investigation, repair, and producing defective products (e.g. [4]). Thus, the main idea is to select the chart parameters to optimally balance the cost due to defective products and the cost of controlling efforts. In the economic design approach, the distribution characteristics (mean and variance) for the process variable in two different states of the process (in-control and out-of-control) should be estimated.

The economic design of EWMA-based mean charts has been investigated by several authors (Ho and Case [5]; Montgomery et al. [6]). Recently, Tolley and English [7] studied the economic design of a control scheme combining both EWMA and $\overline{X}$ charts. Park et al. [8] looked into the economic design of an adaptive EWMA chart in which the user changes the sampling interval and/or sample size dynamically based on the current chart statistic. When the assignable causes lead to changes in both process mean and variance, simultaneous use of mean and dispersion charts is important for detecting the changes quickly. Joint economic design of EWMA charts for process mean and dispersion has been explored in Serel and Moskowitz [9]. In this paper we consider the case where the assignable cause changes only the process mean or dispersion, and correspondingly, we explore the economic design of the single control chart used for monitoring the process parameter (mean or variance) influenced by the assignable cause.

We extend the economic design of EWMA mean charts to the case where quality related costs are computed based on a loss function. In the loss function approach, it is considered that a quality related cost is incurred when a product is not produced on-target. The target is the most desirable (ideal) value for the quality characteristic associated with the product, as specified by the designers. The larger the deviation from the target, the higher is the quality cost. The loss function is a practical approach to estimate the quality costs, and has been previously used in a variety of economic models developed in the context of quality management. We consider linear, quadratic, and exponential loss functions which are commonly used in the literature (e.g. [10–12]). For a given deviation from target, the implied quality cost depends on the loss function used. Using numerical examples, we explore the impact of the form of the loss function on the chart parameters minimizing the overall cost.

Since it is also important to monitor the dispersion of a process, we separately study the economic design of an EWMA-based variance chart. For purposes of comparison, we also present models for determining the economically optimal parameters for the Shewhart $\overline{X}$ and S charts using the quality loss function approach. Due to the complexity of the resulting total cost functions, all optimization models in the paper are solved using a numerical search algorithm.

The remainder of the paper is organized as follows. In Section 2, we describe the Shewhart $\overline{X}$ and EWMA mean charts, and present the total cost function in terms of decision variables. In Section 3, we show how to incorporate the quality loss function into the economic design model. In Section 4, we study the economic design of EWMA variance and S charts. Following illustrative numerical examples in Section 5, concluding remarks are given in Section 6.

2. Economic design of charts for mean

2.1. $\overline{X}$ chart

The observations for the process variable X are assumed to be independent and normally distributed. When the process is in control, the mean and variance of X is μ_0 and σ_0^2, respectively. The lower and upper control limits associated with the Shewhart $\overline{X}$ chart are

$$\mathrm{LCL}_{\overline{X}} = \mu_0 - L\sigma_0/n^{0.5}, \tag{1}$$

and

$$\mathrm{UCL}_{\overline{X}} = \mu_0 + L\sigma_0/n^{0.5}, \tag{2}$$

where L is the control limit parameter and n is the sample size. At any sampling instant t, the sample average $\overline{X_t}$ is compared against these limits, and if it is outside the limits a search for an assignable cause is started. The rough guidelines for setting the $\overline{X}$ chart limits in practice are $n = 4$ or 5, and $L = 3$.

2.2. EWMA mean chart

The chart statistic for the EWMA mean chart at sampling instant t is computed iteratively from:

$$Z_t = \lambda_m\overline{X_t} + (1 - \lambda_m)Z_{t-1}, \tag{3}$$

where λ_m is the smoothing constant associated with the EWMA chart for mean, $0 < \lambda_m \leq 1$, $Z_0 = \mu_0$. The smoothing parameter λ_m determines the extent to which past samples affect the current value of the chart statistic. A smaller λ_m helps to smooth random fluctuations, but it also reduces the responsiveness of the control scheme to shifts in the process mean.

For the EWMA mean chart, the lower and upper control limits (LCL_{ewma} and UCL_{ewma}) are computed based on the asymptotic in-control standard deviation of the EWMA chart statistic Z such that

$$\mathrm{LCL}_{ewma} = \mu_0 - L_m\sigma_z, \tag{4}$$

$$\mathrm{UCL}_{ewma} = \mu_0 + L_m\sigma_z, \tag{5}$$

where $\sigma_z = \sigma_0(\lambda_m/(2-\lambda_m)n)^{0.5}$, and L_m is the control limit parameter (see, e.g. [1,13]). Thus, whenever Z_t is outside the interval (LCL_{ewma}, UCL_{ewma}), the process is considered to be out of control and a search for assignable cause is conducted.

2.3. Total cost function

The cost model used for determining the optimal values of chart parameters is built upon the general cost function of Lorenzen and Vance [14]. It is assumed that the in-control time for the process is distributed exponentially with mean $1/\theta$. When the process is out of control, the mean of X becomes $\mu_0 + \delta\sigma_0$. Eq. (10) in Lorenzen and Vance [14] gives the expected cost per unit time (hour), C, associated with a control chart as:

$$\begin{aligned} C = {} & \{C_0/\theta + C_1(-\tau + nE + h(\text{ARL}_1) + \gamma_1 T_1 + \gamma_2 T_2) + sF/\text{ARL}_0 + W\} \div \{1/\theta + (1-\gamma_1)sT_0/\text{ARL}_0 - \tau \\ & + nE + h(\text{ARL}_1) + T_1 + T_2\} + \{[(a+bn)/h][1/\theta - \tau + nE + h(\text{ARL}_1) + \gamma_1 T_1 + \gamma_2 T_2]\} \\ & \div\{1/\theta + (1-\gamma_1)sT_0/\text{ARL}_0 - \tau + nE + h(\text{ARL}_1) + T_1 + T_2\}, \end{aligned} \tag{6}$$

where,
$h =$ sampling interval (time between two consecutive samples),
$C_0 =$ cost per hour due to nonconformities produced while the process is in control,
$C_1 =$ cost per hour due to nonconformities produced while the process is out of control,
$\tau =$ expected time between the occurrence of the assignable cause and the time of the last sample taken before the assignable cause $= [1-(1+\theta h)\exp(-\theta h)]/[\theta(1-\exp(-\theta h))]$,
$E =$ time to sample and chart one item,
$\text{ARL}_0 =$ average run length while in control,
$\text{ARL}_1 =$ average run length while out of control,
$T_0 =$ expected search time when the signal is a false alarm,
$T_1 =$ expected time to discover the assignable cause,
$T_2 =$ expected time to repair the process,
$\gamma_1 = 1$ if production continues during searches
$= 0$ if production ceases during searches,
$\gamma_2 = 1$ if production continues during repair
$= 0$ if production ceases during repair,
$s =$ expected number of samples taken while in control $= \exp(-\theta h)/[1-\exp(-\theta h)]$,
$F =$ cost per false alarm,
$W =$ cost to locate and repair the assignable cause,
$a =$ fixed cost per sample,
$b =$ cost per unit sampled.

The cost function (6) is generic, and can be used with different types of control charts. The expected cost per hour in (6) is derived from dividing the expected cost per cycle by the expected cycle length. The cycle consists of an in-control phase followed by the out-of-control phase. The corrective action taken against the assignable cause restores the process to the in-control state, and the cycle starts anew. The total cost in a cycle includes sampling inspection, search, and repair costs in addition to the cost due to nonconformities produced. To estimate the quality costs, we use the quality loss function approach, as described in Section 3.

The average run length (ARL) is a measure of the expected number of consecutive samples taken until the sample statistic falls outside the control limits, and it is a function of the current process characteristics. To reduce the total cost, the ARL should be large when the process is in control, and it should be small when the process is out of control. The in-control ARL can be increased by widening the interval between the upper and lower control limits, but this would also cause the out-of-control ARL to increase, unless the sample size is increased as a counter-measure.

It can be shown that the in-control ARL for the Shewhart $\overline{X}$ chart is

$$\text{ARL}_0 = 1/[2\Phi(-L)], \tag{7}$$

and the out-of-control ARL is

$$\text{ARL}_1 = 1/[1-\Phi(L-\delta n^{0.5}) + \Phi(-L-\delta n^{0.5})], \tag{8}$$

where $\Phi(.)$ is the cumulative probability distribution function (cdf) for a standard normal variable [14].

The computation of the average run length for the EWMA mean control chart is more complicated. We use the Markov chain approach which has also been used, among others, by Saccucci and Lucas [15], and Morais and Pacheco [16]. More information regarding the computation of ARL_0 and ARL_1 by the Markov chain method can be found in Appendix A. The ARL of the EWMA mean chart depends on the control limits which, in turn, depend on the chosen sample size, control limit parameter, and smoothing constant.

Thus, the design parameters are n, h and L for the Shewhart $\overline{X}$ chart, and n, h, L_m, and λ_m for the EWMA mean chart. Since analytical solution is difficult, for both types of charts, we use the Nelder–Mead computational optimization method (see, e.g. [17]) to determine the optimal design for a given set of input data. A similar approach has been used by various researchers in the literature for solving the economic models developed for designing the control charts (e.g. [5]).

3. Use of quality loss function in the optimization model

In the traditional formulation of economic design models, the costs due to nonconformities when the process is in-control (C_0) and out-of-control (C_1) have been treated as constants in (6). In recent years, influenced in part by the popularity of Taguchi methods in product design, the quality loss function concept has been incorporated into various statistical decision models where the cost due to poor quality needs to be estimated. In the traditional approach, the upper and lower specification limits are used to classify the quality of the process output as either acceptable or non-acceptable, and products falling outside the specification limits are considered to result in quality costs. In the loss function approach, the probability distribution describing the observations for the quality characteristic is explicitly taken into account in computing the costs resulting from variation of the quality characteristic around its target. It is considered that cost of poor quality is incurred whenever the quality characteristic is not on its target; hence, products that are not produced on-target incur cost even though they may conform to specification limits. Several researchers have applied the loss function approach in the economic design of $\overline{X}$ control charts [10,11]. In this paper, we propose the economic design of EWMA charts based on linear, quadratic, and exponential loss functions.

We note that the appropriate type of the loss function to be used depends on the particular industrial application. It is also possible that the relevant loss function may be different for negative and positive deviations from the target. Cain and Janssen [18] discuss a problem arising in the production of construction panels made of glued and pressed wood chips. The moisture level can be reduced by drying the panels longer in the gas dryers. The longer drying time requires more fuel to be consumed; hence, there is a linear increase in cost as the moisture content decreases. On the other hand, higher moisture increases the press time which increases the total plant operating cost. Thus, the resulting cost function is partly linear and partly quadratic. The cost increases quadratically when the moisture content is higher than planned; however, the cost increases linearly when the moisture content is lower than planned. Although, as in this example, the form of the loss function for the quality characteristic may be region-dependent, in this paper we will focus on the simpler and more common case where the loss function is single-type and symmetric around the target value. But, if needed, these more generalized (mixed type and asymmetric) loss functions can be easily incorporated to our model. We remark that different types of loss functions can also be regarded as reflections of varying risk preferences of the users of control charts. In this case, the quality loss function is related to the user's utility function.

3.1. Linear loss function

We first consider the linear loss function in which quality loss is a linear function of the deviation of the quality characteristic from its target. Let T be the target value for the quality characteristic monitored; we allow the possibility that T can be different from μ_0. Let the probability density function (pdf) of the quality characteristic X be $f(x)$. The quality loss is zero only when the quality characteristic X equals the target T, and the loss increases as the deviation from the target increases. If the loss function $L(x)$ is asymmetric around the target, two different loss coefficients K_1 and K_2 should be estimated such that the loss is calculated as

$$\begin{aligned} L(x) &= K_1(T-x) \quad \text{if } x \le T, \\ &= K_2(x-T) \quad \text{if } x > T. \end{aligned}$$

The expected quality cost per unit of product when the process is in control, J_0, is

$$J_0 = \int_{-\infty}^{\infty} L(x)f(x)\mathrm{d}x = \int_{-\infty}^{\mathrm{T}} K_1(T-x)f(x)\mathrm{d}x + \int_{T}^{\infty} K_2(x-T)f(x)\mathrm{d}x. \tag{9}$$

For a normal random variable with mean μ_0, standard deviation σ_0, and pdf $f(x)$, we have

$$\int_{-\infty}^{\mathrm{T}} xf(x)\mathrm{d}x = \mu_0\Phi(z_0) - \sigma_0\phi(z_0), \tag{10}$$

and

$$\int_{T}^{\infty} xf(x)\mathrm{d}x = \sigma_0\phi(z_0) + \mu_0[1-\Phi(z_0)], \tag{11}$$

where $z_0 = (T-\mu_0)/\sigma_0, f(x) = (2\pi\sigma_0^2)^{-0.5}\exp[-(x-\mu_0)^2/2\sigma_0^2]$, and $\phi(.)$ is the standard normal pdf, $\phi(u) = (2\pi)^{-0.5}\exp(-u^2/2)$. In this paper we consider the symmetric loss functions so we assume that the loss coefficient used for estimating the cost due to nonconformities $K = K_1 = K_2$. Thus, using (10) and (11), we can rewrite (9) as

$$J_0 = \int_{-\infty}^{\infty} K|x-T|f(x)\mathrm{d}x = 2K[\sigma_0\phi(z_0) - (\mu_0 - T)\Phi(z_0)] + K(\mu_0 - T). \tag{12}$$

Let the out-of-control process mean be $\mu_1 = \mu_0 + \delta\sigma_0$. Defining $z_1 = (T - \mu_1)/\sigma_0$, the expected quality cost per unit when the process is out of control, J_1, is

$$J_1 = 2K[\sigma_0\phi(z_1) - (\mu_1 - T)\Phi(z_1)] + K(\mu_1 - T). \tag{13}$$

If p units are produced per hour, we can compute C_0 and C_1 in (6) as: $C_0 = J_0 p$, and $C_1 = J_1 p$. Note that the shift in mean $\delta\sigma_0$ explicitly enters the cost function (6) through the term C_1 when a loss function is used for computing the quality costs.

3.2. Quadratic loss function

The most common loss function used in practice is the symmetric quadratic loss function advocated by Taguchi. The quadratic loss function penalizes the deviations from the target more severely than the linear loss function. Kim and Liao [19] suggest the liquid products in containers such as juice, soda and medicine as potential applications of symmetric quadratic loss function. Product delivery time promised to customers is an example of an asymmetric quadratic loss function [19]. The actual delivery occurring earlier than the promised time incurs a small loss, which is considerably less than the loss associated with a late delivery resulting in customer dissatisfaction. Another example for the asymmetric loss is the contents of a manufactured drug. The low amount of a particular ingredient may make the drug ineffective, but the high level of the same ingredient may have a serious negative effect on users, implying that positive deviation from the target incurs a higher loss than the same amount of deviation below the target [19]. To design control charts based on a symmetric quadratic loss function, we calculate J_0 as

$$\begin{aligned} J_0 &= \int_{-\infty}^{\infty} K(x-T)^2 f(x)\mathrm{d}x = \int_{-\infty}^{\infty} K(x-\mu_0+\mu_0-T)^2 f(x)\mathrm{d}x \\ &= K[\sigma_0^2 + (\mu_0 - T)^2]. \end{aligned} \tag{14}$$

The expected cost per unit under quadratic loss function when the process is out of control is

$$\begin{aligned} J_1 &= \int_{-\infty}^{\infty} K(x-\mu_0-\delta\sigma_0+\mu_0+\delta\sigma_0-T)^2 f(x)\mathrm{d}x \\ &= K[\sigma_0^2 + (\mu_0 - T)^2 + \delta^2\sigma_0^2 - 2\delta\sigma_0(\mu_0 - T)]. \end{aligned} \tag{15}$$

As in the case of linear loss function, the optimal design under quadratic loss function can be found by first finding C_0 and C_1 using (14) and (15), and then substituting them in (6).

3.3. Exponential loss function

Finally we consider the exponential loss function which corresponds to the case of constant risk aversion if we assume that the utility of the decision maker is measured by the negative of the quality loss [11]. The linear loss function is suited to a risk-neutral decision maker whereas the quadratic and exponential loss functions allow incorporation of risk aversion explicitly into the model. The choice of a quadratic loss function implies that the decision maker becomes less risk averse as the deviation of the quality characteristic from the target increases [11]. The exponential loss function implies that the utility of the decision maker decreases exponentially as deviation from the target increases. The expected quality cost per unit when the process is in control, J_0, based on the exponential loss function is

$$J_0 = \int_{-\infty}^{\mathrm{T}} K[\mathrm{e}^{r(T-x)} - 1]f(x)\mathrm{d}x + \int_{T}^{\infty} K[\mathrm{e}^{r(x-T)} - 1]f(x)\mathrm{d}x, \tag{16}$$

where r is a parameter describing the risk aversion of the user. Let $d = |x - T|$. If the utility function is $U(d) = -K(\mathrm{e}^{rd} - 1)$, then the risk aversion defined by $U''(d)/U'(d)$ is equal to r. A higher value of r implies a higher cost penalty for deviations from the target, and correspondingly, a higher level of risk aversion.

To evaluate J_0, note that, using change of variables $u = (x - \mu)/\sigma$,

$$\int_{-\infty}^{\mathrm{T}} \mathrm{e}^{r(T-x)} f(x)\mathrm{d}x = \mathrm{e}^{r(T-\mu)} \int_{-\infty}^{z} \mathrm{e}^{-r\sigma u}\phi(u)\mathrm{d}u, \tag{17}$$

$$\int_{T}^{\infty} \mathrm{e}^{r(x-T)} f(x)\mathrm{d}x = \mathrm{e}^{r(\mu-T)} \int_{z}^{\infty} \mathrm{e}^{r\sigma u}\phi(u)\mathrm{d}u, \tag{18}$$

where $z = (T - \mu)/\sigma$, μ and σ are the mean and standard deviation of a normal random variable with pdf $f(x)$. For the standard normal probability distribution, we have

$$\int_{-\infty}^{z} \mathrm{e}^{-r\sigma u}\phi(u)\mathrm{d}u = \mathrm{e}^{(r\sigma)^2/2}\Phi(z + r\sigma), \tag{19}$$

$$\int_{z}^{\infty} \mathrm{e}^{r\sigma u}\phi(u)\mathrm{d}u = \mathrm{e}^{(r\sigma)^2/2}[1 - \Phi(z - r\sigma)]. \tag{20}$$

Then, substituting (19) and (20) into (17) and (18), J_0 equals

$$J_0 = K\mathrm{e}^{(r\sigma_0)^2/2}[\mathrm{e}^{r(T-\mu_0)}\Phi(z_0 + r\sigma_0) + \mathrm{e}^{r(\mu_0-T)} - \mathrm{e}^{r(\mu_0-T)}\Phi(z_0 - r\sigma_0)] - K. \tag{21}$$

Using $\mu = \mu_1$, the expected quality cost per unit when the process is out of control is given by

$$J_1 = K\mathrm{e}^{(r\sigma_0)^2/2}[\mathrm{e}^{r(T-\mu_1)}\Phi(z_1 + r\sigma_0) + \mathrm{e}^{r(\mu_1-T)} - \mathrm{e}^{r(\mu_1-T)}\Phi(z_1 - r\sigma_0)] - K. \tag{22}$$

4. Economic design of charts for variance

4.1. EWMA variance chart

In this section, we consider the economic design of an EWMA-based chart to be used for monitoring the dispersion of a process. We adopt the approach of several authors including Crowder and Hamilton [20], Gan [13], Acosta-Mejia et al. [21], and Morais and Pacheco [16] who have studied control charts based on EWMA of $\ln S^2$ (sample variance). The chart has the lower and upper control limits

$$\mathrm{LCL}_{ewmavar} = \ln(\sigma_0^2), \tag{23}$$

$$\mathrm{UCL}_{ewmavar} = \ln(\sigma_0^2) + L_v\sigma_y, \tag{24}$$

where $\sigma_y^2 = \lambda_v\psi'[(n-1)/2]/(2-\lambda_v)$, $\psi'(.)$ is the trigamma function, which is the variance of $\ln(S_t^2)$ [20], and L_v is the control limit parameter. The associated chart statistic is

$$Y_t = \max\{\ln(\sigma_0^2), \lambda_v \ln(S_t^2) + (1-\lambda_v)Y_{t-1}\}, \tag{25}$$

where λ_v is the smoothing constant, $0 < \lambda_v \leq 1$, and S_t^2 is the sample variance at sampling instant t defined as

$$S_t^2 = \sum_{i=1}^{n}(X_{it} - \overline{X_t})^2/(n-1). \tag{26}$$

The sample size $n \geq 2$, and $Y_0 = \ln(\sigma_0^2)$. The EWMA variance chart triggers an out-of-control signal when Y_t exceeds $\mathrm{UCL}_{ewmavar}$. We remark that some researchers have proposed a single EWMA chart for simultaneously monitoring the process mean and the process variance [22,23].

We can utilize (6) to economically design the EWMA variance chart based on the assumption that the assignable cause now leads to an increase in process variance. We assume only the variance changes when the process goes out of control, and the process mean does not change. The in-control and out-of-control ARLs can be calculated by using the Markov chain method. More details are given in Appendix B. The expected quality cost per unit when the process is in control is same as that for the mean chart, i.e., for linear loss, J_0 is obtained from (12), for quadratic loss J_0 is determined from (14), and for the case of exponential loss function, J_0 is same as (21). Let $\sigma_1 = \rho\sigma_0$ denote the standard deviation of the process when it is out of control, $\rho \geq 1$. J_1 for the EWMA variance chart under a linear loss function is given by (cf. (13))

$$J_1 = 2K[\sigma_1\phi(z_2) - (\mu_0 - T)\Phi(z_2)] + K(\mu_0 - T), \tag{27}$$

where $z_2 = (T - \mu_0)/\sigma_1$. When a quadratic loss function is used, J_1 is computed from (cf. (14))

$$J_1 = K[\sigma_1^2 + (\mu_0 - T)^2]. \tag{28}$$

If an exponential loss function is used, J_1 is obtained from (cf. (21))

$$J_1 = K\mathrm{e}^{(r\sigma_1)^2/2}[\mathrm{e}^{r(T-\mu_0)}\Phi(z_2 + r\sigma_1) + \mathrm{e}^{r(\mu_0-T)} - \mathrm{e}^{r(\mu_0-T)}\Phi(z_2 - r\sigma_1)] - K. \tag{29}$$

4.2. S chart

An alternative to the EWMA variance chart is the Shewhart S chart used for detecting changes in the process standard deviation. The lower control limit of the S chart is zero, and the upper control limit is

$$\mathrm{UCL}_S = L_S\sigma_0, \tag{30}$$

where L_S is determined from

$$L_S = [\chi^2_{n-1;1-\alpha}/(n-1)]^{0.5}. \tag{31}$$

Table 1
Optimal EWMA mean chart parameters for the foundry example

n	h	λ_m	L_m	C
1	0.28	0.12	3.23	388.78
2	0.88	0.23	2.62	388.91
3	1.09	0.31	2.78	388.25
4	1.27	0.37	2.84	388.29
5	1.57	0.44	2.61	388.19
6	1.94	0.45	2.69	388.18
7	2.36	0.61	2.51	387.80
8	2.56	0.53	2.48	388.02
9	3.24	0.55	2.34	387.89
10	4.22	0.77	2.32	387.39
11	4.04	0.77	2.45	387.38
12	4.72	0.76	2.23	387.65
13	5.38	0.88	2.34	387.70
14	5.54	0.83	2.23	388.01
15	6.11	0.64	2.40	388.95
16	6.07	0.85	2.39	388.51
17	5.38	0.75	2.50	389.12
18	6.87	0.72	2.52	389.97
19	6.43	0.62	2.46	390.42
20	6.40	0.57	2.46	391.16

$\chi^2_{n-1;1-\alpha}$ is the $100(1-\alpha)$th percentile point of the chi-squared probability distribution with $n-1$ degrees of freedom. Thus, the probability of false alarm for the S chart is α. The out-of-control signal is given when the sample standard deviation exceeds UCL_S. The in-control and out-of-control ARLs for the S chart are [24]

$$\mathrm{ARL}_0 = 1/[1 - G((n-1)L_S^2)], \tag{32}$$

and

$$\mathrm{ARL}_1 = 1/[1 - G((n-1)L_S^2\sigma_0^2/\sigma_1^2)], \tag{33}$$

where $G(.)$ is the cdf of the chi-squared probability distribution with n-1 degrees of freedom. For determining the cost-efficient parameters associated with the S chart, we minimize (6) by using J_0 and J_1 given by (12) and (27), (14) and (28) and (21) and (29) in linear loss, quadratic loss, and exponential loss cases, respectively.

5. Numerical examples

As noted previously, if both the mean and dispersion charts are used, in order to minimize the total expected cost, the parameters of the two charts should be determined jointly. Consider the following numerical example: $\gamma_1 = 1$, $\gamma_2 = 0$, $\delta = 2.5$, $F = \$300$, $W = \$150$, $E = 0.5$, $\theta = 0.01$, $T_0 = 2$, $T_1 = 2$, $T_2 = 0$, $a = \$5$, $b = \$1$, $K = 1$, $p = 300$ per hour, $\sigma_0^2 = 1$, and $\mu_0 = T$. Hence, there is an assignable cause that causes a 2.5 standard deviation increase in the process mean. Assuming quadratic loss function and simultaneous use of EWMA mean and variance charts, we determine the best chart parameters as $n = 2$, $h = 0.76$, $\lambda_m = 0.81$, $L_m = 2.76$, $\lambda_v = 0.33$, and $L_v = 4.00$. The resulting hourly cost is \$377.72. Note that simultaneous design of mean and variance charts requires the computation of *joint* ARL (see [9,13,16]). If we use only the EWMA mean chart in this situation, the parameters would be selected as $n = 1$, $h = 0.53$, $\lambda_m = 0.65$, $L_m = 2.70$, with an associated cost of $C = \$376.59$ per hour.

We now present a detailed example to illustrate the design of an EWMA mean chart based on an economic criterion. Lorenzen and Vance [14] describe a foundry operation with an hourly production rate of 84 castings. The cooling curve observed from the molten iron samples is recorded periodically. The cooling curve is related to the carbon-silicate content which influences the tensile strength of the castings, an important product attribute. This example is also used by Prabhu et al. [25]. The cost and process parameters are $\theta = 0.02$, $a = 0$, $b = \$4.22$, $F = \$977.4$, $W = \$1086$, $E = T_0 = T_1 = 5/60$ h, $T_2 = 0.75$ h. Thus, on average the process stays in control for 50 h. The production continues during the search for an assignable cause, but it ceases during repair, i.e., $\gamma_1 = 1$, $\gamma_2 = 0$. The mean of the process increases by 0.86 standard deviation when the process goes out of control. We assume a quadratic loss function with $K = 4$, $\sigma_0^2 = 1$, and $\mu_0 = T$. From (14) and (15), the expected quality cost per casting is \$4 and \$6.96 while producing in control and out of control, respectively.

For a given sample size n, the optimal chart parameters minimizing the expected hourly cost are listed in Table 1. We determine the best design as n changes from 1 to 30 (only the first 20 designs are displayed in Table 1). We observe that the economic design is defined by $n = 11$, $h = 4.04$ h, $\lambda_m = 0.77$, $L_m = 2.45$. Hence, a sample of size 11 should be taken every 4.04 h. Using (4) and (5), the lower and upper control limits are $\mathrm{LCL}_{ewma} = \mu_0 - 0.584$ and $\mathrm{UCL}_{ewma} = \mu_0 + 0.584$. The minimum expected cost is \$387.38 per hour. At these settings, the in-control ARL is 71.5, and the out-of-control ARL is 1.5.

Table 2
Economically designed EWMA mean charts under linear loss

E	F	W	δ	C	n	h	λ_m	L_m	ARL_0	ARL_1
0.05	300	150	0.5	247.70	28	16.66	0.59	1.86	18.1	1.3
			1.5	254.63	8	3.53	0.98	2.82	208.3	1.1
			2.5	261.76	3	1.73	0.81	2.96	327.1	1.1
		900	0.5	254.04	26	19.65	0.57	1.91	20.4	1.4
			1.5	261.86	7	3.33	0.79	2.86	238.3	1.2
			2.5	269.00	3	1.84	0.97	2.91	276.7	1.1
	900	150	0.5	248.42	30	14.52	0.71	2.48	78.8	1.6
			1.5	255.15	8	3.51	0.85	3.10	518.2	1.1
			2.5	262.06	4	1.91	0.87	3.35	1238.7	1.1
		900	0.5	254.65	28	18.68	0.75	2.28	45.6	1.5
			1.5	262.33	9	3.57	0.82	3.11	536.6	1.1
			2.5	269.33	4	1.75	0.99	3.43	1656.8	1.1
0.5	300	150	0.5	249.19	12	9.92	0.52	1.91	21.3	2.2
			1.5	259.22	4	2.45	0.75	2.64	123.0	1.5
			2.5	267.01	2	1.59	0.92	2.69	140.2	1.2
		900	0.5	255.13	13	13.11	0.52	1.84	18.3	2.0
			1.5	266.27	4	2.37	0.75	2.67	134.3	1.5
			2.5	274.21	2	1.59	0.92	2.69	140.2	1.2
	900	150	0.5	250.21	12	7.59	0.35	2.50	104.8	3.3
			1.5	260.51	4	2.06	0.73	3.07	472.4	1.9
			2.5	268.19	2	1.38	0.81	3.16	636.4	1.5
		900	0.5	255.96	13	9.53	0.39	2.52	105.4	3.1
			1.5	267.55	4	2.03	0.73	2.91	280.9	1.7
			2.5	275.38	2	1.31	0.77	3.13	575.9	1.5

In the following examples we explore the sensitivity of the chart parameters to variations in process parameters and loss functions. The following values of parameters are used: $\gamma_1 = 1$, $\gamma_2 = 0$, $F \in \{300, 900\}$, $W \in \{150, 900\}$, $E \in \{0.05, 0.5\}$, $\theta = 0.01$, $T_0 = 2$, $T_1 = 2$, $T_2 = 0$, $a = 5$, $b = 1$, $r = K = 1$, $p = 300$ per hour, $\sigma_0^2 = 1$, and $\mu_0 = T$.

As in Torng et al. [26], we first fix the sample size n and optimize (6) with respect to the other decision variables. After finding best designs by varying n from 1 to 30 for the mean charts, and from 2 to 30 for the variance charts, we identify the design that yields the minimum cost for each chart. The ARLs of EWMA charts are computed using the Markov chain method (with $k = 51$) as described in Appendices A and B. We also impose the following upper limits on the variables: $h \le 40$, $\lambda_m, \lambda_v \le 0.99$, $L, L_m, L_v, L_S \le 4$. The minimum allowable values for λ_m and λ_v are specified as 0.05.

5.1. Optimal EWMA mean and $\overline{X}$ charts

We explore the economic design of EWMA mean and $\overline{X}$ charts under different shift sizes; the chart parameters and the in-control and out-of-control average run lengths resulting from the numerical search for the EWMA mean chart and $\overline{X}$ chart are given in Tables 2–7, respectively.

For a given shift size, the costs show little variation between the EWMA mean and $\overline{X}$ charts. This result is in agreement with Arnold and Collani [27] who stated that the performance of the Shewhart $\overline{X}$ chart cannot be improved significantly by other types of charts when the quality characteristic is normally distributed, and the in-control time is exponentially distributed. Ho and Case [5] have compared the costs associated with economically designed EWMA and $\overline{X}$ charts using 14 examples, and in 12 of these 14 examples the costs were very close (the difference is less than 1%). In the remaining two examples, the difference in costs was approximately 1.4% and 2.1%, respectively.

Most of the time the sample size and sampling interval do not change significantly between the EWMA mean and $\overline{X}$ charts. The differences in sample size and sampling interval of $\overline{X}$ and EWMA mean charts are more notable when the sampling time E and the false alarm cost F are at their high levels. The EWMA mean chart prescribed by the economic design model has characteristics fairly similar to the $\overline{X}$ chart. The value of the smoothing constant λ_m is quite high for all shift sizes, indicating that smoothing of sample means over a large number of past samples is not very useful in our numerical example. A similar pattern has been observed in the examples considered in Ho and Case [5].

5.2. Optimal EWMA variance and S charts

The application of the Nelder–Mead downhill simplex method with various values of ρ yields the chart parameters for the EWMA variance and S chart listed in Tables 8–13, respectively.

Similar to the case of EWMA mean–$\overline{X}$ pair, the performances of the EWMA variance and S charts are also very close to each other. The smoothing constant λ_v in many cases attains its upper bound 0.99, essentially turning the EWMA variance chart into an S chart. Both sample size and sampling interval decrease as the shift parameter ρ increases. Although it can be expected that EWMA-based charts should perform better than Shewhart charts when the shift in mean or variance is

Table 3
Economically designed EWMA mean charts under quadratic loss

E	F	W	δ	C	n	h	λ_m	L_m	ARL_0	ARL_1
0.05	300	150	0.5	313.74	29	10.98	0.79	1.94	19.7	1.3
			1.5	331.79	7	1.99	0.85	2.79	190.7	1.1
			2.5	360.53	3	0.94	0.77	3.01	386.1	1.1
		900	0.5	320.42	29	10.37	0.78	2.00	22.7	1.3
			1.5	339.02	7	1.99	0.85	2.79	190.7	1.1
			2.5	367.83	3	0.94	0.77	3.01	386.1	1.1
	900	150	0.5	315.11	30	9.81	0.78	2.40	62.3	1.5
			1.5	332.79	8	1.86	0.99	3.11	534.5	1.2
			2.5	361.46	4	1.08	0.94	3.34	1193.9	1.1
		900	0.5	321.80	30	10.01	0.63	2.34	55.6	1.5
			1.5	340.02	8	1.97	0.88	3.12	553.8	1.2
			2.5	368.76	4	1.08	0.94	3.34	1193.9	1.1
0.5	300	150	0.5	317.25	8	3.62	0.39	2.11	38.2	3.4
			1.5	342.77	2	0.83	0.51	2.62	126.1	2.5
			2.5	376.59	1	0.53	0.65	2.70	150.0	2.0
		900	0.5	323.76	7	3.20	0.26	2.04	41.0	3.8
			1.5	349.97	2	0.95	0.51	2.60	119.2	2.5
			2.5	383.85	1	0.53	0.65	2.70	150.0	2.0
	900	150	0.5	319.03	8	2.63	0.25	2.70	210.1	5.1
			1.5	345.38	3	0.90	0.58	3.04	439.3	2.3
			2.5	380.09	2	0.71	0.85	3.04	424.0	1.4
		900	0.5	325.36	8	2.77	0.27	2.66	181.4	4.9
			1.5	352.57	3	0.88	0.60	3.08	498.5	2.4
			2.5	387.25	2	0.71	0.79	3.16	637.1	1.5

Table 4
Economically designed EWMA mean charts under exponential loss

E	F	W	δ	C	n	h	λ_m	L_m	ARL_0	ARL_1
0.05	300	150	0.5	551.63	28	6.98	0.78	2.00	22.7	1.3
			1.5	587.43	6	1.40	0.93	2.64	120.8	1.2
			2.5	675.53	3	0.53	0.88	2.94	305.4	1.1
		900	0.5	558.21	25	6.18	0.74	2.04	25.2	1.5
			1.5	594.70	6	1.40	0.93	2.64	120.8	1.2
			2.5	682.85	3	0.53	0.88	2.94	305.4	1.1
	900	150	0.5	553.70	30	7.00	0.78	2.26	43.0	1.4
			1.5	589.02	7	1.32	0.86	3.00	371.5	1.2
			2.5	677.56	3	0.53	0.84	3.15	614.2	1.1
		900	0.5	560.51	29	5.70	0.79	2.54	91.6	1.7
			1.5	596.29	7	1.32	0.86	3.00	371.5	1.2
			2.5	684.88	3	0.53	0.84	3.15	614.2	1.1
0.5	300	150	0.5	557.14	6	2.02	0.29	2.15	49.1	4.4
			1.5	606.09	2	0.65	0.51	2.51	93.0	2.4
			2.5	709.51	1	0.31	0.61	2.70	151.8	2.0
		900	0.5	563.89	7	2.30	0.36	2.10	38.9	3.7
			1.5	613.30	2	0.58	0.46	2.59	119.8	2.5
			2.5	716.79	1	0.31	0.61	2.70	151.8	2.0
	900	150	0.5	559.79	7	1.78	0.20	2.71	243.1	5.8
			1.5	610.06	2	0.45	0.38	3.24	932.1	3.5
			2.5	715.81	1	0.25	0.40	3.22	860.0	2.7
		900	0.5	566.32	7	1.78	0.20	2.71	243.1	5.8
			1.5	617.16	2	0.55	0.37	3.03	472.8	3.2
			2.5	723.09	1	0.25	0.40	3.22	860.0	2.7

small, we have not always observed such a pattern in our numerical experiment. A possible reason for this result is that the optimal sample size is relatively large when the shift is small. In the previously published research, for a given in-control ARL, the difference between the out-of-control ARLs of EWMA and Shewhart charts appears to get smaller as the sample size increases; for example, in Castagliola [28], a comparison of the ARLs for the S chart and an EWMA chart based on ln S^2 indicates that the difference between the ARLs decreases as the sample size increases (regardless of the value of the smoothing constant used in the EWMA chart). In a similar manner, the large sample sizes associated with small shifts in our experiment seem to narrow the gap between the performances of EWMA and Shewhart charts. Another factor to consider is that due to the possibility of multiple local optima on the objective function surface, the search algorithm may not terminate at the global optimal point.

Table 5
Economically designed $\overline{X}$ charts under linear loss

E	F	W	δ	C	n	h	L	ARL_0	ARL_1
0.05	300	150	0.5	247.70	28	15.79	1.89	17.0	1.3
			1.5	254.64	7	3.33	2.66	128.0	1.1
			2.5	261.75	4	1.87	3.15	612.5	1.0
		900	0.5	254.01	28	21.49	1.81	14.2	1.3
			1.5	261.83	7	3.57	2.77	178.4	1.1
			2.5	269.00	3	1.71	2.99	358.5	1.1
	900	150	0.5	248.56	30	18.34	2.21	36.9	1.4
			1.5	255.18	9	3.17	3.12	552.9	1.1
			2.5	262.07	4	1.93	3.30	1034.3	1.1
		900	0.5	254.72	30	21.95	2.25	40.9	1.5
			1.5	262.35	9	3.38	3.28	963.3	1.1
			2.5	269.33	4	1.93	3.30	1034.3	1.1
0.5	300	150	0.5	249.49	15	12.70	1.75	12.5	1.7
			1.5	259.44	4	2.77	2.48	76.1	1.4
			2.5	267.10	2	1.49	2.70	144.2	1.3
		900	0.5	255.29	18	19.17	1.73	12.0	1.5
			1.5	266.48	4	2.77	2.48	76.1	1.4
			2.5	274.31	2	1.49	2.70	144.2	1.3
	900	150	0.5	250.65	24	16.02	2.20	36.0	1.7
			1.5	260.86	5	2.64	2.84	221.7	1.4
			2.5	268.50	2	1.44	3.05	437.0	1.5
		900	0.5	256.20	26	21.98	2.17	33.3	1.5
			1.5	267.91	5	2.06	2.92	285.7	1.5
			2.5	275.69	2	1.44	3.05	437.0	1.5

Table 6
Economically designed $\overline{X}$ charts under quadratic loss

E	F	W	δ	C	n	h	L	ARL_0	ARL_1
0.05	300	150	0.5	313.83	30	9.61	2.09	27.3	1.4
			1.5	331.79	7	1.89	2.83	214.8	1.2
			2.5	360.39	3	0.88	2.94	304.7	1.1
		900	0.5	320.57	30	9.38	2.11	28.7	1.4
			1.5	339.03	7	1.89	2.83	214.8	1.2
			2.5	367.70	3	0.88	2.94	304.7	1.1
	900	150	0.5	315.46	30	9.06	2.23	38.8	1.4
			1.5	332.81	8	1.95	3.02	395.6	1.1
			2.5	361.39	4	0.97	3.37	1330.4	1.1
		900	0.5	322.10	30	12.39	2.16	32.5	1.4
			1.5	340.05	8	1.95	3.02	395.6	1.1
			2.5	368.69	4	0.97	3.37	1330.4	1.1
0.5	300	150	0.5	318.31	13	6.55	1.85	15.6	2.1
			1.5	343.93	3	1.36	2.42	64.4	1.8
			2.5	377.87	2	0.81	2.74	162.8	1.3
		900	0.5	324.62	12	6.55	1.85	15.6	2.2
			1.5	351.14	3	1.12	2.46	72.0	1.8
			2.5	385.11	2	0.81	2.74	162.8	1.3
	900	150	0.5	321.07	18	6.94	2.28	44.2	2.3
			1.5	347.43	4	1.29	2.83	214.8	1.8
			2.5	380.65	2	0.70	3.04	422.7	1.5
		900	0.5	327.12	20	8.76	2.20	36.0	1.9
			1.5	354.53	4	1.29	2.83	214.8	1.8
			2.5	387.89	2	0.70	3.04	422.7	1.5

5.3. *Effect of shift size and loss function*

According to the results displayed in Tables 2–13, for all charts, as the out-of-control mean or variance increases, both sample size and sampling interval decrease. When the shift in mean or variance is small, a large sample size is needed to decide about the state of the process. As the shift size increases, the sample size decreases more rapidly in the EWMA mean and $\overline{X}$ charts compared to the EWMA variance and S charts. For the $\overline{X}$ chart, the observed direction of change in sample size with respect to shift size is consistent with the earlier findings in the literature, as discussed in Montgomery [3]. A similar relationship between sample size and shift size is observed in the numerical examples in Park et al. [8] who studied the economic design of an EWMA mean chart.

For all charts, the sampling interval is shorter in the quadratic loss case compared to the linear loss case. This result can be attributed to the higher cost of defective products under quadratic loss measure. A shorter sampling interval leads

Table 7
Economically designed $\overline{X}$ charts under exponential loss

E	F	W	δ	C	n	h	L	ARL_0	ARL_1
0.05	300	150	0.5	551.82	27	6.52	1.93	18.7	1.3
			1.5	587.33	6	1.15	2.72	153.2	1.2
			2.5	675.63	3	0.53	2.88	251.5	1.1
		900	0.5	558.67	27	7.79	1.92	18.2	1.3
			1.5	594.61	6	1.15	2.72	153.2	1.2
			2.5	682.95	3	0.53	2.88	251.5	1.1
	900	150	0.5	554.00	30	6.94	2.28	44.2	1.5
			1.5	589.12	7	1.20	3.00	370.4	1.2
			2.5	677.61	3	0.49	3.22	780.1	1.2
		900	0.5	560.79	30	6.94	2.28	44.2	1.5
			1.5	596.38	8	1.34	3.04	422.7	1.1
			2.5	684.93	3	0.49	3.22	780.1	1.2
0.5	300	150	0.5	559.42	10	4.06	1.77	13.0	2.4
			1.5	609.73	3	0.90	2.46	72.0	1.8
			2.5	714.05	1	0.43	2.40	61.0	1.9
		900	0.5	565.95	10	4.26	1.89	17.0	2.6
			1.5	616.90	3	0.90	2.46	72.0	1.8
			2.5	721.33	1	0.43	2.40	61.0	1.9
	900	150	0.5	563.96	16	4.36	2.27	43.1	2.5
			1.5	616.12	3	0.64	2.83	214.8	2.5
			2.5	726.05	2	0.45	2.99	358.5	1.4
		900	0.5	570.23	16	5.06	2.20	36.0	2.4
			1.5	623.28	3	0.64	2.83	214.8	2.5
			2.5	733.31	2	0.45	2.99	358.5	1.4

Table 8
Economically designed EWMA variance charts under linear loss

E	F	W	ρ	C	n	h	λ_v	L_v	ARL_0	ARL_1
0.05	300	150	1.5	254.98	19	6.42	0.99	1.54	30.4	1.3
			2	257.62	10	3.56	0.99	1.77	102.3	1.3
			2.5	260.60	8	2.32	0.99	1.90	248.4	1.2
		900	1.5	261.91	18	6.42	0.99	1.54	31.1	1.4
			2	264.78	11	3.39	0.86	1.92	148.3	1.2
			2.5	267.81	7	2.45	0.93	1.76	158.2	1.2
	900	150	1.5	256.23	25	6.28	0.95	1.99	99.8	1.3
			2	258.37	12	3.67	0.91	2.14	346.2	1.3
			2.5	261.15	8	2.29	0.99	2.16	947.6	1.3
		900	1.5	263.11	24	6.28	0.95	1.99	102.4	1.4
			2	265.49	12	3.67	0.91	2.14	346.2	1.3
			2.5	268.34	9	2.53	0.99	2.20	877.2	1.2
0.5	300	150	1.5	259.48	6	2.41	0.58	1.45	55.9	3.1
			2	263.70	4	1.99	0.63	1.36	84.7	2.3
			2.5	267.74	4	1.50	0.89	1.50	177.7	1.7
		900	1.5	266.21	7	3.06	0.50	1.38	38.6	2.6
			2	270.64	5	1.98	0.81	1.57	126.6	2.0
			2.5	274.76	4	1.63	0.84	1.51	182.9	1.8
	900	150	1.5	261.57	8	2.68	0.65	1.84	160.4	3.3
			2	265.13	5	1.71	0.76	1.82	471.1	2.5
			2.5	268.85	4	1.49	0.72	1.69	523.7	2.0
		900	1.5	268.19	8	2.76	0.64	1.79	130.6	3.1
			2	272.09	5	1.70	0.64	1.84	508.0	2.5
			2.5	275.92	4	1.49	0.72	1.69	523.7	2.0

to a decrease in the number of products produced during the out-of-control phase. On the other hand, we do not observe significant differences in sample size and control limits between the quadratic, linear and exponential loss scenarios. Thus, keeping the shift size fixed, the sampling interval is the only design parameter requiring a major adjustment when the cost of poor quality changes.

Notice that in the exponential loss case we assume that the coefficient of risk aversion $r = 1$. The optimal chart parameters depend on the value of r as the cost penalty for deviations from the target increases with r. Thus, in light of the numerical results for linear and quadratic loss cases, it can be expected that higher values of r will lead to shorter sampling intervals.

The in-control ARL values associated with economically optimal designs generally increase with the shift sizes δ and ρ. On the other hand, most of the time the out-of-control ARL decreases as the shift in mean or variance increases. This suggests

Table 9
Economically designed EWMA variance charts under quadratic loss

E	F	W	ρ	C	n	h	λ_v	L_v	ARL_0	ARL_1
0.05	300	150	1.5	331.43	16	2.77	0.99	1.64	44.2	1.5
			2	344.72	9	1.43	0.99	1.51	44.9	1.2
			2.5	361.80	6	0.97	0.99	1.74	211.5	1.3
		900	1.5	338.53	16	2.77	0.99	1.64	44.2	1.5
			2	351.96	9	1.43	0.99	1.78	121.6	1.4
			2.5	369.08	6	0.97	0.99	1.74	211.5	1.3
	900	150	1.5	334.06	20	2.85	0.92	2.01	121.2	1.6
			2	346.52	11	1.62	0.99	2.10	359.9	1.3
			2.5	363.52	8	1.17	0.84	2.10	583.9	1.2
		900	1.5	341.11	22	3.26	0.84	2.05	123.6	1.5
			2	353.74	11	1.64	0.99	2.08	328.6	1.3
			2.5	370.68	8	1.19	0.99	2.10	680.5	1.2
0.5	300	150	1.5	342.41	5	1.31	0.48	1.34	54.2	3.6
			2	362.04	3	0.72	0.69	1.22	107.8	3.1
			2.5	386.03	3	0.68	0.76	1.28	153.6	2.2
		900	1.5	349.34	5	1.18	0.48	1.35	56.3	3.6
			2	369.16	3	0.72	0.69	1.22	107.8	3.1
			2.5	393.19	3	0.68	0.76	1.28	153.6	2.2
	900	150	1.5	347.03	6	1.22	0.51	1.75	196.1	4.4
			2	366.12	4	0.82	0.61	1.61	314.4	2.9
			2.5	389.25	3	0.54	0.82	1.48	663.2	2.7
		900	1.5	353.91	6	1.17	0.38	1.70	177.4	4.3
			2	373.18	4	0.81	0.60	1.62	333.5	2.9
			2.5	396.41	3	0.54	0.82	1.48	663.2	2.7

Table 10
Economically designed EWMA variance charts under exponential loss

E	F	W	ρ	C	n	h	λ_v	L_v	ARL_0	ARL_1
0.05	300	150	1.5	588.65	12	1.68	0.87	1.51	33.6	1.7
			2	659.45	7	0.67	0.95	1.63	92.0	1.5
			2.5	888.40	4	0.26	0.87	1.46	140.9	1.7
		900	1.5	595.78	14	1.76	0.94	1.51	31.5	1.6
			2	666.73	7	0.67	0.95	1.63	92.0	1.5
			2.5	895.72	4	0.26	0.87	1.46	140.9	1.7
	900	150	1.5	593.12	16	1.60	0.80	2.00	131.2	1.9
			2	664.10	7	0.56	0.85	1.93	336.2	1.8
			2.5	894.42	5	0.28	0.85	1.84	560.5	1.7
		900	1.5	600.27	16	1.60	0.80	2.00	131.2	1.9
			2	671.38	7	0.56	0.85	1.93	336.2	1.8
			2.5	901.73	5	0.28	0.85	1.84	560.5	1.7
0.5	300	150	1.5	609.64	4	0.75	0.46	1.20	49.7	4.2
			2	710.96	2	0.25	0.58	0.82	82.7	4.6
			2.5	1017.97	2	0.16	0.70	0.91	155.2	3.5
		900	1.5	616.67	4	0.75	0.46	1.20	49.7	4.2
			2	717.90	2	0.30	0.62	0.82	79.4	4.6
			2.5	1025.27	2	0.16	0.70	0.91	155.2	3.5
	900	150	1.5	618.18	5	0.57	0.47	1.69	238.7	5.4
			2	720.56	3	0.29	0.68	1.45	494.9	4.2
			2.5	1031.76	2	0.14	0.71	1.03	504.1	4.4
		900	1.5	625.13	5	0.58	0.47	1.69	238.7	5.4
			2	727.73	3	0.29	0.68	1.45	494.9	4.2
			2.5	1039.01	2	0.14	0.71	1.03	504.1	4.4

that economic designs with constraints on ARL would differ from the unconstrained economic designs mainly when the shifts in mean or variance are small.

5.4. Effect of cost parameters

We also explore the sensitivity of the chart parameters to several input parameters in our numerical study. The increase in the time to sample and chart one item E is observed to reduce both the optimal sample size and the optimal sampling interval. Keeping other parameters fixed, the false alarm cost F and the sample size n are positively related; a larger F leads to a greater n. The larger sample size helps to reduce the rate of false alarms. The impact of the cost of repair W on the chart parameters is less clear; a significant relationship between W and the chart parameters is not observed in the numerical study. The optimal cost increases as W and/or F increase.

Table 11
Economically designed S charts under linear loss

E	F	W	ρ	C	n	h	L_S	ARL_0	ARL_1
0.05	300	150	1.5	254.99	19	5.47	1.32	38.3	1.4
			2	257.60	10	3.25	1.55	98.5	1.3
			2.5	260.56	7	2.34	1.76	203.1	1.2
		900	1.5	261.93	17	5.54	1.35	43.7	1.5
			2	264.75	10	3.25	1.55	98.5	1.3
			2.5	267.77	8	2.51	1.73	260.1	1.2
	900	150	1.5	256.21	25	6.42	1.34	103.1	1.3
			2	258.39	12	3.10	1.61	370.4	1.3
			2.5	261.13	8	2.50	1.81	571.0	1.2
		900	1.5	263.08	25	7.04	1.34	103.1	1.3
			2	265.52	12	3.71	1.57	226.4	1.2
			2.5	268.32	8	2.50	1.81	571.0	1.2
0.5	300	150	1.5	259.67	7	3.11	1.53	34.3	2.5
			2	263.86	5	2.51	1.73	56.9	1.8
			2.5	267.74	4	1.66	2.00	135.4	1.7
		900	1.5	266.36	8	3.75	1.49	33.7	2.3
			2	270.81	4	1.86	1.88	71.0	2.2
			2.5	274.82	4	1.66	2.00	135.4	1.7
	900	150	1.5	262.27	9	3.25	1.58	95.7	2.8
			2	265.49	5	1.77	2.00	331.2	2.5
			2.5	269.08	4	1.41	2.20	439.3	2.0
		900	1.5	268.84	9	3.57	1.57	87.3	2.8
			2	272.45	5	2.01	1.93	203.5	2.3
			2.5	276.12	4	1.67	2.20	439.3	2.0

Table 12
Economically designed S charts under quadratic loss

E	F	W	ρ	C	n	h	L_S	ARL_0	ARL_1
0.05	300	150	1.5	331.40	16	2.89	1.35	38.3	1.5
			2	344.71	9	1.49	1.59	105.1	1.3
			2.5	361.85	7	1.08	1.78	241.2	1.2
		900	1.5	338.50	16	2.89	1.35	38.3	1.5
			2	351.96	9	1.44	1.62	139.7	1.4
			2.5	369.12	7	1.08	1.78	241.2	1.2
	900	150	1.5	334.10	21	3.09	1.38	115.8	1.5
			2	346.53	11	1.68	1.63	328.3	1.3
			2.5	363.30	7	1.00	1.88	593.7	1.3
		900	1.5	341.15	21	3.11	1.38	115.8	1.5
			2	353.75	11	1.68	1.63	328.3	1.3
			2.5	370.57	7	1.00	1.88	593.7	1.3
0.5	300	150	1.5	343.06	5	1.41	1.65	35.9	3.3
			2	362.67	3	0.75	2.10	82.3	3.0
			2.5	386.28	3	0.63	2.21	132.2	2.2
		900	1.5	349.98	5	1.41	1.65	35.9	3.3
			2	369.79	3	0.75	2.10	82.3	3.0
			2.5	393.46	3	0.63	2.21	132.2	2.2
	900	150	1.5	349.23	6	1.33	1.75	109.8	4.3
			2	367.32	4	0.72	2.14	304.6	3.0
			2.5	390.01	3	0.53	2.49	492.8	2.7
		900	1.5	356.05	6	1.33	1.75	109.8	4.3
			2	374.41	4	0.75	2.16	343.7	3.1
			2.5	397.17	3	0.53	2.49	492.8	2.7

6. Conclusion

Reduction of variation in product performance characteristics is a key element of Six Sigma quality improvement programs applied in large companies such as Motorola and General Electric. Quantifying the quality costs using a loss function is a well-known approach for estimating the economic consequences of variation. We have incorporated linear, quadratic and exponential quality loss functions into the economic design of EWMA control charts. We have explored the impact of the size of shifts in mean and variance on the design parameters. We have also examined the design of Shewhart $\overline{X}$ and S charts based on the concept of quality loss function. Our computational study suggests that using a different type of quality loss function (linear versus quadratic) leads to a significant change in sampling interval while affecting the sample size and control limits very little. It is also observed that the overall costs are insensitive to the choice of Shewhart or EWMA charts.

Table 13
Economically designed S charts under exponential loss

E	F	W	ρ	C	n	h	L_S	ARL_0	ARL_1
0.05	300	150	1.5	588.60	14	1.71	1.38	40.3	1.6
			2	659.60	7	0.62	1.67	97.0	1.5
			2.5	888.69	5	0.33	1.89	155.5	1.5
		900	1.5	595.77	14	1.71	1.38	40.3	1.6
			2	666.88	7	0.62	1.67	97.0	1.5
			2.5	896.00	5	0.33	1.89	155.5	1.5
	900	150	1.5	593.46	18	1.80	1.41	112.2	1.7
			2	664.28	8	0.62	1.76	347.2	1.6
			2.5	895.01	5	0.30	2.04	441.7	1.6
		900	1.5	600.60	18	1.80	1.41	112.2	1.7
			2	671.56	8	0.62	1.76	347.2	1.6
			2.5	902.32	5	0.30	2.04	441.7	1.6
0.5	300	150	1.5	610.93	4	0.76	1.76	39.0	4.0
			2	712.34	2	0.27	2.47	74.0	4.6
			2.5	1019.70	2	0.16	2.68	135.8	3.5
		900	1.5	617.96	4	0.76	1.76	39.0	4.0
			2	719.54	2	0.26	2.58	101.2	5.1
			2.5	1026.95	2	0.16	2.68	135.8	3.5
	900	150	1.5	622.39	5	0.64	1.87	136.3	5.5
			2	724.14	3	0.36	2.34	238.8	3.9
			2.5	1035.87	2	0.13	3.06	451.8	4.5
		900	1.5	629.31	5	0.76	1.82	98.8	4.8
			2	731.30	3	0.36	2.34	238.8	3.9
			2.5	1043.19	2	0.15	2.99	358.5	4.3

Since the economic design approach entails estimation of a number of input parameters (i.e. time and cost estimates for various activities), finding a good design may require solving the model multiple times with different estimates of inputs. Whether the insensitivity of the sample size and control limits to the type of loss function observed in our setting also holds in other problems with different values of input parameters is an open question. Future research may investigate the issue of finding robust solutions to the optimization models studied in this paper.

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Appendix A. Calculating the ARL of the EWMA mean chart

In the Markov chain method we divide the interval between the UCL and LCL into k equally spaced subintervals (k must be an odd integer). For the EWMA mean chart, the subintervals are $R_1 = [u_0, u_1]$, $R_2 = [u_1, u_2], \ldots, R_i = [u_{i-1}, u_i], \ldots, R_k = [u_{k-1}, u_k]$ where $u_i = \mathrm{LCL}_{ewma} + \mathrm{i}\Delta u$ and $\Delta u = (\mathrm{UCL}_{ewma} - \mathrm{LCL}_{ewma})/k$. The subintervals correspond to the transitional states in the Markov chain and the transition probabilities $p_{i,j}$ are found by setting the EWMA statistic Z_t to the midpoint of the subinterval R_i when $u_{i-1} < Z_t \leq u_i$. Hence,

$$p_{i,j} = P(Z_t \in R_j | Z_{t-1} \in R_i) = P(u_{j-1} < Z_t \leq u_j | Z_{t-1} = (u_{i-1} + u_i)/2).$$

Note that $u_{(k+1)/2} = \mu_0$. The transition probabilities can be computed iteratively by using [16]

$$p_{i,j} = f_{i,j} - f_{i,j-1}, \quad i, j = 1, \ldots, k,$$

where $f_{i,j} = \Phi\{[2L_m(j - (1 - \lambda_m)(i - 0.5) - 0.5\lambda_m k)]/(k[\lambda_m(2 - \lambda_m)]^{0.5}) - \delta n^{0.5}\}, i = 1, \ldots, k, j = 0, \ldots, k.$

Recall that $\Phi(.)$ denotes the cumulative distribution function for the standard normal probability distribution. The transient states in the Markov chain are the in-control states and the EWMA statistic Z_t moves to the absorbing state if Z_t falls outside the control limits. The run length distribution of the EWMA mean chart can be found by using the initial probability vector and transition probability matrix. The initial probability vector contains the probabilities of Z starting in each state of the Markov chain. In this paper we use the zero-state ARL, i.e. the starting state for the EWMA statistic is the in-control mean with probability one [1].

Let $\mathbf{P} = [p_{i,j}]$ be the $k \times k$ matrix of transition probabilities $p_{i,j}$. Then, the ARL of the EWMA mean chart when the process mean is $\mu_0 + \delta\sigma_0$ is given by

$$\mathrm{ARL}_{ewma} = \mathbf{p}^{\mathrm{T}}(\mathbf{I} - \mathbf{P})^{-1}\mathbf{1}, \tag{A.1}$$

where $\mathbf{p}$ is the initial probability vector containing the starting state probabilities (i.e. the probability that Z_t starts in state i, $i = 1, \ldots, k$), $\mathbf{I}$ is the $k \times k$ identity matrix, and $\mathbf{1}$ is a column vector of ones [15]. We substitute $\mathbf{p} = \mathbf{e}_i$ in (A.1), $\mathbf{e}_i$ is the ith unit vector with all elements zero except the ith element which is 1. In (A.1), we use $i = (k+1)/2$. To find the in-control ARL, we set $\delta = 0$ in the expression for $f_{i,j}$. Note that, alternatively, the out-of-control ARL can be computed using the steady-state probability vector $\mathbf{p}_s$ as the initial probability vector. The vector $\mathbf{p}_s$ is found by solving $\mathbf{p} = \mathbf{P}^{\mathrm{T}}\mathbf{p}$ subject to $\mathbf{1}^{\mathrm{T}}\mathbf{p} = 1$ [15]. The use of $\mathbf{p}_s$ as the initial probability vector would give the out-of-control ARL in the case where the process stays in control for a long time before the shift occurs. Lucas and Saccucci [1] have found that, for most practical purposes, the difference between zero-state and steady-state ARLs is not significant.

Appendix B. Calculating the ARL of the EWMA variance chart

The average run length of the EWMA variance chart can be determined following a procedure similar to that for the EWMA mean chart. Let $\Delta v = (\mathrm{UCL}_{ewmavar} - \mathrm{LCL}_{ewmavar})/k$. The transition probabilities $q_{i,j}$ in the Markov chain, associated with the EWMA variance chart, can be computed recursively from [16]

$$q_{i,j} = h_{i,j} - h_{i,j-1}, \quad i, j = 1, \ldots, k,$$

where $h_{i,j} = G\{(n-1)\exp([(j-1) - (1-\lambda_v)(i-1.5)]\Delta v/\lambda_v)/\rho^2\}$, $i = 2, \ldots, k$, $j = 1, \ldots, k$, $h_{1,j} = G\{(n-1)\exp[(j-1)\Delta v/\lambda_v]/\rho^2\}$, $j = 1, \ldots, k$, and $h_{i,0} = 0$, $i = 1, \ldots, k$.

Recall that $G(.)$ denotes the cdf for the chi-squared probability distribution with $n-1$ degrees of freedom. Let $\mathbf{Q} = [q_{i,j}]$ be the $k \times k$ matrix of transition probabilities $q_{i,j}$. The ARL of the EWMA variance chart when the process standard deviation is $\rho\sigma_0$ is obtained from

$$\mathrm{ARL}_{ewmavar} = \mathbf{q}^{\mathrm{T}}(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{1}. \tag{B.2}$$

The vector of starting state probabilities $\mathbf{q} = \mathbf{e}_i$, $i = 1$ in (B.2). To find the in-control ARL, we set $\rho = 1$ in the expression for $h_{i,j}$.

References

[1] J.M. Lucas, M.S. Saccucci, Exponentially weighted moving average control schemes: Properties and enhancements, Technometrics 32 (1990) 1–12.
[2] C. Ho, K.E. Case, Economic design of control charts: A literature review for 1981–1991, Journal of Quality Technology 26 (1994) 39–53.
[3] D.C. Montgomery, The economic design of control charts: A review and literature survey, Journal of Quality Technology 12 (1980) 75–87.
[4] M.A. Rahim, A.F.B. Costa, Joint economic design of X-bar and R charts under Weibull shock models, International Journal of Production Research 38 (2000) 2871–2889.
[5] C. Ho, K.E. Case, The economically-based EWMA control chart, International Journal of Production Research 32 (1994) 2179–2186.
[6] D.C. Montgomery, J.C.C. Torng, J.K. Cochran, F.P. Lawrence, Statistically constrained economic design of the EWMA control chart, Journal of Quality Technology 27 (1995) 250–256.
[7] G.O. Tolley, J.R. English, Economic designs of constrained EWMA and combined EWMA-X-bar control schemes, IIE Transactions 33 (2001) 429–436.
[8] C. Park, J. Lee, Y. Kim, Economic design of a variable sampling rate EWMA chart, IIE Transactions 36 (2004) 387–399.
[9] D.A. Serel, H. Moskowitz, Joint economic design of EWMA control charts for mean and variance, European Journal of Operational Research 184 (2008) 157–168.
[10] E.A. Elsayed, A. Chen, An economic design of X-bar control chart using quadratic loss function, International Journal of Production Research 32 (1994) 873–887.
[11] H. Moskowitz, R. Plante, Y.H. Chun, Effect of quality loss functions on the economic design of X-bar process control charts, European Journal of Operational Research 72 (1994) 333–349.
[12] D.A. Serel, M. Dada, H. Moskowitz, R.D. Plante, Investing in quality under autonomous and induced learning, IIE Transactions 35 (2003) 545–555.
[13] F.F. Gan, Joint monitoring of process mean and variance using exponentially weighted moving average control charts, Technometrics 37 (1995) 446–453.
[14] T.J. Lorenzen, L.C. Vance, The economic design of control charts: A unified approach, Technometrics 28 (1986) 3–10.
[15] M.S. Saccucci, J.M. Lucas, Average run lengths for exponentially weighted moving average control schemes using the Markov chain approach, Journal of Quality Technology 22 (1990) 154–162.
[16] M.C. Morais, A. Pacheco, On the performance of combined EWMA schemes for μ and σ: A Markovian approach, Communications in Statistics, Part B-Simulation and Computation 29 (2000) 153–174.
[17] R.L. Rardin, Optimization in Operations Research, Prentice-Hall, NJ, 1998.
[18] M. Cain, C. Janssen, Target selection in process control under asymmetric costs, Journal of Quality Technology 29 (1997) 464–468.
[19] M.W. Kim, W.M. Liao, Estimating hidden quality costs with quality loss functions, Accounting Horizons 8 (1994) 8–18.
[20] S.V. Crowder, M.D. Hamilton, An EWMA for monitoring a process standard deviation, Journal of Quality Technology 24 (1992) 12–21.
[21] C.A. Acosta-Mejia, J.J. Pignatiello, B.V. Rao, A comparison of control charting procedures for monitoring process dispersion, IIE Transactions 31 (1999) 569–579.
[22] G. Chen, S.W. Cheng, H. Xie, Monitoring process mean and variance with one EWMA chart, Journal of Quality Technology 33 (2001) 223–233.
[23] A.F.B. Costa, M.A. Rahim, A single EWMA chart for monitoring process mean and process variance, Quality Technology & Quantitative Management 3 (2006) 295–305.
[24] T.P. McWilliams, E.M. Saniga, D.J. Davis, Economic-statistical design of X-bar and R or X-bar and S charts, Journal of Quality Technology 33 (2001) 234–241.
[25] S.S. Prabhu, D.C. Montgomery, G.C. Runger, Economic-statistical design of an adaptive X-bar chart, International Journal of Production Economics 49 (1997) 1–15.
[26] J.C.C. Torng, J.K. Cochran, D.C. Montgomery, F.P. Lawrence, Implementing statistically constrained economic EWMA charts, Journal of Quality Technology 27 (1995) 257–264.
[27] B.F. Arnold, E. von Collani, Economic process control, Statistica Neerlandica 41 (1987) 89–97.
[28] P. Castagliola, A new S^2-EWMA control chart for monitoring the process variance, Quality and Reliability Engineering International 21 (2005) 781–794.