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Korteweg–de Vries surfaces

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ABSTRACT

We consider 2-surfaces arising from the Korteweg–de Vries (KdV) hierarchy and the KdV equation. The surfaces corresponding to the KdV equation are in a three-dimensional Minkowski (M_3) space. They contain a family of quadratic Weingarten and Willmore-like surfaces. We show that some KdV surfaces can be obtained from a variational principle where the Lagrange function is a polynomial function of the Gaussian and mean curvatures. We also give a method for constructing the surfaces explicitly, i.e., finding their parameterizations or finding their position vectors.

1. Introduction

The connection of curves and surfaces in $\mathbb{R}^3$ to some nonlinear partial differential equations is very well known in differential geometry [1,2]. The motion of curves on two dimensional surfaces in differential geometry lead to some integrable nonlinear differential equations such as the nonlinear Schrödinger equation [3], and modified KdV and KdV equations [4,5]. In the history of differential geometry there are some special subclasses of 2-surfaces such as surfaces of constant Gaussian curvature, surfaces of constant mean curvature, minimal surfaces, developable surfaces, Bianchi surfaces, surfaces where the inverse of the mean curvature is harmonic and the Willmore surfaces. These surfaces arise in many different branches of sciences; in particular, in various parts of theoretical physics (string theory, general theory of relativity), biology and differential geometry [1–8].

Examples of some of these surfaces such as Bianchi surfaces, surfaces where the inverse of the mean curvature is harmonic [9], and the Willmore surfaces [10,11] are very rare. The main reason is the difficulty of solving corresponding differential equations. For this purpose, some indirect methods [12–28] have been developed for the construction of two surfaces in $\mathbb{R}^3$ and in three-dimensional Minkowskian geometries M_3. Among these methods, the soliton surface technique is very effective. In this method, one mainly uses the deformations of the Lax equations of the integrable equations. This way, it is possible to construct families of surfaces corresponding to some integrable equations such as the sine Gordon, Korteweg–de Vries (KdV) equation, modified Korteweg–de Vries (mKdV) equation and Nonlinear Schrödinger (NLS) equation [12–21], belonging to the aforementioned subclasses of 2-surfaces in a three-dimensional flat geometry. For details of integrable equations one may look at [29,30], and the references therein. In particular, using the symmetries of the integrable equations and their Lax equation, we arrive at classes of 2-surfaces. There are many attempts in this direction and examples of new 2-surfaces.

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There are some surfaces derivable from a variational principle. Examples of these surfaces are the minimal surfaces [6,31], surfaces with constant mean curvature, Willmore surfaces [10,11] and surfaces solving the shape equation [32–38]. All these surfaces come from a variational principle where the Lagrange function is a polynomial of degree two in the mean curvature of the surface. There are more general surfaces solving the Euler–Lagrange equations corresponding to more general Lagrange functions of the mean and Gaussian curvatures of the surface [32–34].

In [21], we constructed mKdV surfaces in $\mathbb{R}^3$ using deformation of parameters of the solution for the mKdV equation. We have also constructed the Harry Dym (HD) surfaces in M_3 using spectral deformation. We found new HD surfaces that solve the generalized shape equation. Some of these surfaces belong to Willmore-like and Weingarten surfaces.

In this work, by using the deformation of Lax equations of the KdV equation, we generate some new Weingarten and Willmore-like surfaces. Since the Lax representation of the KdV equation is given in the $\mathfrak{sl}(2,\mathbb{R})$ algebra, the KdV surfaces that we obtain in this work are in the Minkowski space M_3. We also find some KdV surfaces which solve the generalized shape equation. By following Fokas and Gelfand [15], in Section 2, we give the deformation technique in order to construct 2-surfaces. In Section 3, we study the variation of a functional where the Lagrange function is a function of the mean and Gaussian curvatures. Following [32–34], we give the corresponding Euler–Lagrange equations. Solutions of these equations define a family of surfaces extremizing the functional we started with. In Section 4, we give the surfaces corresponding to the KdV hierarchy. In Section 5, we construct surfaces corresponding to the well-known KdV equation by using spectral deformation. These surfaces contain quadratic Weingarten and Willmore-like surfaces. In Section 6, we show that KdV surfaces contain also a subclass of surfaces which extremize families of functionals. For all these surfaces, we find all possible functionals where the Euler–Lagrange equations are exactly solved.

Using the method of deformation of Lax equations, we can obtain the fundamental forms, Gauss and mean curvatures of the surfaces. A parameterization of the position vector of these surfaces cannot be obtained directly. The deformation technique does not produce the surfaces explicitly; we, therefore, give an approach to find a parameterization of the surfaces explicitly. Our approach rests upon solving the Lax equations for a given solution of the KdV equation. Each solution of these linear equations directly gives the position vectors of the corresponding surfaces. The solutions of the KdV equation can be given analytically as we do in this work or numerically. In Section 7, by using our approach we give some surfaces from the traveling wave solutions of the KdV equation. The surfaces arising from the numerical solutions of the KdV equation will be presented in the future. In Section 8, we plot some of the KdV surfaces.

2. Deformation of soliton equations

Surfaces corresponding to integrable equations are called integrable surfaces and a connection formula, relating integrable equations to surfaces, was first established by Sym [12,14]. His formula gives a relation between the family of immersions and Lax pairs defined in a Lie algebra. Here, we shall give a brief introduction (following [17]) of the recent status of the subject and also give some new results.

Let $F : \mathcal{U} \to M_3$ be an isometric immersion of a domain $\mathcal{U} \in M_2$ into M_3, where M_2 and M_3 are two- and three-dimensional pseudo-Riemannian geometries, respectively. Let $(x, t) \in \mathcal{U}$. The surface $F(x, t)$ is uniquely defined up to rigid motions by the first and second fundamental forms. Let $N(x, t)$ be the normal vector field defined at each point of the surface $F(x, t)$. Then the triple $\{F_x, F_t, N\}$ at a point $p \in S$ defines a basis of the tangent space at p, $T_p(S)$, where S is the surface parameterized by $F(x, t)$. The motion of the basis on S is characterized by the Gauss–Weingarten (GW) equations. The compatibility condition of these equations are the well-known Gauss–Mainardi–Codazzi (GMC) equations. The GMC equations are coupled nonlinear partial differential equations for the coefficients $g_{ij}(x, t)$ and $h_{ij}(x, t)$ of the first and the second fundamental forms, respectively. For certain particular surfaces, these equations reduce to a single or to a system of integrable equations. The correspondence between the GMC equations and the integrable equations has been studied extensively, see for example [17].

Let us first give the connection between the integrable equations with a surface in M_3 following Fokas and Gelfand [15]. Let G be a Lie group and $\mathcal{G}$ be the corresponding Lie algebra of dimension 3.

Theorem 2.1 (*Fokas–Gelfand [15]*). *Let $U(x, t; \lambda)$, $V(x, t; \lambda)$, $A(x, t; \lambda)$, $B(x, t; \lambda)$ take values in an algebra $\mathcal{G}$ and let them be differentiable functions of x, t and λ in some neighborhood of $M_2 \times \mathbb{R}$. Assume that these functions satisfy*

$$U_t - V_x + [U, V] = 0, \tag{1}$$

and

$$A_t - B_x + [A, V] + [U, B] = 0. \tag{2}$$

Define $\Phi(x, t; \lambda)$ in a group G and suppose that $F(x, t; \lambda)$ takes values in the algebra $\mathcal{G}$ by the equations

$$\Phi_x = U\,\Phi, \qquad \Phi_t = V\,\Phi, \tag{3}$$

and

$$F_x = \Phi^{-1} A\,\Phi, \qquad F_t = \Phi^{-1} B\,\Phi. \tag{4}$$

Then for each λ, $F(x,t;\lambda)$ defines a 2-dimensional surface in $\mathbb{R}^3$,

$$y_j = F_j(x,t;\lambda), \quad j = 1,2,3, \qquad F = \sum_{k=1}^{3} F_k\, e_k, \tag{5}$$

where e_k, ($k = 1,2,3$) form a basis of $\mathcal{g}$. The first and the second fundamental forms of S are

$$\begin{aligned}(ds_{\mathrm{I}})^2 &\equiv g_{ij}\,dx^i\,dx^j = \langle A,A\rangle\,dx^2 + 2\langle A,B\rangle\,dx\,dt + \langle B,B\rangle\,dt^2,\\ (ds_{\mathrm{II}})^2 &\equiv h_{ij}\,dx^i\,dx^j = -\langle A_x + [A,U], C\rangle\,dx^2 - 2\langle A_t + [A,V], C\rangle\,dx\,dt - \langle B_t + [B,V], C\rangle\,dt^2,\end{aligned} \tag{6}$$

where $i,j = 1,2$, $x^1 = x$ and $x^2 = t$, $\langle A,B\rangle = \frac{1}{2}\mathrm{trace}(AB)$, $[A,B] = AB - BA$, $\|A\| = \sqrt{|\langle A,A\rangle|}$, and $C = \frac{[A,B]}{\|[A,B]\|}$. A frame on this surface S, is

$$\Phi^{-1}A\Phi, \qquad \Phi^{-1}B\Phi, \qquad \Phi^{-1}C\Phi.$$

The Gauss and the mean curvatures of S are given by $K = \det(g^{-1}\,h)$, $H = \frac{1}{2}\mathrm{trace}(g^{-1}\,h)$.

The function Φ, which is defined by equations in Eq. (3) exists if and only if U and V satisfy Eq. (1) [15]. In other words, Eq. (1) is the compatibility condition of Eq. (3). The equations in Eq. (4) define a surface F if and only if A and B satisfy Eq. (2) [15]. Namely, Eq. (2) is the condition to define a surface F in Lie algebra $\mathcal{g}$ which is obtained from Eq. (4). Furthermore, to have regular surfaces, F_x and F_t (or A and B) must be linearly independent at each point of the surface S. This is the regularity condition of the mapping $F : S \to \mathbb{R}^3$. Hence the commutator $[A,B]$ is nowhere zero on the surface. This ensures that the three vectors A, B and C form a triad at each point of the surface.

Here and in what follows, subscripts x and t denote the derivatives of the objects with respect to x and t, respectively. Subscript nx stands for n times x derivative, where n is positive integer. Given U and V, finding A and B from the equation $A_t - B_x + [A,V] + [U,B] = 0$ is in general a difficult task. However, there are some deformations which provide A and B directly. Some of these deformations are given by Sym [12–14], Fokas and Gelfand [15], Fokas et al. [16] and Cieśliński [28]. As an example of such deformations, we shall make use of the λ parameter deformations and gauge symmetries of the Lax equation which are defined as, respectively,

$$A = \frac{\partial U}{\partial \lambda}, \qquad B = \frac{\partial V}{\partial \lambda}, \qquad F = \Phi^{-1}\,\frac{\partial \Phi}{\partial \lambda} \tag{7}$$

and

$$A = M_x + [M,U], \qquad B = M_t + [M,V], \qquad F = \Phi^{-1}M\Phi, \tag{8}$$

where M is any traceless 2×2 matrix.

For the KdV equation the group G is $\mathrm{SL}(2,\mathbb{R})$ and the algebra $\mathcal{g}$ is $\mathfrak{sl}(2,\mathbb{R})$ with the base 2×2 matrices

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{9}$$

Define an inner product on $\mathfrak{sl}(2,\mathbb{R})$ as

$$\langle X,Y\rangle = \frac{1}{2}\,\mathrm{trace}(XY), \tag{10}$$

for $X, Y \in \mathfrak{sl}(2,\mathbb{R})$.

In order to obtain the surfaces using the given technique, we have to find position vector F which is given by $F = \Phi^{-1}\,(\partial\Phi/\partial\lambda)$ [12]. To calculate F explicitly, the Lax equations Eq. (4) need to be solved for a given solution of the KdV equation.

3. Surfaces from a variational principle

Let H and K be the mean and Gaussian curvatures of a 2-surface S (either in M_3 (three-dimensional Minkowski space) or in $\mathbb{R}^3$) then we have the following definition.

Definition 3.1. Let S be a 2-surface with its Gaussian (K) and mean (H) curvatures. A functional $\mathcal{F}$ is defined by

$$\mathcal{F} = \int_S \mathcal{E}(H,K)dA + p\int_V dV \tag{11}$$

where $\mathcal{E}$ is some function of H and K, p is a constant and V is the volume enclosed within the surface S.

The following proposition gives the first variation of the functional $\mathcal{F}$.

Proposition 3.2. *Let $\mathcal{E}$ be a twice differentiable function of H and K. Then the Euler–Lagrange equation for $\mathcal{F}$ reduces to* [32–34]

$$(\nabla^2 + 4H^2 - 2K)\frac{\partial \mathcal{E}}{\partial H} + 2(\nabla \cdot \bar{\nabla} + 2KH)\frac{\partial \mathcal{E}}{\partial K} - 4H\mathcal{E} + 2p = 0. \tag{12}$$

Here and in what follows, $\nabla^2 = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial x^j}\right)$ and $\nabla \cdot \bar{\nabla} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}Kh^{ij}\frac{\partial}{\partial x^j}\right)$, $g = \det(g_{ij})$, g^{ij} and h^{ij} are inverse components of the first and second fundamental forms; $x^i = (x, t)$ and we assume Einstein's summation convention on repeated indices over their ranges.

Example 3.3. The following are some examples:
(i) Minimal surfaces: $\mathcal{E} = 1, p = 0$.
(ii) Constant mean curvature surfaces: $\mathcal{E} = 1$.
(iii) Linear Weingarten surfaces: $\mathcal{E} = aH + b$, where a and b are some constants.
(iv) Willmore surfaces: $\mathcal{E} = H^2$ [10,11].
(v) Surfaces solving the shape equation of lipid membrane: $\mathcal{E} = (H - c)^2$, where c is a constant [35,34].

Definition 3.4. The surfaces obtained from the solutions of the equation

$$\nabla^2 H + aH^3 + bH\,K = 0, \tag{13}$$

are called *Willmore-like* surfaces, where a and b are arbitrary constants.

Remark 3.5. The case $a = -b = 2$ in Eq. (13) corresponds to the Willmore surfaces.

In this work we assume $p = 0$. In addition, for surfaces derivable from a variational principal, we require asymptotic conditions such that H goes to a constant value and K goes to zero asymptotically. This is consistent with the vanishing of boundary terms in obtaining the Euler–Lagrange equation (Eq. (12)). This requires that the soliton equations such as the KdV, mKdV and NLS equations must have solutions decaying rapidly to zero at $|x| \to \pm\infty$. For this purpose, we shall calculate H and K for all surfaces obtained by the KdV equation and look for possible solutions (surfaces) of the Euler–Lagrange equation (Eq. (12)).

4. Surfaces from the KdV hierarchy by using spectral deformation

In this section, we investigate some surfaces arising from the KdV equations. KdV surfaces are embedded in a three-dimensional Minkowski space with signature +1. The following theorem gives the KdV hierarchy [39].

Theorem 4.1 (*Blaszak [39]*)**.** *Let $u(x, t_m) = u$ satisfy the evolution equations*

$$u_{t_m} = Y_m(u) = \varphi^m Y_0, \quad m = 0, 1, 2, \ldots \tag{14}$$

and set

$$U = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \qquad V_m = \begin{pmatrix} \tau_m & \kappa_m \\ \rho_m & -\tau_m \end{pmatrix}, \tag{15}$$

then we have

$$U_{t_m} + (V_m)_x + [U, V_m] = 0, \quad m \geq 0 \tag{16}$$

where $U, V_m \in \mathfrak{sl}(2, \mathbb{R})$, $Y_0 = u_x$, $\varphi(u) = \frac{1}{4}D^2 + u + \frac{1}{2}u_x D^{-1}$

$$\tau_m = -\frac{1}{4}\sum_{i=1}^{m} \lambda^{m-i} Y_{i-1}, \tag{17}$$

$$\kappa_m = \lambda^m + \frac{1}{2}\sum_{i=1}^{m} \lambda^{m-i}\gamma_{i-1}, \tag{18}$$

$$\rho_m = \lambda^m(\lambda - u) - \frac{1}{4}\sum_{i=1}^{m} \lambda^{m-i}\big((Y_{i-1})_x - 2(\lambda - u)\gamma_{i-1}\big), \tag{19}$$

$$D\gamma_j = Y_j. \tag{20}$$

Remark 4.2. U and V_m in Eq. (15) define Lax pairs of the KdV hierarchy Eq. (14).

We present classes of surfaces corresponding to the KdV hierarchy in the following proposition.

Proposition 4.3. *Let u satisfy Eq. (14). The corresponding $\mathfrak{sl}(2,\mathbb{R})$ valued hierarchy of Lax pairs U and V_m are given by (15). The corresponding $\mathfrak{sl}(2,\mathbb{R})$ valued matrices A and B_m are*

$$A = \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}, \qquad B_m = \begin{pmatrix} \mu {\tau'}_m & \mu \kappa'_m \\ \mu \rho'_m & -\mu {\tau'}_m \end{pmatrix}, \tag{21}$$

where $A = \mu\, \partial U/\partial\lambda$, $B = \mu\, \partial V_m/\partial\lambda$, μ and λ are arbitrary constants. Then the surfaces S_m, generated by U, V_m, A and B_m, have the following first and second fundamental forms $(m = 0, 1, 2, \ldots)$

$$(ds_{\mathrm{I}})^2 = \mu^2 \kappa'_m dx\, dt + \mu^2((\tau'_m)^2 + \kappa'_m \rho'_m) dt^2, \tag{22}$$

$$(ds_{\mathrm{II}})^2 = -\mu dx^2 - \mu\kappa_m dxdt - \frac{\mu}{\kappa'_m}\Big(\kappa'_m\big[(\tau'_m)_t + \kappa'_m\rho_m - \kappa_m\rho'_m + 2\tau_m\tau'_m\big] - \tau'_m(\kappa'_m)_t - 2(\tau_m)^2\kappa_m\Big)dt^2, \tag{23}$$

and the corresponding Gaussian and mean curvatures are

$$K_m = \frac{4}{\mu^2(\kappa'_m)^3}\Big(\kappa'_m\Big[(\kappa_m)^2 + (\tau'_m)_t - \kappa_m\rho'_m + \kappa'_m\rho_m + 2\tau_m\tau'_m\Big] - 2\kappa_m(\tau'_m)^2 - \tau'_m(\kappa'_m)_t\Big), \tag{24}$$

$$H_m = -\frac{2}{\mu(\kappa'_m)^2}\Big(\kappa'_m(\kappa_m - \rho'_m) - (\tau'_m)^2\Big) \tag{25}$$

where τ_m, κ_m, ρ_m are respectively (17), (18), (19), primes denote λ partial derivatives and $(\tau'_m)_t = \partial\tau'_m/\partial t$, $(\kappa'_m)_t = \partial\kappa'_m/\partial t$.

5. KdV surfaces from spectral deformations

In this section, we find surfaces arising from the spectral deformation of the Lax pair for the well-known KdV equation. If we let $m = 1$ in Theorem 4.1, we get the KdV equation and its Lax pair as shown in the following example.

Example 5.1. For $m = 1$ in Theorem 4.1 we have the Korteweg–de Vries (KdV) equation

$$u_t = \frac{1}{4}u_{3x} + \frac{3}{2}uu_x = Y_1(u). \tag{26}$$

$\mathfrak{sl}(2,\mathbb{R})$ valued Lax pairs U and V (we use V notation instead of V_1) are

$$U = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \tag{27}$$

$$V = \begin{pmatrix} -\frac{1}{4}u_x & \frac{1}{2}u + \lambda \\ -\frac{1}{4}u_{2x} + \frac{1}{2}\,(2\,\lambda + u)\,(\lambda - u) & \frac{1}{4}u_x \end{pmatrix}. \tag{28}$$

The following proposition gives a class of surfaces that correspond to the KdV equation (Eq. (26)) arising from spectral deformations of Lax pairs.

Proposition 5.2. *Let u satisfy Eq. (26). The corresponding $\mathfrak{sl}(2,\mathbb{R})$ valued Lax pairs U and V of the KdV equation are given by Eqs. (27) and (28). $\mathfrak{sl}(2,\mathbb{R})$ valued matrices A and B are*

$$A = \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}, \tag{29}$$

$$B = \begin{pmatrix} 0 & \mu \\ \frac{\mu}{2}(4\lambda - u) & 0 \end{pmatrix}, \tag{30}$$

where $A = \mu(\partial U/\partial\lambda)$, $B = \mu(\partial V/\partial\lambda)$, λ is the spectral parameter, and μ is a constant. Then the surface S, generated by U, V, A and B, has the following first and second fundamental forms $(i, j = 1, 2)$

$$(ds_{\mathrm{I}})^2 \equiv g_{ij}dx^i\, dx^j = \mu^2 dx\, dt + \frac{\mu^2}{2}(4\lambda - u)dt^2, \tag{31}$$

$$(ds_{\mathrm{II}})^2 \equiv h_{ij}dx^i dx^j = -\mu\, dx^2 - \mu(2\,\lambda + u)dx\, dt - \frac{\mu}{4}\Big(u_{2x} + (u + 2\,\lambda)^2\Big)dt^2, \tag{32}$$

and the corresponding Gaussian and mean curvatures are

$$K = -\frac{u_{2x}}{\mu^2}, \qquad H = \frac{2(\lambda - u)}{\mu}, \tag{33}$$

where $x^1 = x, x^2 = t$.

By using U, V, A and B, we found the first and second fundamental forms, and the Gaussian and mean curvatures of the KdV surfaces corresponding to spectral deformation in Proposition 5.2. The following proposition contains the quadratic Weingarten surfaces which are obtained by considering the travelling wave solutions of the KdV equation, i.e., $u_t + u_x/c = 0$, where c is a constant.

Proposition 5.3. *Let S be the surface obtained in Proposition 5.2 and u satisfy*

$$u_{2x} = -3u^2 - \frac{4}{c}u + 4\beta. \tag{34}$$

Then S is a quadratic Weingarten surface satisfying the relation

$$4\,c\;\mu^2\,K + 4\,\mu\,(2 + 3\,c\,\lambda)\,H - 3\,c\,\mu^2\,H^2 - 4\,(3\,c\,\lambda^2 + 4\,\lambda - 4\,\beta\,c) = 0, \tag{35}$$

where c and β *are constants. Here we assume that* $\mu \neq 0$.

Proposition 5.4. *Let u be the travelling wave solution* ($u_x^2 = -2u^3 + 4\alpha u^2 + 8\beta u + 2\gamma$) *of the KdV equation, then surface S defined in Proposition 5.2 is a Willmore-like surface, i.e., Gaussian and mean curvatures satisfy Eq. (13), where*

$$a = \frac{7}{4}, \qquad b = 1, \tag{36}$$

$$\beta = \frac{1}{20}\left(28\lambda\alpha - 16\alpha^2 - 21\lambda^2\right), \tag{37}$$

$$\gamma = \frac{1}{5}\left(16\alpha^3 - 56\lambda\alpha^2 + 70\alpha\lambda^2 - 28\lambda^3\right). \tag{38}$$

$\alpha = -1/c$ ($c \neq 0$), λ *and c are arbitrary constants.*

Proposition 5.5. *By using the travelling wave solution of the KdV equation and Proposition 5.2, one can show that the mean curvature of the KdV surface S satisfies a more general differential equation*

$$\nabla^2 H = -\frac{1}{2\mu^3}\Big[5\mu^3 H^3 + 2\mu^2(2\alpha - 3\lambda)H^2 + 4\mu(12\alpha\lambda - 9\lambda^2 - 8\alpha^2 - 12\beta)H + 56\lambda^3 - 112\lambda^2\alpha + 64\alpha^2\lambda - 32\lambda\beta + 64\alpha\beta + 16\gamma\Big]. \tag{39}$$

6. KdV surfaces from a variational principle

It is important to search surfaces which solve the generalized shape equation or in other words surfaces arising from a variational principle [8,32–36,38]. The following proposition gives a class of the KdV surfaces that solve the Euler–Lagrange equation (Eq. (12)).

Proposition 6.1. *Let u be the travelling wave solution of the KdV equation, i.e.,* $u_x^2 = -2u^3 + 4\alpha u^2 + 8\beta u + 2\gamma$. *Then there are KdV surfaces, defined in Proposition 5.2, that satisfy the generalized shape equation (Eq. (12)) when* $\mathcal{E}$ *is a polynomial function of H and K.*

Here are several examples:

Example 6.2. Let $\deg(\mathcal{E}) = N$, then

(i) for $N = 3$: $\mathcal{E} = a_1 H^3 + a_2 H^2 + a_3 H + a_4 + a_5 K + a_6 KH$,

$$a_1 = -\frac{11\,p\,\mu^4}{64\,\Omega_1}, \qquad a_2 = -\frac{15}{32\,\Omega_1}\,p\,\mu^3\big(2\,\alpha - 3\,\lambda\big),$$

$$a_3 = -\frac{p\,\mu^2}{16\,\Omega_1}\big(33\,\lambda^2 - 44\,\alpha\,\lambda + 8\,\alpha^2 - 20\,\beta\big),$$

$$a_4 = \frac{p\,\mu}{8\,\Omega_1}\left(47\,\lambda^3 - 94\,\alpha\,\lambda^2 + 4\,(10\,\alpha^2 - 17\,\beta)\,\lambda + 40\,\alpha\,\beta - 2\,\gamma\right),$$
$$a_6 = \frac{7\,p\,\mu^4}{16\,\Omega_1}$$

where $\Omega_1 = 12\,\lambda^4 - 32\,\alpha\,\lambda^3 + (20\,\alpha^2 - 36\,\beta)\lambda^2 + (40\,\alpha\,\beta - 3\,\gamma)\lambda + 2\,\alpha\,\gamma + 16\,\beta^2$, $\mu \neq 0$, $p \neq 0$, $\lambda, \alpha, \beta, \gamma$ and a_5 are arbitrary constants, but λ, α, β and γ cannot be zero at the same time.

(ii) for $N = 4$:

$$\mathcal{E} = a_1H^4 + a_2H^3 + a_3H^2 + a_4H + a_5 + a_6K + a_7KH + a_8K^2 + a_9KH^2,$$
$$a_1 = -\frac{1}{64}\,(34\,a_9 + 15\,a_8)$$
$$a_2 = \frac{1}{56\,\mu}\left[(210\,\lambda - 140\,\alpha)\,a_9 + (195\,\lambda - 130\,\alpha)\,a_8 - 22\,\mu\,a_7\right]$$
$$\begin{aligned}a_3 \;=\; & \frac{1}{56\,\mu^2}\Big[\Big(1512\,\alpha\,\lambda - 308\,\alpha^2 - 1134\,\lambda^2 + 588\,\beta\Big)a_9 + \Big(546\,\beta - 718\,\alpha^2 \\ & - 2025\,\lambda^2 + 2700\,\alpha\,\lambda\Big)a_8 + 60\,\mu\,\Big(3\,\lambda - 2\,\alpha\Big)a_7\Big],\end{aligned}$$
$$\begin{aligned}a_4 \;=\; & \frac{1}{14\,\mu^3}\Big[\Big(1414\,\lambda^3 - 2828\,\lambda^2\alpha + \big(1652\,\alpha^2 - 700\,\beta\big)\,\lambda + 392\,\beta\,\alpha - 28\,\gamma \\ & - 280\,\alpha^3\Big)a_9 + \Big(2265\,\lambda^3 - 4530\,\lambda^2\alpha + \big(2702\,\alpha^2 - 954\,\beta\big)\,\lambda \\ & - 42\,\gamma + 524\,\beta\,\alpha - 484\,\alpha^3\Big)a_8 - 2\,\Big(33\lambda^2 - 20\,\beta + 8\,\alpha^2 - 44\,\alpha\,\lambda\Big)\mu\,a_7\Big],\end{aligned}$$
$$\begin{aligned}a_5 \;=\; & \frac{1}{28\,\mu^3}\Big[\Big(19960\,\lambda^3\alpha - 7485\,\lambda^4 + \big(2844\,\beta - 19012\,\alpha^2\big)\,\lambda^2 \\ & + \big(96\,\gamma + 7664\,\alpha^3 - 3536\,\beta\,\alpha\big)\,\lambda + 784\,\beta^2 - 1008\,\alpha^4 + 1616\,\alpha^2\beta \\ & - 64\,\alpha\,\gamma\Big)a_8 + \Big(9744\,\lambda^3\alpha - 3654\,\lambda^4 + \big(168\,\beta - 9688\,\alpha^2\big)\,\lambda^2 \\ & + \big(4256\,\alpha^3 - 224\,\beta\,\alpha\big)\,\lambda + 224\,\beta^2 + 224\,\alpha^2\beta - 672\,\alpha^4\Big)a_9 \\ & + 8\,\mu\,a_7\Big(47\,\lambda^3 - 94\,\lambda^2\alpha + \big(-68\,\beta + 40\,\alpha^2\big)\,\lambda - 2\,\gamma + 40\,\beta\,\alpha\Big)\Big],\end{aligned}$$
$$\begin{aligned}a_7 \;=\; & \frac{1}{16\,\mu\,\Omega_2}\Big[-672\,\Big(4\,\alpha\,\lambda - \alpha^2 + \beta - 3\,\lambda^2\Big)\Big(7\,\lambda^3/6 - 7\,\lambda^2\alpha/3 \\ & + \big(\alpha^2 - 5\,\beta/3\big)\,\lambda + \beta\,\alpha - \gamma/24\Big)a_9 + \Big(4680\,\lambda^5 - 15600\,\lambda^4\alpha \\ & + \big(17576\,\alpha^2 - 9672\,\beta\big)\lambda^3 - \big(7664\,\alpha^3 + 414\,\gamma - 18240\,\beta\,\alpha\big)\lambda^2 \\ & + \big(552\,\alpha\,\gamma + 1008\,\alpha^4 + 3216\,\beta^2 - 9280\,\alpha^2\beta\big)\,\lambda - 170\,\alpha^2\gamma \\ & + 42\,\gamma\,\beta - 2032\,\alpha\,\beta^2 + 1008\,\beta\,\alpha^3\Big)a_8 + 7\,p\,\mu^5\Big]\end{aligned}$$

where $\Omega_2 = 4\,\lambda^3\,(3\,\lambda - 8\,\alpha) + 4\,\big(5\,\alpha^2 - 9\,\beta\big)\,\lambda^2 + (-3\,\gamma + 40\,\beta\,\alpha)\,\lambda + 2\,\alpha\,\gamma + 16\,\beta^2$, and $\mu \neq 0$, $p \neq 0$, $\lambda, \alpha, \beta, \gamma, a_6, a_8$ and a_9 are arbitrary constants, but λ, α, β and γ cannot be zero at the same time.

(iii) for $N = 5$:

$$\begin{aligned}\mathcal{E} \;=\; & a_1\,H^5 + a_2\,H^4 + a_3\,H^3 + a_4\,H^2 + a_5\,H + a_6 + a_7\,K + a_8\,K\,H \\ & + a_9\,K^2 + a_{10}\,K\,H^2 + a_{11}\,K^2\,H + a_{12}\,K\,H^3,\end{aligned}$$

$a_1, a_2, a_3, a_4, a_5, a_6, a_8$ can be written in terms of $a_9, a_{10}, a_{11}, a_{12}, \alpha, \beta, \gamma, \mu, p$ and λ.

For general $N \geq 3$, from the above examples, the polynomial function $\mathcal{E}$ takes the form

$$\mathcal{E} = \sum_{n=0}^{N} H^n \sum_{l=0}^{\left\lfloor \frac{(N-n)}{2} \right\rfloor} a_{nl} K^l,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and a_{nl} are constants.

7. The parameterized form of the KdV surfaces

In the previous section, possible surfaces satisfying certain equations are found without giving the F functions explicitly. In this section, we find the position vector

$$\mathbf{y} = (y_1(x,t), y_2(x,t), y_3(x,t)) \tag{40}$$

of the KdV surfaces for a given solution of the KdV equation and the corresponding Lax pairs. Our method of constructing the position vector $\mathbf{y}$ of integrable surfaces consists of the following steps:

(i) Find a solution $u = u(x,t)$ of the KdV equation with a given symmetry:

Here, we use travelling wave solutions $u_t = -u_x/c$. Using this assumption, we get

$$u_x^2 = -2u^3 - \frac{4}{c}u^2 + 8\beta u + 2\gamma, \tag{41}$$

where $c \neq 0$, β and γ are arbitrary constants.

(ii) Find a solution of the Lax equations (Eq. (3)) for given U and V:

In our case, corresponding $\mathfrak{sl}(2,\mathbb{R})$ valued Lax pairs of the KdV equation U and V are given by Eqs. (27) and (28). Consider the 2×2 matrix Φ

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}. \tag{42}$$

By using Φ and U, we can write $\Phi_x = U\Phi$ in matrix form as

$$\begin{pmatrix} (\Phi_{11})_x & (\Phi_{12})_x \\ (\Phi_{21})_x & (\Phi_{22})_x \end{pmatrix} = \begin{pmatrix} \Phi_{21} & \Phi_{22} \\ (\lambda - u)\Phi_{11} & (\lambda - u)\Phi_{12} \end{pmatrix}. \tag{43}$$

Using $(\Phi_{11})_x = \Phi_{21}$ and $(\Phi_{21})_x = (\lambda - u)\Phi_{11}$, we have

$$(\Phi_{11})_{xx} - (\lambda - u)\Phi_{11} = 0. \tag{44}$$

Similarly we have an equation for Φ_{12} as

$$(\Phi_{12})_{xx} - (\lambda - u)\Phi_{12} = 0. \tag{45}$$

By solving Eqs. (44) and (45), we determine the explicit x dependence of Φ_{11}, Φ_{12} and also Φ_{21}, Φ_{22}. By using $\Phi_t = V\Phi$, we get

$$(\Phi_{11})_t = -\frac{1}{4}u_x\Phi_{11} + \left(\frac{1}{2}u + \lambda\right)\Phi_{21}, \tag{46}$$

$$(\Phi_{21})_t = \left[-\frac{1}{4}u_{2x} + \frac{1}{2}\,(2\lambda + u)\,(\lambda - u)\right]\Phi_{11} + \frac{1}{4}u_x\Phi_{21}, \tag{47}$$

and

$$(\Phi_{12})_t = -\frac{1}{4}u_x\Phi_{12} + \left(\frac{1}{2}u + \lambda\right)\Phi_{22}, \tag{48}$$

$$(\Phi_{22})_t = \left[-\frac{1}{4}u_{2x} + \frac{1}{2}\,(2\lambda + u)\,(\lambda - u)\right]\Phi_{12} + \frac{1}{4}u_x\Phi_{22}. \tag{49}$$

Hence solving these equations, we determine the explicit t dependence of Φ_{11}, Φ_{21}, Φ_{12} and Φ_{22}. Thus we find a solution Φ of the Lax equations.

(iii) Find F: If Φ depends on λ explicitly, F can be found directly from

$$F = \Phi^{-1}\frac{\partial\Phi}{\partial\lambda} = y_1\mathrm{e}_1 + y_2\mathrm{e}_2 + y_3\mathrm{e}_3. \tag{50}$$

If Φ can be obtained for a fixed value of λ, then we can use Eq. (4) to find F. For our case, A and B are given by Eqs. (29) and (30), respectively. Integrating equations (Eq. (4)) we obtain F. We get the components of the vector $\mathbf{y}$ by writing F as a linear combination of e_1, e_2 and e_3, and collecting the coefficients of e_i.

Remark 7.1. The approach given above is easily applicable for simple solutions such as the travelling wave solutions of the KdV equation. For other solutions such as the two soliton solutions of the KdV equation it is very hard to find the parameterization of S analytically. It is however possible to solve the Lax equation numerically and plot the corresponding surfaces.

Remark 7.2. All KdV surfaces have the first and second fundamental forms given in Eqs. (31) and (32), respectively. Their Gaussian and mean curvatures are given in Eq. (33). Local and global properties of these surfaces depend on the function $u(x,t)$, a solution of the KdV equation. We assume that the function $u(x,t)$ is differentiable with respect to x three times and with respect to t once. This implies that the corresponding surfaces are locally smooth enough. For the global properties of these surfaces, we need asymptotic conditions of $u(x,t)$. For instance if $u(x,t)$ is an asymptotically decaying solution of the KdV equation, such as the one soliton solution, then the corresponding surfaces are asymptotically flat surfaces, i.e., Gaussian curvature goes to zero and mean curvature goes to a constant asymptotically. Furthermore, depending on the function $u(x,t)$, some KdV surfaces are shown to be Weingarten, some of them are Willmore-like and some of them extremize the most general functional where the Lagrange function is a polynomial of the Gaussian and mean curvatures.

Example 7.3. Let $u = u_0 = \frac{2}{3}(\alpha \pm \sqrt{\alpha^2 + 3\beta})$ be a constant solution of the integrated form $u_x^2 + 2u^3 - 4\alpha u^2 - 8\beta u - 2\gamma = 0$ of the KdV equation (Eq. (26)), where $\alpha = -1/c$, $c \neq 0$. Denoting $\lambda - u_0 = m^2$ and $(2\lambda + u_0)/2 = n$, we solve the Lax equations $\Phi_x = U\Phi$ and $\Phi_t = V\Phi$ for the given Lax pairs U and V by Eqs. (27) and (28), respectively. The components of Φ are

$$\Phi_{11} = C_1\, e^{m(nt+x)} + D_1\, e^{-m(nt+x)}, \tag{51}$$

$$\Phi_{12} = C_2\, e^{m(nt+x)} + D_2\, e^{-m(nt+x)}, \tag{52}$$

$$\Phi_{21} = m(C_1\, e^{m(nt+x)} - D_1\, e^{-m(nt+x)}), \tag{53}$$

$$\Phi_{22} = m(C_2\, e^{m(nt+x)} - D_2\, e^{-m(nt+x)}) \tag{54}$$

where C_1, gC_2, D_1 and D_2 are arbitrary constants.

Here we find that $\det(\Phi) = 2m(C_2D_1 - C_1D_2) \neq 0$.

By using A, B, and Φ, we solve Eq. (4) and write F as

$$F = \Phi^{-1}\frac{\partial\Phi}{\partial\lambda} = y_1 \mathrm{e}_1 + y_2 \mathrm{e}_2 + y_3 \mathrm{e}_3, \tag{55}$$

where $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3$ are basis elements of $\mathfrak{sl}(2,\mathbb{R})$ and

$$y_1 = -\left(\frac{D_1C_2 + C_1D_2}{D_1C_2 - C_1D_2}\right)\frac{(4\lambda - u_0)t + x}{2\sqrt{\lambda - u_0}}, \tag{56}$$

$$y_2 = \left(\frac{D_1C_1 - D_2C_2}{D_1C_2 - D_2C_1}\right)\frac{(4\lambda - u_0)t + x}{2\sqrt{\lambda - u_0}}, \tag{57}$$

$$y_3 = -\left(\frac{D_1C_1 + D_2C_2}{D_1C_2 - D_2C_1}\right)\frac{(4\lambda - u_0)t + x}{2\sqrt{\lambda - u_0}}. \tag{58}$$

Thus we find the position vector $\mathbf{y} = (y_1(x,t), y_2(x,t), y_3(x,t))$, where y_1, y_2 and y_3 are given by Eqs. (56)–(58), respectively. This solution corresponds to a plane in M_3.

Example 7.4. Let $u = 2\,k^2\,c^2 \mathrm{sech}^2 k(t - cx)$ be a one soliton solution of the KdV equation, where $k^2 = -1/c^3$. By denoting $k(t - cx) = \xi$, we find the solutions of Eqs. (44) and (45) as

$$\Phi_{11} = A_1(t)\mathrm{sech}\,\xi + B_1(t)[\sinh\xi + \xi\,\mathrm{sech}\,\xi], \tag{59}$$

$$\Phi_{12} = A_2(t)\mathrm{sech}\,\xi + B_2(t)[\sinh\xi + \xi\,\mathrm{sech}\,\xi], \tag{60}$$

and

$$\Phi_{21} = (\Phi_{11})_x = kc\,A_1(t)\,\mathrm{sech}\,\xi\ \tanh\xi + kc\,B_1(t)\,[\xi\,\mathrm{sech}\,\xi\ \tanh\xi - \cosh\xi - \mathrm{sech}\,\xi], \tag{61}$$

$$\Phi_{22} = (\Phi_{12})_x = kc\,A_2(t)\,\mathrm{sech}\,\xi\ \tanh\xi + kc\,B_2(t)\,[\xi\,\mathrm{sech}\,\xi\ \tanh\xi - \cosh\xi - \mathrm{sech}\,\xi], \tag{62}$$

for $\lambda = k^2c^2$. Using these functions and considering Eqs. (46)–(49) with $u_x = 4k^3c^3\,\mathrm{sech}^2\xi\ \tanh\xi$, $u_{2x} = 4k^3c^3\left(2\mathrm{sech}^2\xi\ \tanh^2\xi - \mathrm{sech}^4\xi\right)$, we get

$$B_1(t) = B_1 \quad \text{and} \quad A_1(t) = 2B_1kt + C_1, \tag{63}$$

$$B_2(t) = B_2 \quad \text{and} \quad A_2(t) = 2B_2kt + C_2, \tag{64}$$

where B_1, B_2, C_1 and C_2 are arbitrary constants. Thus the components of Φ are

$$\Phi_{11} = B_1\,(2kt\,\mathrm{sech}\,\xi + \sinh\xi + \xi\,\mathrm{sech}\,\xi) + C_1\mathrm{sech}\,\xi, \tag{65}$$

$$\Phi_{12} = B_2\,(2kt\,\mathrm{sech}\,\xi + \sinh\xi + \xi\,\mathrm{sech}\,\xi) + C_2\mathrm{sech}\,\xi, \tag{66}$$

$$\Phi_{21} = kc\Big[B_1\Big(2kt\,\mathrm{sech}\,\xi\,\tanh\xi - \cosh\xi - \mathrm{sech}\,\xi + \xi\,\mathrm{sech}\,\xi\,\tanh\xi\Big) + C_1\mathrm{sech}\,\xi\,\tanh\xi\Big], \tag{67}$$

$$\Phi_{22} = kc\Big[B_2\Big(2kt\,\mathrm{sech}\,\xi\,\tanh\xi - \cosh\xi - \mathrm{sech}\,\xi + \xi\,\mathrm{sech}\,\xi\,\tanh\xi\Big) + C_2\mathrm{sech}\,\xi\,\tanh\xi\Big]. \tag{68}$$

Here we find that $\det(\Phi) = 2\,k\,c(C_2B_1 - C_1B_2) \neq 0$.

By inserting Φ, A, and B into Eq. (4), and solving the resultant equation, we find the immersion function F explicitly as

$$F = y_1\mathrm{e}_1 + y_2\mathrm{e}_2 + y_3\mathrm{e}_3, \tag{69}$$

where

$$y_1 = \frac{2E_1}{\zeta_1}\Big[\Big(R_1\zeta_2 + R_2\zeta_1 + \zeta_3\Big)E_2 + R_3\zeta_2E_3 + E_4\Big], \tag{70}$$

$$y_2 = \frac{E_1}{\zeta_1}\Big[\Big(R_4\zeta_2 + R_2\zeta_1 + \zeta_4\Big)E_5 + \Big(R_5\zeta_2 + R_6\Big)E_6 + E_7\Big], \tag{71}$$

$$y_3 = \frac{E_1}{\zeta_1}\Big[\Big(R_4\zeta_2 + R_2\zeta_1 + \zeta_4\Big)E_8 + \Big(R_5\zeta_2 + R_6\Big)E_9 + E_{10}\Big], \tag{72}$$

and e_1, e_2, e_3 are basis elements of $\mathfrak{sl}(2,\mathbb{R})$. Here $\zeta_i,\ i = 1, 2, 3, 4$, $R_j,\ j = 1, 2, \ldots, 6$, and $E_l,\ l = 1, 2, \ldots, 10$ are given as

$$\zeta_1 = 1 + \mathrm{e}^{-2\xi}, \qquad \zeta_2 = \mathrm{e}^{-2\xi} - 1, \qquad \zeta_3 = c^3(\mathrm{e}^{-4\xi} - 1 - 2\sinh(2\xi)), \tag{73}$$

$$\zeta_4 = \zeta_3 + 288t^2, \qquad R_1 = -8(cx + 3t)^2, \qquad R_2 = 4kc^3(9t - cx), \tag{74}$$

$$R_3 = 8kc^3(3t - cx), \qquad R_4 = -8(c^2x^2 - 6tcx - 9t^2), \tag{75}$$

$$R_5 = -16kc^3(cx + 3t), \qquad R_6 = -192kc^3t, \tag{76}$$

$$E_1 = \mu/32c^2\,(B_1C_2 - B_2C_1)\,, \qquad E_2 = B_1B_2, \qquad E_3 = C_1B_2 + C_2B_1, \tag{77}$$

$$E_4 = -16c^3C_1C_2, \qquad E_5 = B_2^2 - B_1^2, \qquad E_6 = B_2C_2 - B_1C_1, \tag{78}$$

$$E_7 = 16c^3\left(C_1^2 - C_2^2\right), \qquad E_8 = B_1^2 + B_2^2, \qquad E_9 = B_1C_1 + B_2C_2, \tag{79}$$

$$E_{10} = -16c^3(C_1^2 + C_2^2), \tag{80}$$

where $\zeta_i,\ i = 1, 2, 3, 4$ and $R_j,\ j = 1, 2, \ldots, 6$ are functions of x and t, and $E_l,\ l = 1, 2, \ldots, 10$ are constants given in terms of arbitrary constants B_1, B_2, C_1, and C_2.

The position vector $\mathbf{y} = (y_1(x,t), y_2(x,t), y_3(x,t))$ of the KdV surface in M_3 corresponding to a one soliton solution of the KdV equation is given by Eqs. (70)–(72). Here y_3 is the time like and y_1 and y_2 are space like coordinates in M_3.

8. Graphs of some of the KdV surfaces

In this section, we will plot some of the surfaces given in Example 7.4 for special values of the constants.

Example 8.1. Taking $\mu = 1, k = 11.18, c = -0.2, B_1 = 2, B_2 = 1, C_1 = 1, C_2 = 1$ in Eqs. (70)–(72), we get the surface given in Fig. 1.

The components of the position vector are

$$\begin{aligned} y_1 \;&=\; \frac{1}{1+\mathrm{e}^{-2\xi}}\Big[\Big(-x^2 + (-0.89 - 30\,t)\,x - 225\,t^2 - 20.12\,t\Big)\,\mathrm{e}^{-2\xi} \\ &\quad + 0.05\,\sinh(2\xi) - 0.02\,\mathrm{e}^{-4\xi} + x^2 + (0.45 + 30\,t)\,x + 225\,t^2 + 0.22\Big], \end{aligned} \tag{81}$$

$$\begin{aligned} y_2 \;&=\; \frac{1}{1+\mathrm{e}^{-2\xi}}\Big[\Big(-168.75\,t^2 + (22.5\,x + 4.19)\,t + 0.75\,x^2 + 0.39\,x\Big)\,\mathrm{e}^{-2\xi} \\ &\quad - 0.04\,\sinh(2\xi) + 0.02\,\mathrm{e}^{-4\xi} - 506.25\,t^2 + (-2.51 - 22.5\,x)\,t - 0.75\,x^2 - 0.056\,x - 0.019\Big], \\ y_3 \;&=\; \frac{1}{1+\mathrm{e}^{-2\xi}}\Big[\Big(281.25\,t^2 + (-37.5\,x - 2.51)\,t - 1.25\,x^2 - 0.95\,x\Big)\,\mathrm{e}^{-2\xi} \\ &\quad + 0.06\,\sinh(\xi) - 0.03\,\mathrm{e}^{-4\xi} + 843.75\,t^2 + (17.61 + 37.5\,x)\,t + 1.25\,x^2 + 0.39\,x + 0.23\Big], \end{aligned} \tag{82}$$

where $\xi = 11.18t + 2.25x$.

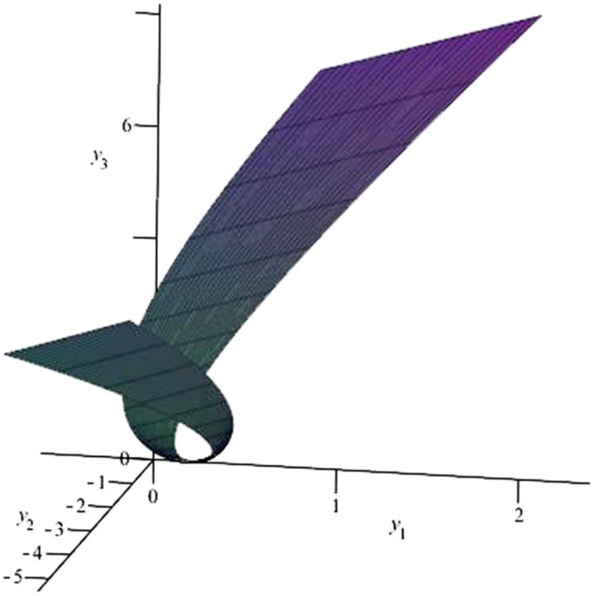

Fig. 1. $(x, t) \in [-0.1, 0.1] \times [-0.1, 0.1]$.

Example 8.2. Taking $\mu = 1$, $k = 0.19$, $c = -3$, $B_1 = 1$, $B_2 = 0$, $C_1 = 0$, $C_2 = 1$ in Eqs. (70)–(72), we get the surface given in Fig. 2. The components of the position vector are

$$y_1 = \frac{\ }{1 + \mathrm{e}^{-2\xi}} \left[-0.85\,(x + t)\left(\mathrm{e}^{-2\xi} - 1\right) \right], \tag{83}$$

$$\begin{aligned} y_2 = {} & \frac{1}{288\left(1 + \mathrm{e}^{-2\xi}\right)} \Big[\left(72\,x^2 + 144\,tx - 72\,t^2\right)\left(\mathrm{e}^{-2\xi} - 1\right) + 405 \\ & + 20.52\ (9\,t + 3\,x)\left(1 + \mathrm{e}^{-2\xi}\right) + 27\,\mathrm{e}^{-4\xi} - 54\,\sinh(2\xi) - 288\,t^2 \Big], \end{aligned} \tag{84}$$

$$\begin{aligned} y_3 = {} & \frac{1}{288\left(1 + \mathrm{e}^{-2\xi}\right)} \Big[\left(-72\,x^2 - 144\,tx + 72\,t^2\right)\left(\mathrm{e}^{-2\xi} - 1\right) + 459 \\ & - 20.52\ (9\,t + 3\,x)\left(1 + \mathrm{e}^{-2\xi}\right) - 27\,\mathrm{e}^{-4\xi} + 54\,\sinh(2\xi) + 288\,t^2 \Big], \end{aligned} \tag{85}$$

where $\xi = 0.19t + 0.57x$.

9. Conclusion

In this work, we considered families of surfaces such as the Willmore-like surfaces, the Weingarten surfaces and the surfaces derivable from a variational principle by using the KdV equation. Willmore-like surfaces, except for some particular values of the parameters, do not arise from a variational problem. To construct these families of surfaces, we use the method of deformation of the Lax equations corresponding to nonlinear partial differential equations, specifically the KdV equation. Any surface obtained through this method is called the *integrable surface*. This method allows us to find the first and second fundamental forms, the Gaussian and mean curvatures of these surfaces. By solving the corresponding Lax equations of integrable equations, it is possible to find explicit locations, i.e., position vectors of these surfaces. As an application we used the KdV equation and its Lax equation. Corresponding to these equations, we have found several families of Willmore-like surfaces and a hierarchy of surfaces arising from a variational problem, where the Lagrange function is a polynomial of the Gaussian and mean curvatures of these surfaces. We have plotted some of the KdV surfaces for special values of the parameters.

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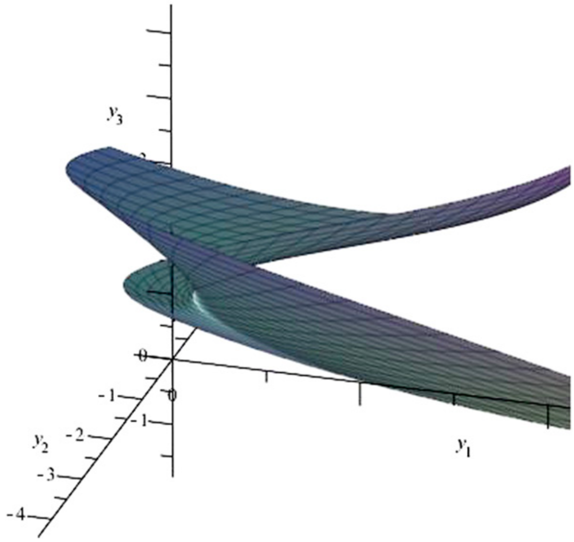

Fig. 2. $(x, t) \in [-2, 2] \times [-2, 2]$.

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