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Primary decomposition of the J-groups of complex projective and lens spaces

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Abstract

We determine the decomposition of J-groups of complex projective and lens spaces as a direct-sum of cyclic groups.

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1. Introduction

This paper is a continuation of [1] whose results we briefly summarize. For a finite-dimensional CW-complex X, let $J(X)$ denote the finite Abelian group of stable fibre homotopy classes of vector-bundles over X and for a prime p, $J_p(X)$ the p-summand of $J(X)$. For $n, k \in \mathbb{Z}^+$, let $P_n(\mathbb{C}) = S^{2n+1}/U(1)$ and $L^n(p^k) = S^{2n+1}/\mathbb{Z}_{p^k}$ denote the complex projective space of (complex)-dimension n and the associated lens space respectively. In [1] $J_p(P_n(\mathbb{C}))$ and $J(L^n(p^k))$ are determined by means of a set of generators and a complete set of relations. Let r_n be the greatest integer such that $p^{r_n} \leqslant n/p - 1$. Then for $0 \leqslant s \leqslant r_n$ and $0 \leqslant j \leqslant r_n - s$ we defined a decreasing sequence by $t_j^s = [n - p^s(p^j - 1)/p^{s+j}(p-1)]$ where for a real number x, $[x]$ denotes the greatest integer less than or equal to x. Put $t_j = t_j^0$. We let ω denote the realification of the reduction of the Hopf bundle over $P_n(\mathbb{C})$. Let $\psi_{\mathbb{R}}^k$ denote the Adams operation acting on $\widetilde{K}_{\mathbb{R}}(P_n(\mathbb{C}))$ and also on $J(P_n(\mathbb{C}))$ and $\rho_{\mathbb{R}}^k$ the associated characteristic class taking values in $1 + \widetilde{K}_{\mathbb{R}}(P_n(\mathbb{C})) \otimes Q_k$ where Q_k is the sub-ring of rationals whose denominators are powers of k. $m \in \mathbb{Z}$ is defined to be a singular s-exponent if and only if the coefficient of ω^m in the power series $\rho_{\mathbb{R}}^p(\psi_{\mathbb{R}}^{p^k}(\omega))^{p^{t_k}}$ is not integral (i.e. fractional) for some $k \in \mathbb{Z}^+$. The j-index, ε_m^j of a singular s-exponent m is the exponent of p in the denominator of the coefficient of ω^m in the expansion of $\rho_{\mathbb{R}}^p(\psi_{\mathbb{R}}^{p^j}(\omega))^{p^{t_j}}$. $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{r_n})$ is an s-admissible sequence if and only if Congruence 1: $\sum_j \frac{\alpha_j}{p^{\varepsilon_m^j}} \in \mathbb{Z}$ is satisfied by all singular s-exponents m. We let $\Phi_s^0 = \{t_k^s \mid t_k^s \equiv 0 \pmod{p}\}$ and $\mathcal{M}^s =$ set of all singular s-exponents. Then [1, Proposition 4.4.4] states that there is a bijection $\sigma_s : \Phi_s^0 \to \mathcal{M}^s$ ($\mathcal{M} = \mathcal{M}^0$) given by $\sigma_s(t_k^s) = 1/2(p-1)p^k t_k^s$. If $t_k^s \in \Phi_s^0$, $t_k^s = p^\nu \Delta$ ($\nu \geqslant 1$,

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$(\Delta, p) = 1)$) and $m = \sigma_s(t_k^s)$ then $\varepsilon_m^j = p^{k+\nu-j}\Delta + k + \nu - j - t_j^s$ [1, Proposition 6.2.7] reduces the question of relations in $J_p(P_n(\mathbb{C}))$ to s-admissibility; in particular, proves that a relation: $\sum_{j\geqslant 0}\alpha_j p^{t_j^s}\Psi_{\mathbb{R}}^{p^{s+j}}(\omega) = 0$ $(0 \leqslant s \leqslant r_n)$ exists in $J_p(P_n(\mathbb{C}))$ if and only if $\alpha = \{\alpha_j\}$ is an s-admissible sequence. In [1, Section 5.2] two different sequences called α- and β- sequences are constructed for each $0 \leqslant s \leqslant r_n$ where $\alpha_j^s = \mp 1$ or 0 and they are proved to be s-admissible. In [1, Proposition 6.2.8 and 6.2.9] we obtain the corresponding set of (r_n+1)-relations in $J_p(P_n(\mathbb{C}))$ which are proved to be complete. Hence in [1] $J_p(P_n(\mathbb{C}))$ is determined by generators and a complete set of relations. Analogous relations are then obtained for the J-groups of lens spaces.

However, the determination of the structure of a finite Abelian group is far from being over unless its primary decomposition into cyclic groups is uncovered and it is the purpose of the present paper to determine the primary decompositions of $J_p(P_n(\mathbb{C}))$ and $J(L^n(p^k))$. Using the framework of [1] the primary decomposition of $J_p(P_n(\mathbb{C}))$ is reduced to the solution of the following problem in elementary number theory. For a prime p and a rational q, let $v_p(q)$ denote the exponent of p in the prime factorization of q.

Problem. Let $k, d \in \mathbb{Z}^+$, $\{k_i\}$ and τ_i are strictly-decreasing sequences such that $0 \leqslant k_i \leqslant \tau_i \leqslant k$. Given integers $\{\varepsilon_i^j\}$ $(1 \leqslant i \leqslant d,\ j \leqslant \tau_i)$ such that

- (i) For fixed i, ε_i^j is a strictly-increasing sequence in j for $j \leqslant k_i - 1$, $\varepsilon_i^{k_i-1} \leqslant \varepsilon_i^{k_i}$, $\varepsilon_i^{k_i} = \varepsilon_i^{k_i+1}$ and $\{\varepsilon_i^j\}$ is a strictly-decreasing sequence in j for $k_i + 1 \leqslant j \leqslant \tau_i$.
- (ii) For fixed i, $\varepsilon_i^j \geqslant 1$ for at least one $0 \leqslant j \leqslant k$.
- (iii) $\varepsilon_i^{j'} - \varepsilon_i^j > \varepsilon_{i'}^{j'} - \varepsilon_{i'}^j$ for $i < i'$ and $j < j'$.

Find the least $v_p(\alpha_k)$ for a solution $\alpha = (\alpha_0, \ldots, \alpha_k)$ $(\alpha_i \in \mathbb{Z})$ of Congruence 1; i.e.

$$\frac{\alpha_k}{p^{\varepsilon_i^k}} + \frac{\alpha_{k-1}}{p^{\varepsilon_i^{k-1}}} + \cdots + \frac{\alpha_0}{p^{\varepsilon_i^0}} \in \mathbb{Z} \quad (1 \leqslant i \leqslant d). \tag{1}$$

The main effort of this paper is concentrated in giving a solution to this problem.

Let $\mathcal{H}_k = \{\Omega \in 2^{[1,2,\ldots,k]}$ either $\Omega = \varphi$, or, $\Omega = \{i_1, \ldots, i_r\}$, $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant d$, $\varepsilon_{i_t}^{k-t+1} - \varepsilon_{i_t}^{k-t} > 0$ and $\varepsilon_{i_t}^{k-t} > 0$, $1 \leqslant t \leqslant r\}$. For each $\Omega \in \mathcal{H}_k$ we define the associated set $\Phi = \{\{j_1, \ldots, j_s\} \mid 1 \leqslant j_1 < \cdots < j_s \leqslant k$, $(\varepsilon_{j_h}^{k-j_h+j} - \varepsilon_{j_h}^{k-j_h+j-1}) > 0$, $1 \leqslant j \leqslant s\}$. Then we observe that $\Omega \cup \Phi = \{1, 2, \ldots, l\}$, $l \leqslant d$. For each element $\Omega = (i_1, \ldots, i_r) \in \mathcal{H}_k$, we put $v_p(\alpha_{k-t+1}) = v_p(\alpha_{k-t}) + (\varepsilon_{i_t}^{k-t+1} - \varepsilon_{i_t}^{k-t})$ $(1 \leqslant t \leqslant r)$ and $\alpha_{k-r} = 1$, and obtain a vector $\alpha = (\alpha_0, \ldots, \alpha_k)$ with $\alpha_0 = \cdots = \alpha_{k-r-1} = 0$ and $\alpha_k = p^{\sum_{t=1}^r(\varepsilon_{i_t}^{k-t+1} - \varepsilon_{i_t}^{k-t})}$. The Congruence Theorem (e.g. Theorem 2.14) which is original (and proved in this paper for the first time) shows that the vector α so defined is a solution of the system of Congruences 1 with respect to $\Omega \cup [l+1, l+2, \ldots, k]$. We then require all terms of Congruence 1 with respect to $j_1, j_2, \ldots, j_s$ be integral. This necessitates that $v_p(\alpha_{k-j_h+h}) \geqslant \varepsilon_{j_h}^{k-j_h+h}$ $(1 \leqslant h \leqslant s)$; or, equivalently, that $v_p(\alpha_k) \geqslant \sum_{t=1}^{j_h-h}(\varepsilon_{i_t}^{k-t+1} - \varepsilon_{i_t}^{k-t}) + \varepsilon_{j_h}^{k-j_h+h}$. Hence if we define

$$u(\Omega) = \max\left(\sum_{t=1}^{r}\left(\varepsilon_{i_t}^{k-t+1} - \varepsilon_{i_t}^{k-t}\right);\ \max_{\substack{1\leqslant h\leqslant s\\ \varepsilon_{j_h}^{k-j_h+h} > 0}}\left(\sum_{t=1}^{j_h-h}\left(\varepsilon_{i_t}^{k-t+1} - \varepsilon_{i_t}^{k-t}\right) + \varepsilon_{j_h}^{k-j_h+h}\right)\right). \tag{2}$$

We obtain a solution $\alpha = (\alpha_0, \ldots, \alpha_k)$ with $v_p(\alpha_k) = u(\Omega)$. We then define a unique element $\Omega^0 \in \mathcal{H}_k$ and define (e.g. Definition 2.5) $u_k = u(\Omega^0)$. Hence a solution to the system of Congruences 1 exists with $v_p(\alpha_k) = u_k$. (Actually, $\alpha_k = p^{u_k}$.) We then show (e.g. Proposition 2.6) that u_k is minimal among $u(\Omega)$ as Ω varies over $\mathcal{H}_k$. The observation that when all the terms of Congruence 1 are not integral there are at least two terms with highest denominator with positive p-exponent leads to Lemmas 2.8, 2.9, Corollary 2.10 and Lemma 2.11 and Lemma 2.11 together with Proposition 2.6 yield Proposition 2.12 which states that for any solution $\alpha = (\alpha_0, \ldots, \alpha_k)$ of the system of Congruence 1, $v_p(\alpha_k) \geqslant u_k$. Hence the problem stated is solved in full. This completes the elementary number theory.

The proof of Theorem 2.21 which is the main result of the paper is straightforward algebra. It combines [1, Proposition 6.2.14] with the solution of the above problem to deduce the primary decomposition of $J_p(P_n(\mathbb{C}))$.

Thus, relations, $\beta_0\omega + \cdots + \beta_k\psi_{\mathbb{R}}^{p^k}(\omega) = 0$ with minimal $v_p(\beta_k) = t_k + u_k$ (actually, $\beta_k = p^{t_k+u_k}$) exist. It follows from elementary algebra in a straightforward way that $J_p(P_n(\mathbb{C}))$ has a primary decomposition with invariants $p^{t_k+u_k}$ for $0 \leqslant k \leqslant r_n$. The first summand in this decomposition is generated by w and it is proved that it has order $p^{t_0+u_0} = M_{n+1,p} = p$-component of the Atiyah–Todd number M_{n+1}. We then extend this to J-groups of lens spaces. Let $G(p,n,r)$ be the sub-group of $J(L^n(p^r))$ generated by powers of w. The index functions ε_m^j are replaced by a certain reduction $\varepsilon_m^j(r)$ with respect to r. The whole theory goes over with u_k^r defined analogously with u_k in terms of $\varepsilon_m^j(r)$ and $G(p,n,r)$ has a primary decomposition with invariants $p^{t_k+u_k^r}$ for $1 \leqslant k \leqslant r-1$. From this we recover the decomposition of $J(L^n(p^r))$ into a direct-sum of cyclic groups. The first summand generated by w has order $p^{t_0+u_0^r} = M_{n+1}(p^r)$ where $M_{n+1}(p^r)$ is defined in [1, Definition 7.3.4]. We also recover the primary decompositions of $J_p(P_n(\mathbb{C}))$ and $J(L^n(p^r))$ when $n = (p-1)p^k + s$ $(0 \leqslant s \leqslant p-2)$. In [1] as a demonstration we wrote down the α- and β-relations in $J_2(P_{164}(\mathbb{C}))$. In this paper we write down, very quickly, the primary decomposition of $J_2(P_{164}(\mathbb{C}))$.

These primary decompositions are existential in the sense that they only give the invariants of $J_p(P_n(\mathbb{C}))$ and $J(L^n(p^r))$ (i.e. the orders of the cyclic groups in this decomposition) without an explicit expression for their generators except that of the first summand which is generated by w. For this reason, the previous paper [1] is essential to those who seek explicit relations in these J-groups.

With [1] and the present paper the algebraic structure of the J-groups of complex projective and lens spaces is completely determined and there is nothing more to do on the algebraic side. However, J-groups have two different structures compatible with each other; the algebraic structure and the degree-function on them defined by stable co-degrees of vector bundles just as vector spaces with a norm. The algebraic structure thus determined, the way is opened to the determination of the degree-function and it is hoped that the infrastructure of this problem is laid down in these two papers. The degree-function q on negative multiples of the complex Hopf bundle is the complex stable James number which is the order of the obstruction to cross-sectioning a certain Stiefel fibration. Let $1+\sum_{i\geqslant 1} a_i^n x^i = (1+\sum_{i\geqslant 1}\frac{x^{p^i-1}}{p^i})^n$. Then a folklore conjecture states that the p-primary component $q_p(n\eta_{k-1})$ of $q(n\eta_{k-1})$ is equal to $LCD\{a_i^n\colon 1 \leqslant i \leqslant k-1\}$ for either p odd or $p = 2$ and n even.

2. Primary decomposition of $J_p(P_n(\mathbb{C}))$

2.1. *Background material from [1]*

p is a fixed prime throughout. We define a decreasing sequence $(t_k)_{0\leqslant k\leqslant r_n}$ by $t_k = [\frac{n-p^k+1}{p^k(p-1)}]$. Let $\Phi^0 = \{t_k\colon t_k \equiv 0 \pmod p\}$. m is a singular exponent (i.e. $m \in \mathcal{M}$) if the coefficient of ω^m in the expansion of $\rho_{\mathbb{R}}^p(\psi_{\mathbb{R}}^{p^k}(\omega))^{p^{t_k}}$ is not integral for some $0 \leqslant k \leqslant r_n$. [1, Proposition 4.4.4] states that there is a bijection, $\sigma : \Phi^0 \to \mathcal{M}$ given by $\sigma(t_k) = \frac{1}{2}(p-1)p^k t_k$; i.e. if $t_k = p^\nu\Delta$ $(\nu \geqslant 1,\ (\Delta,p)=1)$ then $m = \frac{1}{2}(p-1)p^i\Delta$ where $i = k+\nu$. $t_j = p^{i-j}\Delta - 1$, $(k+1 \leqslant j \leqslant i)$ and if we let $T_m = \bigcup_{j=k}^{i} t_j$ then for consecutive elements, $m, m' \in \mathcal{M}$ $(m' > m)$, $T_{m'} \cap T_m$ is either empty or equal to $\{t_{i'}\} = \{t_k\}$ and the latter is always the case if $p = 2$. The j-index, ε_m^j of m, defined for $j \leqslant i$, is the p-exponent of the denominator of the coefficient of ω^m in the expansion of $\rho_{\mathbb{R}}^p(\psi_{\mathbb{R}}^{p^j}(\omega))^{p^{t_j}}$ when that coefficient is not integral and is given by the formula, $\varepsilon_m^j = p^{i-j}\Delta + i - j - t_j$. $\{\varepsilon_m^j\}_{k\leqslant j\leqslant i} = (\nu\ \nu\ \nu-1\ \nu-2\ \ldots\ 2\ 1)$ and $(\varepsilon_m^j)_{j\leqslant k-1}$ is a strictly-increasing sequence bounded above by ν. $\varepsilon_m^{k-1} = \nu$ iff $t_{k-1} = p^{\nu+1}\Delta + 1$.

2.2. Lemma. *Let $m, m' \in \mathcal{M}$, $m' > m$, $m = \sigma(t_k) = \sigma(p^\nu\Delta) = \frac{1}{2}(p-1)p^i\Delta$, $m' = \sigma(t_{k'}) = \sigma(p^{\nu'}\Delta') = \frac{1}{2}(p-1)p^{i'}\Delta'$ $((\Delta,p) = (\Delta',p) = 1,\ i = k+\nu,\ i' = k'+\nu')$. If $j' < j \leqslant i'$ then $(\varepsilon_m^j - \varepsilon_m^{j'}) > (\varepsilon_{m'}^j - \varepsilon_{m'}^{j'})$.*

Proof.

$$\begin{aligned}&(\varepsilon_m^j - \varepsilon_m^{j'}) - (\varepsilon_{m'}^j - \varepsilon_{m'}^{j'})\\ &\quad = (p^{i-j}\Delta + i - j - t_j) - (p^{i-j'}\Delta + i - j' - t_{j'}) - [(p^{i'-j}\Delta' + i' - j - t_j) - (p^{i'-j'}\Delta' + i' - j' - t_{j'})]\end{aligned}$$

$$= \left(p^i \Delta - p^{i'} \Delta'\right)\left(p^{-j} - p^{-j'}\right)$$
$$= \frac{2}{(p-1)}(m-m')\left(\frac{1}{p^j} - \frac{1}{p^{j'}}\right) > 0. \qquad \square$$

2.3. Definition. For $0 \leqslant k \leqslant r_n$, $\mathcal{M}_k$ is the set $m \in \mathcal{M}$ such that the coefficient of ω^m in the expansion of $\rho_{\mathbb{R}}^p(\psi_{\mathbb{R}}^{p^j}(\omega))^{p^{t_j}}$ is not integral for some $0 \leqslant j \leqslant k$. Let $\mathcal{M}_k = \{m_1, \ldots, m_d\}$, $m_1 < m_2 < \cdots < m_d$.

2.4. Observation. Let $m, m' \in \mathcal{M}$, $m' > m$. If $(\varepsilon_m^i - \varepsilon_m^{i-1})$ is either undefined, or, $(\varepsilon_m^i - \varepsilon_m^{i-1}) \leqslant 0$ $(i \geqslant 1)$ then either $(\varepsilon_{m'}^{i-1} - \varepsilon_{m'}^{i-2})$ in undefined, or, $(\varepsilon_{m'}^{i-1} - \varepsilon_{m'}^{i-2}) \leqslant 0$.

Proof. It follows from 2.1 that if $m = \sigma(t_k)$ and $m' = \sigma(t_{k'})$ Then $k' \leqslant k-1$ and $k \leqslant i$ and hence $k' \leqslant i-1$. Thus, $(\varepsilon_{m'}^{i-1} - \varepsilon_{m'}^{i-2})$ is either undefined, or, $(\varepsilon_{m'}^{i-1} - \varepsilon_{m'}^{i-2}) \leqslant 0$. $\square$

2.5. Definition. Let $\mathcal{H}_k = \{\Omega \in 2^{\mathcal{M}_k}$ either $\Omega = \phi$, or, $\Omega = (m_{i_1}, \ldots, m_{i_r})$, $1 < i_1 < i_2 < \cdots < i_r \leqslant d$, $(\varepsilon_{i_t}^{(k-t+1)} - \varepsilon_{i_t}^{k-t}) > 0$ and $\varepsilon_{i_t}^{k-t} > 0$, $1 \leqslant t \leqslant r\}$. Define the associated set $\Phi = \{\Omega \in \mathcal{M}_k - \Omega \mid$ either $m < m_{i_r}$, or, $m > m_{i_r}$ and $(\varepsilon_m^{(k-r)} - \varepsilon_m^{k-r-1}) > 0\}$:

(i) Let $m \in \Phi$, $m_{i_t} < m < m_{i_{t+1}}$. Then $(\varepsilon_m^{(k-t+1)} - \varepsilon_m^{k-t}) > 0$ (since if we assume the contrary then $(\varepsilon_{i_t}^{(k-t+1)} - \varepsilon_{i_t}^{k-t})$ is either undefined, or, $(\varepsilon_{i_t}^{(k-t+1)} - \varepsilon_{i_t}^{k-t}) \leqslant 0$ by Observation 2.4 which is a contradiction).

(ii) Let $m_\ell = \sup(\Omega \cup \Phi)$. It follows from the Observation 2.4 that $\Omega \cup \Phi = \{m_1, m_2, \ldots, m_\ell\}$, $\ell \leqslant d$ and $(\varepsilon_m^{(k-r)} - \varepsilon_m^{k-r-1})$ is either undefined, or $(\varepsilon_m^{(k-r)} - \varepsilon_m^{k-r-1}) \leqslant 0$ for $m > m_\ell$.

For $\Omega \in \mathcal{H}_k - \{\emptyset\}$ we write down the corresponding set of inequalities $I(\Omega)$ for the index functions ε_i^j. Let $\Omega = (m_{i_1}, \ldots, m_{i_r})$, $\Phi = (m_{j_1}, \ldots, m_{j_s})$ $(r + s = \ell)$.

For $1 \leqslant t \leqslant r$, $\varepsilon_{i_t}^{(k-t)} > \sum_{u=t+1}^r (\varepsilon_{i_u}^{(k-u+1)} - \varepsilon_{i_u}^{(k-u)})$ and $\varepsilon_{i_t}^{(k-t)} > \sum_{u=t+1}^r (\varepsilon_{i_u}^{(k-u+1)} - \varepsilon_{i_u}^{(k-u)}) + \varepsilon_{j_h}^{(k-j_h+h)}$ for $j_h - h \geqslant t$, $\varepsilon_{j_h}^{(k-j_h+h)} > 0$.

There is a 1–1 correspondence between $\Omega \in \mathcal{H}_k - \{\phi\}$ and the set of inequalities $I(\Omega)$. The set of inequalities $\{I(\Omega);\ \Omega \in \mathcal{H}_k - \{\phi\}\}$ are disjoint. (Two of which cannot hold simultaneously.) Hence at most one set of inequalities is satisfied.

We let $\Omega^0 = \Omega$ if $I(\Omega)$ is satisfied for some $\Omega \in \mathcal{H}_k$ and $\Omega^0 = \phi$ if none of the set of inequalities $I(\Omega)$ is satisfied.

For $\Omega = \{m_{i_1}, \ldots, m_{i_r}\} \in \mathcal{H}_k$ and $\Phi = \{m_{j_1}, \ldots, m_{j_s}\}$ we define

$$u(\Omega) = \max\left(\sum_{t=1}^{r}\left(\varepsilon_{i_t}^{(k-t+1)} - \varepsilon_{i_t}^{(k-t)}\right);\ \max_{\substack{1 \leqslant h \leqslant s \\ \varepsilon_{i_t}^{k-j_h+h} > 0}} \left(\sum_{t=1}^{j_h - h}\left(\varepsilon_{i_t}^{(k-t+1)} - \varepsilon_{i_t}^{(k-t)}\right) + \varepsilon_{j_h}^{k-j_h+h}\right)\right). \tag{3}$$

We note that $u(\phi) = \max(\varepsilon_m^k\colon m \in \mathcal{M}_k,\ (\varepsilon_m^k - \varepsilon_m^{k-1}) > 0)$. We then define $u_k = u(\Omega^0)$ where Ω^0 is the unique element of $\mathcal{H}_k$ defined above.

2.6. Proposition. *Let $\Omega \in \mathcal{H}_k$. Then $v_p(u(\Omega)) \geqslant u_k$.*

Proof. Let $\Omega \in \mathcal{H}_k$ and Φ be the associated set. Let $\Omega^0 \in \mathcal{H}_k$ be the unique element of $\mathcal{H}_k$ defined in 2.5 such that $u_k = u(\Omega^0)$. Let $\Omega^0 = \{m_{i_1}, \ldots, m_{i_r}\}$, $\Phi^0 = \{m_{j_1}, \ldots, m_{j_s}\}$, $\Omega^0 \cup \Phi^0 = \{m_1, \ldots, m_\ell\}$. Then by Definition 2.5,

$$u_k = \max\left(\sum_{t=1}^{r}\left(\varepsilon_{i_t}^{(k-t+1)} - \varepsilon_{i_t}^{(k-t)}\right);\ \max_{\substack{1 \leqslant h \leqslant s \\ \varepsilon_{j_h}^{k-j_h+h} > 0}} \left(\sum_{t=1}^{j_h - h}\left(\varepsilon_{i_t}^{(k-t+1)} - \varepsilon_{i_t}^{(k-t)}\right) + \varepsilon_{j_h}^{k-j_h+h}\right)\right). \tag{4}$$

Let $\varepsilon_{j_h}^{k-j_h+h} > 0$ for some $1 \leqslant h \leqslant s$. Suppose $m_{i_1}, \ldots, m_{i_p} \in \Omega$ and $m_{i_{p+1}} \notin \Phi$ $(1 \leqslant p \leqslant j_h - h)$

$$\begin{aligned}
&\sum_{t=1}^{j_h-h}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right)+\varepsilon_{j_h}^{k-j_h+h} \\
&\quad=\sum_{t=1}^{p}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right)+\sum_{t=p+1}^{j_h-h}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right) \\
&\quad=\sum_{t=1}^{p}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right)+\varepsilon_{i_{p+1}}^{k-p-1}-\sum_{t=p+2}^{j_h-h}\left(\varepsilon_{i_{p+1}}^{k-p}-\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right)-\varepsilon_{j_h}^{k-j_h+h}\right) \\
&\quad<\sum_{t=1}^{p}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right)+\varepsilon_{i_{p+1}}^{k-p}<u(\Omega)
\end{aligned}\tag{5}$$

since by definition of Ω^0 we have $\varepsilon_{i_{p+1}}^{k-p-1}<\sum_{t=p+2}^{j_h-h}(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t})+\varepsilon_{j_h}^{k-j_h+h}$. Also,

$$\begin{aligned}
\sum_{t=1}^{r}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right)&=\sum_{t=1}^{p}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right)+\varepsilon_{i_{p+1}}^{k-p}-\left(\varepsilon_{i_{p+1}}^{k-p-1}-\sum_{t=p+2}^{r}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right)\right) \\
&<\sum_{t=1}^{p}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right)+\varepsilon_{i_{p+1}}^{k-p}<u(\Omega). \qquad \square
\end{aligned}\tag{6}$$

2.7. Definition. Motivated by Proposition 2.6, we can give an alternative definition of u_k. Let $\mathcal{H}_k$ be defined in Definition 2.5. If $\Omega\in\mathcal{H}_k$ define $u(\Omega)$ as in 2.5. Then define $u_k=\min u(\Omega)$. Proposition 2.6, shows the equivalence of Definitions 2.5 and 2.7.

If we use Definition 2.5, we have to check out the set of inequalities $I(\Omega)$ and hence pick the unique Ω^0 and with Definition 2.7 we have to check out $u(\Omega)$ for $\Omega\in\mathcal{H}_k$ and take its minimum and the two require equal labour. However, Definition 2.5 gives more insight.

2.8. Lemma. *If all the terms in Congruence* 1; *i.e.* $\sum_{j=0}^{k}\frac{\alpha_j}{p^{\varepsilon_m^j}}\in\mathbb{Z}$ *are not integral there exists a subset* U *of* $(1,2,\dots,k)$ *containing at least two elements such that*

(i) $v_p(\alpha_i)<\varepsilon_m^i\ \forall i\in U$;
(ii) $v_p(\alpha_i)-v_p(\alpha_j)<\varepsilon_m^i-\varepsilon_m^j\ \forall i,j\in U$; *i.e.* $(v_p(\alpha_i)-\varepsilon_m^i\mid i\in U)$ *is a constant* k_U;
(iii) $v_p(\alpha_i)-v_p(\alpha_j)<\varepsilon_m^i-\varepsilon_m^j$ *for* $0\leqslant j<i\leqslant k$, $i\notin U$, $j\in U$; *i.e.* $(v_p(\alpha_i)-\varepsilon_m^i)>k_U\ \forall i\notin U$.

Proof. Assume the contrary. Then either (a) There is no pair (i,j), $0<j<i\leqslant k$ such that $v_p(\alpha_i)-v_p(\alpha_j)=\varepsilon_m^i-\varepsilon_m^j$, or, (b) For each subset U of $(0,1,\dots,k)$ with the property that $v_p(\alpha_i)-v_p(\alpha_j)<\varepsilon_m^i-\varepsilon_m^j\ \forall i,j\in U$ then there exists $s\notin U$ such that $v_p(\alpha_s)-v_p(\alpha_j)=\varepsilon_m^s-\varepsilon_m^j\ \forall j\in U$.

In either case, let $0\leqslant i_0\leqslant k$ be such that $v_p(\frac{\alpha_{i_0}}{p^{\varepsilon_m^{i_0}}})=v_p(\alpha_{i_0})-\varepsilon_m^{i_0}=\min(v_p(\alpha_i)-\varepsilon_m^i)$ Then we have strict inequality; i.e. $v_p(\alpha_{i_0})-\varepsilon_m^{i_0}<v_p(\alpha_i)-\varepsilon_m^i\ \forall i\neq i_0$. If $v_p(\alpha_{i_0})-\varepsilon_m^{i_0}\geqslant 0$ then $v_p(\alpha_i)-\varepsilon_m^i\geqslant 0$ $(0\leqslant i\leqslant k)$ and all the terms of Congruence 1 would be integral which is a contradiction. Hence $v_p(\alpha_{i_0})-\varepsilon_m^{i_0}<0$. Thus, Congruence 1 does not hold which is a contradiction. $\square$

2.9. Lemma. *Let* $\alpha=(\alpha_0,\dots,\alpha_k)$ *be an admissible sequence such that all the terms of Congruence* 1 *with respect to* $(m_{j_1},\dots,m_{j_s})$ *be integral and* $(m_{i_1},\dots,m_{i_r})$ *be the remaining elements of* $\mathcal{M}_k$ *with respect to which not all the terms of Congruence* 1 *are integral. Let* U_t *be the set defined in Lemma* 2.8 *for each* m_{i_t} $(1\leqslant t\leqslant r)$ *and let* $V_t=(0,1,\dots,k)-U_t$ *be its complement. Let* $k_t=\sup U_t$ *and* $l_t=\inf U_t$. *Then* $\{0\leqslant i\leqslant k,\ i>l_t\}\subseteq V_{t+1}$ *and* $U_{t+1}\subseteq\{i;\ 0\leqslant i\leqslant k,\ i\leqslant l_t\}$. *Hence either* $l_t=k_{t+1}$, *or,* $l_t\in V_{t+1}$ $(1\leqslant t\leqslant r-1)$.

Proof. Let $i \in U_{t+1}$ for $i > l_t$. Then either $i \in U_t$ and $v_p(\alpha_i) - v_p(\alpha_{l_t}) = \varepsilon_{i_t}^{i} - \varepsilon_{i_t}^{l_t}$, or, $i \in V_t$ and $v_p(\alpha_i) - v_p(\alpha_{l_t}) > \varepsilon_{i_t}^{i} - \varepsilon_{i_t}^{l_t}$, and hence in either case, $v_p(\alpha_i) - v_p(\alpha_{l_t}) \geqslant \varepsilon_{i_t}^{i} - \varepsilon_{i_t}^{l_t}$. By Lemma 2.2, $\varepsilon_{i_t}^{i} - \varepsilon_{i_t}^{l_t} > \varepsilon_{i_{t+1}}^{i} - \varepsilon_{i_{t+1}}^{l_t}$. Thus, $v_p(\alpha_i) - \varepsilon_{i_{t+1}}^{i} > v_p(\alpha_{l_t}) - \varepsilon_{i_{t+1}}^{l_t}$. By (iii) of Lemma 2.8, $\min_{0 \leqslant j \leqslant k}(v_p(\alpha_j) - \varepsilon_{i_{t+1}}^{j}) = k_{U_{t+1}}$ and hence $i \notin U_{t+1}$, i.e. $i \in V_{t+1}$. □

2.10. Corollary. *Let $\alpha = (\alpha_0, \ldots, \alpha_k)$ be an admissible sequence such that all the terms of Congruence 1 with respect to $(m_{j_1}, \ldots, m_{j_s})$ be integral, $j_1 < j_2 < \cdots < j_s$ (i.e. $v_p(\alpha_i) \geqslant \varepsilon_{j_h}^{i}$, $0 \leqslant i \leqslant k$, $1 \leqslant h \leqslant s$) and $(m_{i_1}, \ldots, m_{i_r})$ are the remaining elements $i_1 < i_2 < \cdots < i_r$, with respect to which not all the terms of Congruence 1 are integral. Then there exists a strictly-decreasing sequence $(l_t)_{0 \leqslant t \leqslant r}$ in the interval $[0, k]$ with $l_0 = k$ and $l_t \geqslant k - i_t$ such that $v_p(\alpha_{l_t}) < \varepsilon_{i_t}^{l_t}$; $\varepsilon_{i_t}^{l_{t-1}} - \varepsilon_{i_t}^{l_t} > 0$ and $\varepsilon_{i_t}^{l_t} > 0$ $(1 \leqslant t \leqslant r)$. If we define*

$$K_\alpha = \max\left(\sum_{t=1}^{r}\left(\varepsilon_{i_t}^{l_{t-1}} - \varepsilon_{i_t}^{l_t}\right); \max_{\substack{1 \leqslant h \leqslant s \\ \varepsilon_{j_h}^{l_{j_h-h}} > 0}} \left(\sum_{t=1}^{j_h-h}\left(\varepsilon_{i_t}^{l_{t-1}} - \varepsilon_{i_t}^{l_t}\right) + \varepsilon_{j_h}^{l_{j_h-h}}\right)\right), \tag{7}$$

then $v_p(\alpha_k) \geqslant K_\alpha$.

Proof. Let U_t be the set defined in Lemma 2.8 and let $l_t = \inf U_t$ $(1 \leqslant t \leqslant r)$. Put $l_0 = k$. It follows from Lemma 2.9 that $\{l_t\}$ forms a strictly-decreasing sequence such that $(\varepsilon_{i_t}^{l_{t-1}} - \varepsilon_{i_t}^{l_t}) > 0$ and $\varepsilon_{i_t}^{l_t} > 0$. Either $k = \sup U_1$ and $v_p(\alpha_k) - v_p(\alpha_{l_1}) = \varepsilon_{i_1}^{k} - \varepsilon_{i_1}^{l_1}$, or, $k \notin U_1$ and $v_p(\alpha_k) - v_p(\alpha_{l_1}) > \varepsilon_{i_1}^{k} - \varepsilon_{i_1}^{l_1}$. In either case,

1. $v_p(\alpha_k) - v_p(\alpha_{l_1}) \geqslant \varepsilon_{i_1}^{k} - \varepsilon_{i_1}^{l_1}$. Similarly, we have
2. $v_p(\alpha_{l_1}) - v_p(\alpha_{l_2}) \geqslant \varepsilon_{i_2}^{l_1} - \varepsilon_{i_2}^{l_2}$

$\vdots$

t. $v_p(\alpha_{l_{t-1}}) - v_p(\alpha_{l_t}) \geqslant \varepsilon_{i_t}^{l_{t-1}} - \varepsilon_{i_t}^{l_t}$. Summing up these inequalities for $1 \leqslant t \leqslant r$ we obtain

$$v_p(\alpha_k) \geqslant \sum_{t=1}^{j_h-h}\left(\varepsilon_{i_t}^{l_{t-1}} - \varepsilon_{i_t}^{l_t}\right). \tag{8}$$

For $1 \leqslant h \leqslant s$, summing up the first $t = j_h - h$ inequalities above together with the inequality; $v_p(\alpha_{l_{j_h-h}}) \geqslant \varepsilon_{j_h}^{l_{j_h-h}}$ we obtain

$$v_p(\alpha_k) \geqslant \sum_{t=1}^{j_h-h}\left(\left(\varepsilon_{i_t}^{l_{t-1}} - \varepsilon_{i_t}^{l_t}\right) + \varepsilon_{j_h}^{l_{j_h-h}}\right). \qquad \square \tag{9}$$

2.11. Lemma. *Let $\alpha = (\alpha_0, \ldots, \alpha_k)$ be an admissible sequence and K_α be as defined in Corollary 2.10. Then $K_\alpha \geqslant u(\Omega)$ for some $\Omega \in \mathcal{H}_k$.*

Proof. Let $\alpha = (\alpha_0, \ldots, \alpha_k)$ be an admissible sequence. Let the elements $(m_{j_1}, \ldots, m_{j_s})$ and $(m_{i_1}, \ldots, m_{i_r})$ of $\mathcal{M}_k$ and the number K_α be defined as in Corollary 2.10. Define $\Omega = \{m_{i_t} \mid (\varepsilon_{i_t}^{k-t+1} - \varepsilon_{i_t}^{k-t}) > 0$ and $\varepsilon_{i_t}^{k-t} > 0\}$. Then $\Omega \in \mathcal{H}_k$. Let Φ be the associated set to Ω as defined in 2.5. Then $\Phi \subseteq (m_{j_1}, \ldots, m_{j_s})$. Let $\Omega = \{m_{i_1}, \ldots, m_{i_{r'}}\}$ and $\Phi = (m_{j_1}, \ldots, m_{j_{s'}})$ where $r' \leqslant r$ and $s' \leqslant s$. Then by repeated application of Lemma 2.2 and for $t = j_h - h$ $(1 \leqslant h \leqslant s')$ we obtain

$$\begin{aligned} K_\alpha &\geqslant \left(\varepsilon_{i_1}^{k} - \varepsilon_{i_1}^{l_1}\right) + \left(\varepsilon_{i_2}^{l_1} - \varepsilon_{i_2}^{l_2}\right) + \cdots + \left(\varepsilon_{i_t}^{l_{t-1}} - \varepsilon_{i_t}^{l_t}\right) + \varepsilon_{j_h}^{l_t} \\ &\geqslant \left(\varepsilon_{i_1}^{k} - \varepsilon_{i_1}^{k-1}\right) + \left(\varepsilon_{i_2}^{k-1} - \varepsilon_{i_2}^{k-2}\right) + \cdots + \left(\varepsilon_{i_t}^{k-t+1} - \varepsilon_{i_t}^{k-t}\right) + \left(\varepsilon_{j_h}^{k-t} - \varepsilon_{j_h}^{l_t}\right) + \varepsilon_{j_h}^{l_t} \\ &= \left(\varepsilon_{i_1}^{k} - \varepsilon_{i_1}^{k-1}\right) + \left(\varepsilon_{i_2}^{k-1} - \varepsilon_{i_2}^{k-2}\right) + \cdots + \left(\varepsilon_{i_t}^{k-t+1} - \varepsilon_{i_t}^{k-t}\right) + +\varepsilon_{j_h}^{l_t}. \end{aligned}$$

Similarly, $K_\alpha \geqslant \sum_{t=1}^{r'}(\varepsilon_{i_t}^{k-t+1} - \varepsilon_{i_t}^{k-t})$. Hence $K_\alpha \geqslant u(\Omega)$. □

2.12. Proposition. *Let $\alpha = (\alpha_0, \ldots, \alpha_k)$ be an admissible sequence. Then $v_p(\alpha_k) \geqslant u_k$.*

Proof. $v_p(\alpha_k) \geqslant u_\alpha$ by Corollary 2.10. $K_\alpha \geqslant u(\Omega)$ for some $\Omega \in K$ by Lemma 2.11 and $u(\Omega) \geqslant u_k$ by Proposition 2.6. □

We shall prove (e.g. Proposition 2.18) that there exists an admissible sequence $\alpha = (\alpha_0, \ldots, \alpha_k)$ with $\alpha_k = p^{u_k}$ and we need the Congruence Theorem for this purpose. The Congruence Theorem is an original contribution of this paper.

2.13. Remark. Consider the system of congruences,

$$\frac{\beta_k}{p^{\varepsilon_i^k}} + \frac{\beta_{k-1}}{p^{\varepsilon_i^{k-1}}} + \cdots + \frac{\beta_0}{p^{\varepsilon_i^0}} \in \mathbb{Z} \quad (1 \leqslant i \leqslant d).$$

If we define $h = \max(\varepsilon_i^j\colon\ 0 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant d-1)$ then any simultaneous solution $\beta = (\beta_0, \ldots, \beta_k)$ is determined in $\beta \in (\mathbb{Z}_{p^h})^{k+1}$ (i.e. $\beta_j \in \mathbb{Z}_{p^h}, 0 \leqslant j \leqslant k$).

2.14. Theorem *(Congruence Theorem). Let p be a prime and $\varepsilon_i^j \in \mathbb{Z}$ $(0 \leqslant j \leqslant k, 1 \leqslant i \leqslant d)$ and $\varepsilon_i^k < \cdots < \varepsilon_i^{k-i+1} = \varepsilon_i^{k-i} > \varepsilon_i^{k-i-1} > \cdots > \varepsilon_i^0$. Let $h = \max(\varepsilon_i^j\colon\ 0 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant d)$ and $\beta \in \mathbb{Z}_{p^h}$. Then there exists a simultaneous solution $\beta = (\beta_0, \ldots, \beta_k) \in (\mathbb{Z}_{p^h})^{k+1}$ of the system of congruences*:

$$\frac{\beta_k}{p^{\varepsilon_i^k}} + \frac{\beta_{k-1}}{p^{\varepsilon_i^{k-1}}} + \cdots + \frac{\beta_0}{p^{\varepsilon_i^0}} \in \mathbb{Z} \quad (1 \leqslant i \leqslant d) \text{ with } \beta_k = \beta.$$

If β is a unit in $\mathbb{Z}_{p^h}$ so are β_j $(0 \leqslant j \leqslant k)$.

Proof. Define $d_i^j = (\varepsilon_i^j - \varepsilon_i^{j-1}) \geqslant 1$ $(0 \leqslant j \leqslant i-1)$ and $\bar{d}_i^j = (\varepsilon_i^j - \varepsilon_i^{j+1}) \geqslant 1$ $(i \leqslant j \leqslant k)$. Let $1 \leqslant r \leqslant h$. We shall show by induction on r that the following system of congruences have a unique simultaneous solution in $(\mathbb{Z}_{p^r})^{k+1}$ with $\beta_k = \beta$.

$$\begin{aligned}
0.&\ \ \beta_i + \beta_{i-1} \equiv p^{\bar{d}_i^i}\bar{\gamma}_i^i + p^{d_i^{i-1}}\gamma_i^{i-1} \pmod{p^r},\\
1.&\ \ \gamma_i^{i-1} + \beta_{i-2} \equiv p^{d_i^{i-2}}\gamma_i^{i-2} \pmod{p^r},\\
&\ \ \vdots\\
(i-2).&\ \ \gamma_i^2 + \beta_1 \equiv p^{d_i^1}\gamma_i^1 \pmod{p^r},\\
(i-1).&\ \ \gamma_i^1 + \beta_0 \equiv 0 \pmod{p^r},\\
\bar{i}.&\ \ \bar{\gamma}_i^i + \beta_{i+1} \equiv p^{\bar{d}_i^{i+1}}\bar{\gamma}_i^{i+1} \pmod{p^r},\\
\overline{i+1}.&\ \ \bar{\gamma}_i^{i+1} + \beta_{i+2} \equiv p^{\bar{d}_i^{i+2}}\bar{\gamma}_i^{i+2} \pmod{p^r},\\
&\ \ \vdots\\
\overline{(k-1)}.&\ \ \bar{\gamma}_i^{k-1} + \beta_k \equiv 0 \pmod{p^r}.
\end{aligned}$$

For $r = 1$, $\beta_i + \beta_{i-1} \equiv p^{\bar{d}_i^i}\bar{\gamma}_i^i + p^{d_i^{j-1}}\gamma_i^{i-1} \equiv 0 \pmod{p}$, i.e. $\beta_i \equiv -\beta_{i-1} \pmod{p}$ $(1 \leqslant j \leqslant k-1)$. Let $\beta_k = \beta$ and this determines $\beta_j \equiv (-1)^{k-j}\beta \pmod{p}$. $\gamma_i^j + \beta_{j-1} \equiv p^{d_i^{j-1}}\gamma_i^{j-1} \equiv 0 \pmod{p}$ and thus $\gamma_i^j \equiv (-1)^{k-j}\beta \pmod{p}$ $(j \leqslant i-1)$ and, similarly, $\bar{\gamma}_i^j \equiv (-1)^{k-j}\beta \pmod{p}$ $(i \leqslant j \leqslant k-1)$. Let $r > 1$ and assume the induction-hypothesis for $(r-1)$. $\beta_i + \beta_{i-1} \equiv p^{\bar{d}_i^i}\bar{\gamma}_i^i + p^{d_i^{i-1}}\gamma_i^{i-1} \pmod{p^r}$ $(\bar{d}_i^i, d_i^{i-1} \geqslant 1)$ where $\bar{\gamma}_i^i, \gamma_i^{i-1}$ are the unique solutions mod p^{r-1}. Hence $\beta_i + \beta_{i-1}$ is uniquely determined mod p^r $(1 \leqslant i \leqslant k)$. From this and the fact that $\beta_k = \beta$, all the β_i $(0 \leqslant i \leqslant k)$ are uniquely determined in $\mathbb{Z}_{p^r}$. From the equation, $\gamma_i^j + \beta_{j-1} \equiv p^{d_i^{j-1}}\gamma_i^{j-1} \pmod{p^r}$ and the fact that γ_i^{j-1} is determined mod p^{r-1}, $d_i^{j-1} > 1$, β_{j-1} is determined mod p^r, the variables γ_i^j are uniquely determined mod p^r $(2 \leqslant j \leqslant i-1)$. γ_i^1 is uniquely determined in $\mathbb{Z}_{p^r}$ from the equation, $\gamma_i^1 + \beta_0 \equiv 0 \pmod{p^r}$, $\bar{\gamma}_i^j$ is uniquely determined in $\mathbb{Z}_{p^r}$ $(i \leqslant j \leqslant k-1)$ from the equation, $\bar{\gamma}_i^j + \beta_{j+1} \equiv p^{\bar{d}_i^{j+1}}\bar{\gamma}_i^{j+1} \pmod{p^r}$ and the fact that the class of $\bar{\gamma}_i^{j+1}$ is determined in $\mathbb{Z}_{p^{r-1}}$, $\bar{d}_i^{j+1} > 1$, and that β_{j+1} is determined in $\mathbb{Z}_{p^r}$. The variable $\bar{\gamma}_i^{k-1}$ is determined uniquely in $\mathbb{Z}_{p^r}$ from the equation, $\bar{\gamma}_i^{k-1} + \beta_k \equiv 0 \pmod{p^r}$. Hence all the variables are uniquely determined in $\mathbb{Z}_{p^r}$. Let $\gamma_i^j \equiv x_i^j \pmod{p^{r-1}}$ and $\bar{\gamma}_i^j \equiv \bar{x}_i^j \pmod{p^{r-1}}$. Then $\gamma_i^j = x_i^j + k_i^j p^{r-1}$. The equations $0, 1, \dots, i-2, \bar{i}, \overline{i+1}, \dots, (\overline{k-1})$ are determined with $\gamma_i^j = x_i^j$ and $\bar{\gamma}_i^j = \bar{x}_i^j$ on the RHS of the congruence and if we now put γ_i^j and $\bar{\gamma}_i^j$ instead of x_i^j and $\bar{x}_i^j$, the RHS of the congruences differ by elements of the form $k_i^j p^{d_i^j} p^{r-1}$, or, $\bar{k}_i^j p^{\bar{d}_i^j} p^{r-1}$ which are congruent to 0 (mod p^r). Hence the uniquely determined variables satisfy the given system of congruences in $\mathbb{Z}_{p^r}$. The variables $(\beta_0, \dots, \beta_k)$ of the system of congruences for $r = h$ is a solution of the original system of congruences with $\beta_k = \beta$. $\square$

2.15. Remark. $\alpha = (\alpha_0, \dots, \alpha_k)$ is called admissible with respect to a subset S of $\mathcal{M}_k$ if and only if Congruence 1 is satisfied for all $m \in S$.

2.16. Proposition. *Let $\Omega = \{m_{i_1}, \dots, m_{i_r}\} \in \mathcal{H}_k$ $(1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant d)$ and $\Phi = \{m_{j_1}, \dots, m_{j_s}\}$ $(1 \leqslant j_1 < j_2 < \cdots < j_s)$, be its associated set as in Definition 2.5. Then there exists an admissible sequence $\alpha = (\alpha_0, \dots, \alpha_k)$ with respect to the set $\Omega \cup (\mathcal{M}_k - (\Omega \cup \Phi))$, with $\alpha_k = p^{\sum_{t=1}^{r} (\varepsilon_{i_t}^{k-t+1} - \varepsilon_{i_t}^{k-t})}$.*

Proof. Let $\mathcal{M}_k = \{m_1, \dots, m_d\}$, $\Omega \cup \Phi = \{m_1, \dots, m_l\}$, $l \leqslant d$. Suppose $(\varepsilon_{l+j}^{k-r-j+1} - \varepsilon_{l+j}^{k-r-j}) \leqslant 0$ $(1 \leqslant j \leqslant e)$ and $(\varepsilon_{l+j}^{k-r-j+1} - \varepsilon_{l}^{k-r-j})$ is undefined for $j > e$. Let $\alpha_{k-t+1} = p^{\sum_{u=t}^{r} (\varepsilon_{i_u}^{k-u+1} - \varepsilon_{i_u}^{k-t})} \beta_{k-t+1}$ $(1 \leqslant t \leqslant r)$, $\alpha_{k-r} = \beta_{k-r}$, $\alpha_{k-r-i} = p^{\sum_{j=1}^{i} (\varepsilon_{l+j}^{k-r-j} - \varepsilon_{l+j}^{k-r-j+1})} \beta_{k-r-i}$ $(1 \leqslant i \leqslant e)$ and $\alpha_j = \beta_j = 0$ $(k - r - e < j \leqslant k)$. Let $i_{r+j} = l + j$ $(1 \leqslant j \leqslant e)$. Then Congruence 1; i.e. $\sum_{j=0}^{k} \frac{\alpha_j}{p^{\varepsilon_m^j}} \in \mathbb{Z}$ $(m \in \Omega \cup (\mathcal{M}_k - (\Omega \cup \Phi)))$ can be written down as Congruence 2; i.e.

$$\frac{\beta_k}{p^{\delta_{i_t}^k}} + \frac{\beta_{k-1}}{p^{\delta_{i_t}^{k-1}}} + \cdots + \frac{\beta_{k-r-e}}{p^{\delta_{i_t}^{k-r-e}}} \in \mathbb{Z} \quad (1 \leqslant t \leqslant r + e). \tag{10}$$

We claim the

Statement. $\delta_{i_t}^k < \delta_{i_t}^{k-1} < \cdots < \delta_{i_t}^{k-t+1} = \delta_{i_t}^{k-t} > \delta_{i_t}^{k-t+1} > \cdots > \delta_{i_t}^{k-r-e}$ $(1 \leqslant t \leqslant r + e)$.

Proof. (i) For $1 \leqslant j \leqslant t \leqslant r$,

$$\begin{aligned}
\delta_{i_t}^{k-j+1} - \delta_{i_t}^{k-(j-1)+1} &= \delta_{i_t}^{k-j+1} - \delta_{i_t}^{k-j+2} \\
&= \left[\varepsilon_{i_t}^{k-j+1} - \sum_{s=j}^{r} \left(\varepsilon_{i_s}^{k-s+1} - \varepsilon_{i_s}^{k-s}\right)\right] - \left[\varepsilon_{i_t}^{k-j+2} - \sum_{s=j-1}^{r} \left(\varepsilon_{i_s}^{k-s+1} - \varepsilon_{i_s}^{k-s}\right)\right] \\
&= \left(\varepsilon_{i_{j-1}}^{k-j+2} - \varepsilon_{i_{j-1}}^{k-j+1}\right) - \left(\varepsilon_{i_t}^{k-j+2} - \varepsilon_{i_t}^{k-j+1}\right) > 0 \quad \text{by Lemma 2.2}
\end{aligned}$$

$$\text{i.e. } \delta_{i_t}^{k-j+1} > \delta_{i_t}^{k-(j-1)+1}. \tag{11}$$

(ii) For $1 \leqslant t \leqslant r$,

$$\begin{aligned}
\delta_{i_t}^{k-t+1} - \delta_{i_t}^{k-t} &= \left[\varepsilon_{i_t}^{k-t+1} - \sum_{s=t}^{r} \left(\varepsilon_{i_s}^{k-s+1} - \varepsilon_{i_s}^{k-s}\right)\right] - \left[\varepsilon_{i_t}^{k-t} - \sum_{s=t+1}^{r} \left(\varepsilon_{i_s}^{k-s+1} - \varepsilon_{i_s}^{k-s}\right)\right] \\
&= \left(\varepsilon_{i_t}^{k-t+1} - \varepsilon_{i_t}^{k-t}\right) - \left(\varepsilon_{i_t}^{k-t+1} - \varepsilon_{i_t}^{k-t}\right) = 0
\end{aligned}$$

$$\text{i.e. } \delta_{i_t}^{k-t+1} = \delta_{i_t}^{k-t}. \tag{12}$$

(iii) For $1 \leqslant t \leqslant r$, $t < j \leqslant r - 1$,

$$\begin{aligned}\delta_{i_t}^{k-j+1}-\delta_{i_t}^{k-(j+1)+1}&=\delta_{i_t}^{k-j+1}-\delta_{i_t}^{k-j}\\&=\left[\varepsilon_{i_t}^{k-j+1}-\sum_{s=j}^{r}\left(\varepsilon_{i_s}^{k-s+1}-\varepsilon_{i_s}^{k-s}\right)\right]-\left[\varepsilon_{i_t}^{k-j}-\sum_{s=j+1}^{r}\left(\varepsilon_{i_s}^{k-s+1}-\varepsilon_{i_s}^{k-s}\right)\right]\\&=\left(\varepsilon_{i_t}^{k-j+1}-\varepsilon_{i_t}^{k-j}\right)-\left(\varepsilon_{i_j}^{k-j+1}-\varepsilon_{i_j}^{k-j}\right)>0\quad\text{by Lemma 2.2,}\end{aligned}$$

i.e. $\delta_{i_t}^{k-j+1}>\delta_{i_t}^{k-(j+1)+1}$. (13)

(iv) For $1\leqslant t\leqslant r$, $0\leqslant j\leqslant e-1$,

$$\begin{aligned}\delta_{i_t}^{k-r-j+1}-\delta_{i_t}^{k-r-(j+1)+1}&=\delta_{i_t}^{k-r-j+1}-\delta_{i_t}^{k-r-j}\\&=\left[\varepsilon_{i_t}^{k-r-j+1}-\sum_{s=1}^{j-1}\left(\varepsilon_{l+s}^{k-r-s}-\varepsilon_{l+s}^{k-r-s+1}\right)\right]-\left[\varepsilon_{i_t}^{k-r-j}-\sum_{s=1}^{j}\left(\varepsilon_{l+s}^{k-r-s}-\varepsilon_{l+s}^{k-r-s+1}\right)\right]\\&=\left(\varepsilon_{i_t}^{k-r-j+1}-\varepsilon_{i_t}^{k-r-j}\right)-\left(\varepsilon_{l+j}^{k-r-j+1}-\varepsilon_{l+j}^{k-r-j}\right)>0\quad\text{by Lemma 2.2.}\end{aligned}\tag{14}$$

(v) For $1\leqslant t\leqslant r$, $\delta_{l+j}^{k-t+1}>\delta_{l+j}^{k-t}$ and the proof is similar to that of (i).
(vi) For $1\leqslant j\leqslant i\leqslant e$,

$$\begin{aligned}\delta_{l+i}^{k-l-j+1}-\delta_{l+i}^{k-l-(j-1)+1}&=\delta_{l+i}^{k-l-j+1}-\delta_{l+i}^{k-l-j+2}\\&=\left[\varepsilon_{l+i}^{k-l-j+1}-\sum_{s=1}^{j-1}\left(\varepsilon_{l+s}^{k-r-s}-\varepsilon_{l+s}^{k-r-s+1}\right)\right]-\left[\varepsilon_{l+i}^{k-r-j+2}-\sum_{s=1}^{j-2}\left(\varepsilon_{l+s}^{k-r-s}-\varepsilon_{l+s}^{k-r-s+1}\right)\right]\\&=\left(\varepsilon_{l+j-1}^{k-r-j+2}-\varepsilon_{l+j-1}^{k-r-j+1}\right)-\left(\varepsilon_{l+j}^{k-r-j+2}-\varepsilon_{l+j}^{k-r-j+1}\right)>0\quad\text{by Lemma 2.2.}\end{aligned}\tag{15}$$

(vii) For $0\leqslant i\leqslant e-1$,

$$\begin{aligned}\delta_{l+i}^{k-r-i+1}-\delta_{l+i}^{k-r-i}&=\left[\varepsilon_{l+i}^{k-r-i+1}-\sum_{s=1}^{i-1}\left(\varepsilon_{l+s}^{k-r-s}-\varepsilon_{l+s}^{k-r-s+1}\right)\right]-\left[\varepsilon_{l+i}^{k-r-i}-\sum_{s=1}^{i}\left(\varepsilon_{l+s}^{k-r-s}-\varepsilon_{l+s}^{k-r-s+1}\right)\right]\\&=\left(\varepsilon_{l+i}^{k-r-i+1}-\varepsilon_{l+i}^{k-r-i}\right)-\left(\varepsilon_{l+i}^{k-r-i+1}-\varepsilon_{l+i}^{k-r-i}\right)=0.\end{aligned}\tag{16}$$

(viii) For $1\leqslant i<j\leqslant e$,

$$\begin{aligned}\delta_{l+i}^{k-r-j+1}-\delta_{l+i}^{k-r-(j+1)+1}&=\delta_{l+i}^{k-r-j+1}-\delta_{l+i}^{k-r-j}\\&=\left[\varepsilon_{l+i}^{k-r-j+1}-\sum_{s=1}^{j-1}\left(\varepsilon_{l+s}^{k-r-s}-\varepsilon_{l+s}^{k-r-s+1}\right)\right]-\left[\varepsilon_{l+i}^{k-r-j}-\sum_{s=1}^{j}\left(\varepsilon_{l+s}^{k-r-s}-\varepsilon_{l+s}^{k-r-s+1}\right)\right]\\&=\left(\varepsilon_{l+i}^{k-r-j+1}-\varepsilon_{l+i}^{k-r-j}\right)-\left(\varepsilon_{l+j}^{k-r-j+1}-\varepsilon_{l+j}^{k-r-j}\right)>0\quad\text{by Lemma 2.2.}\qquad\square\end{aligned}\tag{17}$$

Hence Congruence 2 satisfies the hypothesis of the Congruence Theorem and we deduce from the Congruence Theorem that there exists a solution $\beta=(\beta_0,\ldots,\beta_k)$ with $\beta_k=1$. Thus, Congruence 1 admits a solution $\alpha=(\alpha_0,\ldots,\alpha_k)$ with respect to $m\in\Omega\cup(\mathcal{M}_k-(\Omega\cup\Phi))$ with $\alpha_k=p^{\sum_{t=1}^{r}(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t})}$ which by definition is an admissible sequence with respect to $\Omega\cup(\mathcal{M}_k-(\Omega\cup\Phi))$. $\square$

2.17. Proposition. *Let $\Omega=(m_{i_1},\ldots,m_{i_r})\in\mathcal{H}_k$, $i_1<i_2<\cdots<i_r$. Then there exists an admissible sequence $\alpha=(\alpha_0,\ldots,\alpha_k)$ with $\alpha_k=p^{u(\Omega)}$.*

Proof. Let $\Phi=(m_{j_1},\ldots,m_{j_s})$ be the associated set to Ω so that $\Omega\cup\Phi=(m_1,m_2,\ldots,m_l)$.

By Proposition 2.16 there exists an admissible sequence $\alpha'=(\alpha'_0,\ldots,\alpha'_k)$ with respect to the set $\Omega\cup\{m_{l+1},\ldots,m_d\}$ such that $\alpha'_k=p^{\sum_{t=1}^{r}(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t})}$. By definition

$$u(\Omega)=\max\left(\sum_{t=1}^{r}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right);\ \max_{\substack{1\leqslant h\leqslant s\\ \varepsilon_{j_h}^{k-j_h-h}>0}}\left(\sum_{t=1}^{r}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right)+\varepsilon_{j_h}^{k-j_h-h}\right)\right)$$
$$=\lambda+\sum_{t=1}^{r}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right)\quad\text{for } \lambda\geqslant 0.$$

Define $\alpha=(\alpha_0,\dots,\alpha_k)$ by $\alpha_j=p^{\lambda}\alpha'_j$ $(k-r\leqslant j\leqslant k)$. Then since Congruence 1 is homogeneous with respect to the variables α_j, it follows that α is also admissible with respect to the set $\Omega\cup\{m_{l+1},\dots,m_d\}$, If we substitute

$$\alpha_{k-t+1}=p^{\sum_{u=t}^{r}(\varepsilon_{i_u}^{k-u+1}-\varepsilon_{i_u}^{k-u})}\beta_{k-t+1}\quad(1\leqslant t\leqslant r),\quad \alpha_{k-r}=\beta_{k-r},$$
$$\alpha_{k-r-i}=p^{\sum_{j=1}^{i}(\varepsilon_{l+j}^{k-r-j}-\varepsilon_{l+j}^{k-r-j+1})}\beta_{k-r-i}\quad(1\leqslant i\leqslant e)\tag{18}$$

and $\alpha_j=\beta_j=0$ $(k-r-e<j\leqslant k)$ as in the proof of Proposition 2.16, Congruence 1 takes the form 2. $\sum_{j=0}^{k}\frac{\beta_j}{p^{\delta_m^j}}\in\mathbb{Z}$. Then by precisely the same arguments as used in the proof of Proposition 2.16, we can establish the inequalities:

$$\delta_{j_h}^{k}<\delta_{j_h}^{k-1}<\cdots<\delta_{j_h}^{k-j_h+h}>\delta_{j_h}^{k-j_h+h-1}>\cdots>\delta_{j_h}^{0}\quad(1\leqslant h\leqslant s)$$
$$\delta_{j_h}^{k-j_h+h}=\varepsilon_{j_h}^{k-j_h+h}-\lambda-\sum_{t=j_h-h+1}^{r}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right)$$
$$=\varepsilon_{j_h}^{k-j_h+h}-\left(\lambda+\sum_{t=1}^{r}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right)\right)+\sum_{t=1}^{j_h-h}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right)$$
$$=\varepsilon_{j_h}^{k-j_h+h}-u(\Omega)+\sum_{t=1}^{j_h-h}\left(\varepsilon_{i_t}^{k-t+1}-\varepsilon_{i_t}^{k-t}\right)\leqslant 0\tag{19}$$

by definition of $u(\Omega)$. Thus, $\delta_{j_h}^{i}<\delta_{j_h}^{k-j_h+h}\leqslant 0$ for $i\neq k-j_h+h$, by the above inequality. Hence all terms of Congruence 2 and hence of Congruence 1 with respect to m_{j_h} are integral $(1\leqslant h\leqslant s)$. It follows that α is an admissible sequence with $\alpha_k=p^{u(\Omega)}$. $\square$

2.18. Proposition. *There exists an admissible sequence $\alpha=(\alpha_0,\dots,\alpha_k)$ with $\alpha_k=p^{u(\Omega)}$.*

Proof. Take $\Omega=\Omega^0$ in Proposition 2.17. $\square$

2.19. Lemma. *Let $m\in\mathcal{M}$, $m=\sigma(t_k)$, $t_k=p^{\nu}\Delta$ $(\nu\geqslant 1,\ (\Delta,p))=1$, $i=k+\nu$. Then for $j\leqslant k+1$, $p^{i-j}\Delta+i-j<t_{j-1}$.*

Proof. $t_{j-1}-(p^{i-j}\Delta+i-j)\geqslant p^{i+j+1}\Delta+\frac{p^{k-j+1}\Delta-1}{(p-1)}-(p^{i-j}\Delta+i-j)$ by [1, Lemma 4.4.6] $\geqslant p^{i-j+1}\Delta+k-j+1-p^{i-j}\Delta-i+j=(p-1)p^{i-j}\Delta-\nu+1\geqslant(p-1)p^{i-k-1}-(\nu-1)=(p-1)p^{\nu-1}\Delta-(\nu-1)>0$. $\square$

2.20. Corollary. *For $j<k$, $t_k+u_k<t_j$.*

Proof. Since t_j is a strictly-decreasing sequence, it suffices to prove that $t_k+u_k<t_{k-1}$. Let $\Phi=\{m_1,m_2,\dots,m_l\}$ be the associated set to the empty-set $\phi\in\mathcal{H}_k$. Then $(\varepsilon_j^k-\varepsilon_j^{k-1})>0$; i.e. if $m_j=\sigma(t_{k_j})$ then $k\leqslant k_j$. Let $t_{k_j}=p^{\nu_j}\Delta_j$ $(\nu_j\geqslant 1,\ (\Delta_j,\ p)=1)$, $i_j=k_j+\nu_j$, $m_j=\frac{1}{2}(p-1)p^{i_j}\Delta_j$ $(1\leqslant j\leqslant l)$. Then $\varepsilon_j^k=p^{i_j-k}\Delta_j+i_j-k-t_k<t_{k-1}-t_k$ by Lemma 2.19. Thus, $u_k\leqslant u(\phi)=\max_{1\leqslant j\leqslant l}\varepsilon_j^k<t_{k-1}-t_k$ i.e. $t_k+u_k<t_{k-1}$. $\square$

2.21. Theorem. $J_p(P_n(\mathbb{C}))=\bigoplus_{k=0}^{r_n}\mathbb{Z}_{p^{t_k+u_k}}$. *The order of the first summand generated by ω is the p-component, $M_{n+1,p}$ of the Atiyah–Todd number M_{n+1}.*

Proof. By Proposition 2.18, for each $0 \leqslant k \leqslant r_n$, there exists an admissible sequence $\alpha = (\alpha_0, \dots, \alpha_k)$ with $\alpha_k = p^{u_k}$. It follows from [1, Proposition 6.2.14] that there exists in $J_p(P_n(\mathbb{C}))$ a relation, $p^{t_0}\alpha_0\omega + p^{t_1}\alpha_1\psi_{\mathbb{R}}^{p}(\omega) + \cdots + p^{t_k+u_k}\psi_{\mathbb{R}}^{p^k}(\omega) = 0$ By Corollary 2.20, $t_j > t_k + u_k$ $(0 \leqslant j \leqslant k-1)$ and hence if we let $x_k = p^{t_0-(t_k+u_k)}\alpha_0\omega + p^{t_1-(t_k+u_k)}\alpha_1\psi_{\mathbb{R}}^{p}(\omega) + \cdots + p^{t_{k-1}-(t_k+u_k)}\alpha_{k-1}\psi_{\mathbb{R}}^{p^{k-1}}(\omega) + \psi_{\mathbb{R}}^{p^k}(\omega)$ then the above relation can be written down as: $p^{t_k+u_k}x_k = 0$ $(0 \leqslant k \leqslant r_n)$. Since $\{\omega, \psi_{\mathbb{R}}^{p}(\omega), \dots, \psi_{\mathbb{R}}^{p^{r_n}}(\omega)\}$ spans $J_p(P_n(\mathbb{C}))$ and that the coefficient of $\psi_{\mathbb{R}}^{p^k}(\omega)$ in the expansion of x_k is 1, it follows that $\{x_0, x_1, \dots, x_{r_n}\}$ spans $J_p(P_n(\mathbb{C}))$. Suppose $\beta_0x_0 + \cdots + \beta_{r_n}x_{r_n} = 0$. We claim the following statement:

Statement. *If* $\beta_0x_0 + \cdots + \beta_kx_k = 0$ $(1 \leqslant k \leqslant r_n)$ *then* $\beta_kx_k = 0$ *and* $\beta_0x_0 + \cdots + \beta_{k-1}x_{k-1} = 0$.

Proof. Substituting for x_j in terms of $\psi_{\mathbb{R}}^{p^i}(\omega)$ we obtain a relation, $\alpha_0\omega + \alpha_1\psi_{\mathbb{R}}^{p}(\omega) + \cdots + \alpha_{k-1}\psi_{\mathbb{R}}^{p^{k-1}}(\omega) + \beta_k\psi_{\mathbb{R}}^{p^k}(\omega) = 0$. By [1, Proposition 6.2.14], $\alpha_j = p^{t_j}\alpha'_j$ $(0 \leqslant j \leqslant k-1)$ and $\beta_k = p^{t_k}\alpha'_k$ where $\alpha' = (\alpha'_0, \dots, \alpha'_k)$ is an admissible sequence. By Proposition 2.12, $p^{u_k} | \alpha'_k$ and thus $p^{t_k+u_k} | \beta_k$. Hence $\beta_kx_k = 0$ and thus, $\beta_0x_0 + \cdots + \beta_{k-1}x_{k-1} = 0$, proving the statement. □

It follows from the statement by induction on k starting with $k = r_n$ that $\beta_0x_0 = \beta_1x_1 = \cdots = \beta_{r_n}x_{r_n} = 0$. This proves the desired primary decomposition. As for the second part of the theorem, $t_0 = [\frac{n}{p-1}]$, $\mathcal{H}_0 = \{\phi\}$. $\Phi = \mathcal{M}_0$. $m \in \mathcal{M}_0$ is of the form $m = \frac{1}{2}(p-1)p^i\Delta$, $(\Delta, p) = 1$. $\varepsilon_m^0 = p^i\Delta + i - t_0 > 0$. Let $r_m = p^i\Delta = \frac{2m}{(p-1)} \leqslant [\frac{n}{p-1}]$. Then $r_m + v_p(r_m) - t_0 > 0$.

$$\begin{aligned}
u_0 &= \max_{m \in \mathcal{M}_0}\bigl[r_m + v_p(r_m) - t_0\bigr] = \max\left[r + v_p(r) - t_0\colon\ 1 \leqslant r \leqslant \left[\frac{n}{p-1}\right],\ r + v_p(r) \geqslant \left[\frac{n}{p-1}\right]\right],\\
t_0 + u_0 &= \max\left[r + v_p(r)\colon\ 1 \leqslant r \leqslant \left[\frac{n}{p-1}\right],\ r + v_p(r) \geqslant \left[\frac{n}{p-1}\right]\right]\\
&= \max\left[r + v_p(r)\colon\ 1 \leqslant r \leqslant \left[\frac{n}{p-1}\right]\right] = v_p(M_{n+1}). \qquad \square
\end{aligned}$$

2.22. Remark. The second part of the theorem is the solution of the complex analogue of the vector field problem and the simplest proof so far has been provided.

As a corollary to Theorem 2.21, we recover [1, Proposition 6.2.12], i.e.

2.23. Corollary. *Let* $n = p^k(p-1) + r'$ $(0 \leqslant r' \leqslant p-2)$. *Then*

$$J_p\bigl(P_n(\mathbb{C})\bigr) = \mathbb{Z}_{p^{(p^k+k)}} \oplus \mathbb{Z}_{p^{(p^{k-1}-1)}} \oplus \mathbb{Z}_{p^{(p^{k-2}-1)}} \oplus \cdots \oplus \mathbb{Z}_{p^{p-1}}.$$

Proof. $m_i = m = \{m\}$, $m = \frac{1}{2}(p-1)p^k$ $(1 \leqslant i \leqslant k)$ $\varepsilon_m^j = p^{k-j} - 1$ $(1 \leqslant j \leqslant i)$, i.e. $\varepsilon_m^i < \varepsilon_m^{i-1} < \cdots < \varepsilon_m^0$. Thus, $\mathcal{H}_i = \{\phi\}$ $(1 \leqslant i \leqslant k)$ and the corresponding set Φ to ϕ is empty. Thus, $u_i = 0$ $(1 \leqslant i \leqslant k)$ and hence the ith-summand has order $p^{t_i} = p^{p^{k-i}-1}$ $(1 \leqslant i \leqslant k)$. The order of the first summand follows from the definition of the Atiyah–Todd number. □

2.24. Example. As a demonstration we wrote down in [1, Example 6.2.13] the α- and β-relations for $J_2(P_{164}(\mathbb{C}))$. We now obtain the primary decomposition of $J_2(P_{164}(\mathbb{C}))$.

t_j:	{164	81	40	19	9	4	1	0}
ε_{82}^j:	2	2	1					
ε_{80}^j:	1	3	3	3	2	1		
ε_{64}^j:				1	2	2	2	1

$$k=1:\quad \mathcal{M}_1=(80,82),\ \tau=\begin{bmatrix}2&2\\3&1\end{bmatrix},\ \mathcal{H}_1=\{(80),\phi\},\ u((80))=4,\ u(\phi)=3,\ u_1=3;$$

$$k=2:\quad \mathcal{M}_2=(80,82),\ \tau=\begin{bmatrix}1&2&2\\3&3&1\end{bmatrix},\ \mathcal{H}_2=\{\phi\},\ u(\phi)=3,\ u_2=3;$$

$$k=3:\quad \mathcal{M}_3=(64,80,82),\ \tau=\begin{bmatrix}&1&2&2\\3&3&3&1\\1&&&\end{bmatrix},\ \mathcal{H}_3=\{(64)\},\ u((64))=1,\ u_3=1;$$

$$k=4:\quad \mathcal{M}_4=(64,80,82),\ \tau=\begin{bmatrix}&&1&2&2\\2&3&3&3&1\\2&1&&&\end{bmatrix},\ \mathcal{H}_4=\{(64),\phi\},\ u((64))=1,\ u((\phi))=2,\ u_4=1;$$

$$k=5:\quad \mathcal{M}_5=(64,80,82),\ \tau=\begin{bmatrix}&&&1&2&2\\1&2&3&3&3&1\\2&2&1&&&\end{bmatrix},\ \mathcal{H}_5=\{\phi\},\ u(\phi)=0,\ u_5=0;$$

$$k=6:\quad \mathcal{M}_6=(64,80,82),\ \tau=\begin{bmatrix}&&&&1&2&2\\&1&2&3&3&3&1\\2&2&2&1&&&\end{bmatrix},\ \mathcal{H}_6=\{\phi\},\ u(\phi)=0,\ u_6=0;$$

$$k=7:\quad \mathcal{M}_7=(64,80,82),\ \tau=\begin{bmatrix}&&&&&1&2&2\\&&1&2&3&3&3&1\\1&2&2&2&1&&&\end{bmatrix},\ \mathcal{H}_7=\{\phi\},\ u(\phi)=0,\ u_7=0;$$

$M_{165,2}=2^{166}$. According to Theorem 1.15,

$$J_2\big(P_{164}(\mathbb{C})\big)=\mathbb{Z}_{2^{166}}\oplus\mathbb{Z}_{2^{84}}\oplus\mathbb{Z}_{2^{40}}\oplus\mathbb{Z}_{2^{20}}\oplus\mathbb{Z}_{2^{10}}\oplus\mathbb{Z}_{2^{4}}\oplus\mathbb{Z}_{2^{1}}.$$

According to this primary decomposition, $|J_2(P_{164}(\mathbb{C}))|=2^{325}$. We know from [1] that $|J_2(P_{164}(\mathbb{C}))|=2^{\sum_{k=0}^{7}[\frac{164}{2^k}]}=2^{164+82+41+20+10+5+2+1}=2^{325}$. It checks.

3. Primary decomposition of $J(L^n(p^r))$

3.1. Definition. Let $n\in\mathbb{Z}^+$, $r\leqslant r_n$. Then $\mathcal{J}(p,n,r)=\psi_{\mathbb{R}}^{p^r}(J_p(P_n(\mathbb{C})))=$ subgroup of $J_p(P_n(\mathbb{C}))$ generated by $\psi_{\mathbb{R}}^{p^r}(\omega),\psi_{\mathbb{R}}^{p^{r+1}}(\omega),\dots,\psi_{\mathbb{R}}^{p^{rn}}(\omega)$. Let $G(p,n,r)$ be the subgroup of $J(L^n(p^r))$ generated by the powers of ω. Then it follows from [1] that $G(p,n,r)$ is the quotient, $G(p,n,r)=J_p(P_n(\mathbb{C}))/\mathcal{J}(p,n,r)$. For details refers to [1, Definition 5.1.8 and Section 7.1].

We now define reduced index functions $\varepsilon_m^j(r)$ which will play the same role for lens spaces as index functions ε_m^j play for complex projective spaces.

3.2. Definition. Let $n\in\mathbb{Z}^+$, $r\leqslant r_n$ and $m=\frac{1}{2}(p-1)p^i\Delta\in\mathcal{M}$ $((\Delta,p)=1)$. Then $\varepsilon_m^j(r)=p^{i-j}\Delta+\min(i,r-1)-j-t_j$ $(j\leqslant i)$.

3.3. Lemma. *Let $n\in\mathbb{Z}^+$, $r\leqslant r_n$ and $m=\sigma(t_k)=\sigma(p^\nu\Delta)=\frac{1}{2}(p-1)p^i\Delta\in\mathcal{M}$, $(\Delta,p)=1$, $i=k+\nu$. Then*

$$\varepsilon_m^j(r)=\begin{cases}\varepsilon_m^j & \text{if } i<r,\\ \varepsilon_m^j-\varepsilon_m^r & \text{if } k+1<r\leqslant i,\\ r-j & \text{if } k+1<r\leqslant j\leqslant i,\\ \text{non-positive} & \text{if } k+1\geqslant r.\end{cases}$$

Proof. (i) If $i<r$, it follows from its definition that $\varepsilon_m^j(r)=\varepsilon_m^j$.

(ii) If $k+1<r\leqslant i$, $\varepsilon_m^j(r)-[(\varepsilon_m^j-\varepsilon_m^r)]=p^{i-j}\Delta+r-1-j-t_j-(p^{i-j}\Delta+i-j-t_j)+(p^{i-r}\Delta+i-r-t_r)=p^{i-r}\Delta-1-t_r=0$ by [1, Lemma 4.4.3].

(iii) If $k+1\geqslant r$, $\varepsilon_m^j(r)=p^{i-j}\Delta+r-1-t_j\leqslant p^{i-j}\Delta+k-t_j\leqslant 0$ by [1, Lemma 4.4.6]. □

We now state a slight variation of [1, Proposition 5.1.7].

3.4. Proposition. *If $m' > m$ are consecutive elements in $\mathcal{M}$, $m' = \sigma(t_{k'}) = \sigma(p^{\nu'}\Delta')$, $m = \sigma(t_k) = \sigma(p^\nu \Delta)$, $(\Delta, p) = (\Delta', p) = 1$. Let $k' + \nu' \leqslant s \leqslant k$ and $(\alpha_0, \dots, \alpha_s)$ be admissible in $\mathcal{M}^{m'}$ (in the sense of [1, Definition 5.1.3]). Then there exist integers $(\alpha_j)_{j \geqslant k+1}$ such that $(\alpha_0, \dots, \alpha_s, 0 \dots, 0, \alpha_{k+1}, \dots, \alpha_{k+\nu}, \dots, \alpha_{r_n})$ is an admissible sequence.*

Proof. Identical with that of [1, Proposition 5.1.7]. □

3.5. Proposition. *There exists a relation, $\beta_0\omega + \cdots + \beta_s \psi_{\mathbb{R}}^{p^s}(\omega) = 0$ in $G(p,n,r)$ $(s \leqslant r-1)$ iff $\beta_j = p^{t_j}\alpha_j$, $0 \leqslant j \leqslant s$ where $\alpha = (\alpha_0, \dots, \alpha_s)$ is an admissible sequence with respect to $\varepsilon_m^j(r)$.*

Proof. Suppose $\beta_0\omega + \cdots + \beta_s \psi_{\mathbb{R}}^{p^s}(\omega) = 0$ in $G(p,n,r)$. Then $\beta_0\omega + \cdots + \beta_s \psi_{\mathbb{R}}^{p^s}(\omega) = 0$ in $J_p(P_n(\mathbb{C}))$ mod $\mathcal{J}(p,n,r)$; i.e. there exist integers $\beta_r, \beta_{r+1}, \dots, \beta_{r_n}$ such that $\beta_0\omega + \cdots + \beta_s \psi_{\mathbb{R}}^{p^s}(\omega) + \beta_r \psi_{\mathbb{R}}^{p^r}(\omega) + \cdots + \beta_{r_n} \psi_{\mathbb{R}}^{p^{rn}}(\omega) = 0$ in $J_p(P_n(\mathbb{C}))$. By [1, Proposition 6.2.14], $\beta_j = p^{t_j}\alpha_j$ $(0 \leqslant j \leqslant s,\ r \leqslant j \leqslant r_n)$ where $\alpha = (\alpha_0, \dots, \alpha_s, 0, \dots, 0, \alpha_r, \dots, \alpha_{r_n})$ is an admissible sequence with respect to the index functions, ε_m^j. Suppose that $m = \sigma(t_k) = \sigma(p^\nu\Delta) \in \mathcal{M}$

(i) $k + \nu < r$. Then by Lemma 3.3, $\varepsilon_m^j(r) = \varepsilon_m^j$ $(0 \leqslant j \leqslant k+\nu)$ and thus

$$\sum_{j=0}^{\min(s,k+\nu)} \frac{\alpha_j}{p^{\varepsilon_m^j(r)}} = \sum_{j=0}^{\min(s,k+\nu)} \frac{\alpha_j}{p^{\varepsilon_m^j}} \in \mathbb{Z}.$$

(ii) $k + 1 < r \leqslant k + \nu$. By Lemma 3.3, $\varepsilon_m^j(r) = \varepsilon_m^j - \varepsilon_m^r$ $(0 \leqslant j \leqslant r)$ and $\varepsilon_m^j = r - j$ $(r \leqslant j \leqslant k+\nu)$.

$$\left(\sum_{j=0}^{s} + \sum_{j=r}^{k+\nu}\right) \frac{\alpha_j}{p^{\varepsilon_m^j}} = \beta \in \mathbb{Z}.$$

Multiplying by $p^{\varepsilon_m^r}$, we obtain:

$$\sum_{j=0}^{s} \frac{\alpha_j}{p^{\varepsilon_m^j - \varepsilon_m^r}} + \sum_{j=r}^{k+\nu} \frac{\alpha_j}{p^{\varepsilon_m^j - \varepsilon_m^r}} = \beta p^{\varepsilon_m^r} \in \mathbb{Z}; \quad \text{i.e.} \quad \sum_{j=0}^{s} \frac{\alpha_j}{p^{\varepsilon_m^j(r)}} + \sum_{j=r}^{k+\nu} \frac{\alpha_j}{p^{\varepsilon_m^j(r)}} \in \mathbb{Z},$$

$$\frac{\alpha_j}{p^{\varepsilon_m^j(r)}} = \frac{\alpha_j}{p^{r-j}} = \alpha_j p^{j-r} \quad (r \leqslant j \leqslant k+\nu).$$

Thus, $\sum_{j=0}^{s} \frac{\alpha_j}{p^{\varepsilon_m^j(r)}} \in \mathbb{Z}$.

(iii) $r \leqslant k + 1$. By Lemma 3.3, $\varepsilon_m^j(r) \leqslant 0$ and thus $\sum_{j=0}^{s} \frac{\alpha_j}{p^{\varepsilon_m^j(r)}} \in \mathbb{Z}$.

Hence $(\alpha_0, \dots, \alpha_s)$ is an admissible sequence with respect to the reduced index functions $\varepsilon_m^j(r)$.

Conversely, let $(\alpha_0, \dots, \alpha_s)$ be an admissible sequence with respect to $\varepsilon_m^j(r)$ and $\beta_j = p^{t_j}\alpha_j$ $(0 \leqslant j \leqslant s)$. Let $m = \sigma(t_k) = \sigma(p^\nu\Delta) \in \mathcal{M}$ $((\Delta, p) = 1)$ be such that $k + \nu < r$. Then by Lemma 3.3, $\varepsilon_m^j(r) = \varepsilon_m^j$ and $\sum_{j=0}^{\min(s,k+\nu)} \frac{\alpha_j}{p^{\varepsilon_m^j}} = \sum_{j=0}^{\min(s,k+\nu)} \frac{\alpha_j}{p^{\varepsilon_m^j(r)}} \in \mathbb{Z}$. Suppose there exists no $m \in \mathcal{M}$ such that $k + 1 < r \leqslant k + \nu$. Let $m = \sup\{m' = \sigma(t_{k'}) = \sigma(p^{\nu'}\Delta')\colon k' + \nu' < r\}$. Then $(\alpha_0, \dots, \alpha_s)$ is admissible in $\mathcal{M}^m$. It follows from [1, Proposition 5.1.7] if $s \leqslant k + \nu$ and from Proposition 3.4 if $s > k + \nu$ that $(\alpha_0, \dots, \alpha_s)$ extends to an admissible sequence $(\alpha_0, \dots, \alpha_s, 0, \dots, 0, \alpha_r, \dots, \alpha_{r_n})$. If there exists $m = \sigma(t_k) = \sigma(p^\nu\Delta) \in \mathcal{M}$ $((\Delta, p) = 1)$ such that $k + 1 < r \leqslant k + \nu$ then put $-\alpha_r = \sum_{j=0}^{s} \frac{\alpha_j}{p^{\varepsilon_m^j(r)}} \in \mathbb{Z}$ and $\alpha_{r+1} = \cdots = \alpha_{k+\nu} = 0$,

$$\sum_{j=0}^{s} \frac{\alpha_j}{p^{\varepsilon_m^j - \varepsilon_m^r}} + \alpha_r = 0, \quad \text{or,} \quad \sum_{j=0}^{s} \frac{\alpha_j}{p^{\varepsilon_m^j}} + \frac{\alpha_r}{p^{\varepsilon_m^r}} = 0, \text{ i.e.}$$

$(\sum_{j=0}^{s}+\sum_{j=r}^{k+\nu})\frac{\alpha_j}{p^{\varepsilon_m^j}} \in \mathbb{Z}$. By [1, Proposition 5.1.7], $(\alpha_0, \ldots, \alpha_s, 0, \ldots, 0, \alpha_r, \ldots, \alpha_{k+\nu})$ extends to an admissible sequence, $(\alpha_0, \ldots, \alpha_s, 0, \ldots, 0, \alpha_r, \ldots, \alpha_{r_n})$ with respect to ε_m^j. Put $\beta_j = p^{t_j}\alpha_j$ $(r \leqslant j \leqslant r_n)$ and we obtain from [1, Proposition 6.2.14], the relation, $(\sum_{j=0}^{s}+\sum_{j=r}^{r_n})\beta_j\psi_{\mathbb{R}}^{p^j}(\omega) = 0$ in $J_p(P_n(\mathbb{C}))$. Thus, $\sum_{j=0}^{s}\beta_j\psi_{\mathbb{R}}^{p^j}(\omega) = 0$ in $G(p,n,r)$. $\square$

3.6. Definition. We define the invariant u_k^r by replacing the index functions ε_m^j in the definition of u_k by the reduced index functions $\varepsilon_m^j(r)$.

We obtain for $G(p,n,r)$ the analogue of Theorem 2.21 for $J_p(P_n(\mathbb{C}))$. Let $M_{n+1}(p^r)$ be as defined in [1, Definition 7.3.4].

3.7. Theorem. $G(p,n,r) = \bigoplus_{k=0}^{(r-1)} \mathbb{Z}_{p^{t_k+u_k^r}}$. *The first summand generated by ω has order $M_{n+1}(p^r)$.*

From Theorem 3.7 we write down the decomposition of $J(L^n(p^r))$ into cyclic groups; i.e.

3.8. Theorem.

$$J\big(L^n\big(p^r\big)\big) = \begin{cases} \bigoplus_{k=0}^{r-1}\mathbb{Z}_{p^{t_k+u_k^r}} & \textit{if } p \textit{ is odd and } n \not\equiv 0 \pmod 4,\\ \bigoplus_{k=0}^{r-1}\mathbb{Z}_{p^{t_k+u_k^r}} \oplus \mathbb{Z}_2 & \textit{if } p \textit{ is odd and } n \equiv 0 \pmod 4,\\ \bigoplus_{k=0}^{r-2}\mathbb{Z}_{2^{t_k+u_k^r}} \oplus \mathbb{Z}_{2^{t_{r-1}+u_{r-1}^r+1}} & \textit{if } p = 2. \end{cases}$$

The first summand generated by ω has order $M_{n+1}(p^r)$. As a corollary to Theorem 3.8, we recover [1, Proposition 7.3.8], i.e.

3.9. Corollary. *Let* $n = p^k(p-1)+r'$ $(0 \leqslant r' \leqslant p-2)$ $(1 \leqslant r \leqslant k)$. *Then*

$$J\big(L^n\big(p^r\big)\big) = \begin{cases} \mathbb{Z}_{p^{p^k+r-1}} \oplus \mathbb{Z}_{p^{(p^{k-1}-1)}} \oplus \mathbb{Z}_{p^{p^{k-2}-1}} \oplus \cdots \oplus \mathbb{Z}_{p^{(p^{k-r+1}-1)}}\\ \quad \textit{if } p \textit{ is odd and } n \not\equiv 0 \pmod 4,\\ \mathbb{Z}_{p^{p^k+r-1}} \oplus \mathbb{Z}_{p^{(p^{k-1}-1)}} \oplus \mathbb{Z}_{p^{p^{k-2}-1}} \oplus \cdots \oplus \mathbb{Z}_{p^{(p^{k-r+1}-1)}} \oplus \mathbb{Z}_2\\ \quad \textit{if } p \textit{ is odd and } n \equiv 0 \pmod 4,\\ \mathbb{Z}_{2^{2^k+r-1}} \oplus \mathbb{Z}_{2^{(2^{k-1}-1)}} \oplus \mathbb{Z}_{2^{2^{k-2}-1}} \oplus \cdots \oplus \mathbb{Z}_{2^{2^{k-r+1}}}\\ \quad \textit{if } p = 2. \end{cases}$$

Proof. $\varepsilon_m^i(r) < \varepsilon_m^{i-1}(r) < \cdots < \varepsilon_m^0(r)$ and the set $\mathcal{H}_i^r$ defined in analogy with $\mathcal{H}_i$ consists, merely of ϕ and the associated set Φ to $\emptyset$ is empty and hence $u_i^r = 0$ $(r-1 \leqslant i \leqslant k)$. The order of the first summand follows from the definition of the number $M_{n+1}(p^r)$. $\square$

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