THE EQUITY PREMIUM IN CONSUMPTION AND PRODUCTION MODELS

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In this paper we use a simple model with a single Cobb–Douglas firm and a consumer with a CRRA utility function to show the difference between the equity premia in the production-based Brock model and the consumption-based Lucas model. With this simple example we show that the equity premium in the production-based model *exceeds* that of the consumption-based model with probability 1.

1. INTRODUCTION

The equity premium puzzle that was raised by Mehra and Prescott (1985) has spawned a vast literature over the past quarter century in which the authors attempt to explain the magnitude of the difference between the returns on equity and the returns on risk-free assets. As Cochrane (2008, p. 261) puts it, "The ink spilled on the equity premium puzzle would sink the Titanic . . . " See Mehra (2006) and the articles in Mehra (2008) for many references to some of this ink.

As a point of departure from this literature, Akdeniz and Dechert (2007) took the asset pricing model in Brock (1982) and numerically simulated the returns on equity and a risk-free asset. They showed that there were indeed parameterizations for which the equity premia were much higher than for the corresponding consumption-based model. Contrast this with Mehra and Prescott (1985), who concluded that they were unable to replicate the large historical equity premium on U.S. securities with the Lucas (1978) asset pricing model.[1]

What then is the source of the difference between the findings of Mehra and Prescott and those of Akdeniz and Dechert? The consumption-based asset pricing model takes the production process as given (as in an endowment economy) and assumes that the output is paid out to the consumer as dividends, which in turn equals consumption. The formula of Lucas (1978) is then used to price the production process (which is thought of as an asset owned by the consumer). This methodology relies on the basic idea in Mehra's theorem (2006, p. 70), which

states that adding production to the model "... does not increase the set of joint equilibrium processes on consumption and asset prices." This is certainly true if the asset that is being priced is a net cash flow of $N_t = Y_t - I_t \equiv C_t$. That is, two different production processes, $\{Y_t\}$ and $\{Y'_t\}$, will have *identical* asset prices, provided that their respective investment processes, $\{I_t\}$ and $\{I'_t\}$, are such that $Y_t - I_t = Y'_t - I'_t$ for all t. This is the essence of Mehra's theorem: there is an invariance in the consumption-based asset prices in all models that produce the same consumption stream.

In Brock's model 1982 the net cash flow that is priced is the stream of per period profits, which *does not equal* consumption. As Brock shows in his appendix, the difference in these two concepts is that when *the firms own* the capital stock, the net cash flow is output minus investment, which equals consumption. If *the consumer owns* the capital, then the net cash flow is output minus rents, which *does not* equal consumption. In a more realistic framework, there are inputs of production that are owned by firms and inputs that are rented. (For instance, the latter include labor and certain types of resources, among others.) In this highly aggregated framework, these two models represent polar extremes.

In this paper we perform the following thought experiment: use the real side of the Brock (1982) model to generate the sequences of consumption, investment and output. Then price the consumption sequence (output minus investment) as in Lucas (1978) and the profit sequence as in Brock (1982) and compare the equity premia for the two models. As we show in Section 3, there is a direct analytical link between these two prices, with which we can derive a formula for the difference between the equity premia. In Section 4 we use a simple model with a logarithmic utility function and Cobb–Douglas production function to show that the equity premium on the profit sequence can be strictly larger than it is on the consumption sequence.

2. THE ONE-FIRM MODEL

Consider the Brock and Mirman (1972) growth model:

$$\max_{\{x_t\}} \quad \mathsf{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

$$\text{subject to:} \quad c_t + x_t = y_t$$

$$y_{t+1} = f(x_t, \xi_{t+1}),$$

where c_t and x_t are the consumption and investment at time t and ξ_{t+1} is the production shock that affects output, y_{t+1}, at date $t+1$. In analyzing this model, we make the same assumptions on u and f as Brock and Mirman: the utility function, u, is concave, differentiable and $\lim_{c \downarrow 0} u'(c) = \infty$. Similarly, with probability 1 the production function, $f(x, \xi)$, is concave and differentiable in x and the Inada conditions hold, $\lim_{x \downarrow 0} f'(x, \xi) = \infty$ and $\lim_{x \to \infty} f'(x, \xi) = 0$.

2.1. Model 1: Pricing Profits

Following the method of Brock (1982), we can also use this growth model as the foundation for two asset pricing models. In the first model, which is in the text of Brock (pp. 14–15), the consumer owns the capital stock and rents it to the firm on a per period basis. We will refer to this as model 1. In this model the consumer's budget constraint is

$$c_t + x_t + P_t S_t \leq (\pi_t + P_t)S_{t-1} + r_t x_{t-1},$$

where P_t is the share price at date t, S_t is the number of shares held at date t, r_t is the rental on capital received at date t, and π_t is the profits paid as dividends at date t. The consumer treats profits (in this model all profits are paid out as dividends) and rents, π_t and r_t, as exogenous to his/her consumption, rental, and share purchase decisions, c_t, x_t, and S_t. In a rational expectations equilibrium the following hold:

$$\pi_t = f(x_{t-1}, \xi_t) - f'(x_{t-1}, \xi_t)x_{t-1},$$
$$r_t = f'(x_{t-1}, \xi_t),$$
$$S_t = 1,$$
$$c_t + x_t = y_t.$$

It is notationally useful to define the (random) intertemporal marginal rate of substitution by

$$\Gamma_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}. \tag{1}$$

The Euler equations with respect to x_t and S_t for this model are

$$1 = \mathsf{E}_t \left[\Gamma_{t+1} r_{t+1} \right], \tag{2}$$

$$P_t = \mathsf{E}_t[\Gamma_{t+1}(P_{t+1} + \pi_{t+1})]. \tag{3}$$

Any asset that is in zero net supply can be priced within this model. The price, B_t, of a one period bond that pays one unit of the consumption good at date $t+1$ will satisfy at date t

$$u'(c_t)B_t = \mathsf{E}_t[\beta u'(c_{t+1})]$$

or, using equation (1),

$$B_t = \mathsf{E}_t \left[\Gamma_{t+1} \right]. \tag{4}$$

The equity premium is the expected return on equity minus the return on the risk-free asset:

$$\mathsf{E}_t \left[\frac{P_{t+1}}{P_t} + \frac{\pi_{t+1}}{P_t} \right] - \frac{1}{B_t}, \tag{5}$$

where the first term in brackets is the return on the appreciation of the price of the equity and the second is the dividend return. Because the risk-free bond pays out one unit of consumption at date $t+1$, its return is the final term in equation (5).

2.2. Model 2: Pricing Consumption

In the second model, which is in the appendix of Brock (1982, p. 36–37), the firm owns the capital stock and maximizes the present discounted value of its net cash flow. We will refer to this as model 2. In this model, the firm has the traditional capital budget,

$$\tilde{P}_t(S_t - S_{t-1}) + y_t = x_t + d_t S_{t-1}, \tag{6}$$

where $\tilde{P}_t$ is the share price at time t (not necessarily the same as P_t in model 1) and d_t are the dividends per share declared at time t and paid on the shares outstanding prior to the firm's operations in the equity market, $S_t - S_{t-1}$. (In the example below, which has an analytical solution, capital fully depreciates in one period. Hence, gross investment is $x_t - x_{t-1} + \delta x_{t-1} = x_t$. We will assume $\delta = 1$ throughout because it does not affect the basic point that we make.)

The firm is assumed to maximize the present discounted value of its net cash flow. As is shown in Brock (1982), this leads to the valuation formula

$$\tilde{P}_t S_t = E_t[\Gamma_{t+1}(\tilde{P}_{t+1} S_{t+1} + N_{t+1})], \tag{7}$$

where

$$N_{t+1} = y_{t+1} - x_{t+1} = c_{t+1}.$$

If we let $V_t = \tilde{P}_t S_t$, then the valuation of the firm turns out to be

$$V_t = E_t [\Gamma_{t+1} (V_{t+1} + c_{t+1})], \tag{8}$$

which is precisely the Lucas (1978) formula for the price of the asset in an endowment economy in which the consumption stream is the dividends of the asset.

That the two models give a different value of the firm (and in the example below a different equity premium) stems from the fact that in model 1, the firms rent capital from the consumers (who own it), whereas in model 2 the firms own the capital. These are the two polar cases.

3. THE COMPARISON OF EQUITY PREMIA

The equity premium in the Lucas model is

$$E_t \left[\frac{V_{t+1}}{V_t} + \frac{c_{t+1}}{V_t} \right] - \frac{1}{B_t},$$

where, similarly to the Brock model, the first term in brackets is the return to the appreciation of the price of the asset and the second is the return on dividends, which, recall, equals consumption. The difference between the equity premium in the two models is

$$\Delta_t = E_t \left[\frac{P_{t+1} + \pi_{t+1}}{P_t} - \frac{V_{t+1} + c_{t+1}}{V_t} \right]. \tag{9}$$

To complete the analysis to see whether there is, in fact, a difference between the equity premia in the two models, we need to solve for the relationship between V_t and P_t. Let the optimal solution in model 1 have $\{c_t\}$ as the consumption sequence. Price this sequence with the model 2 model. Its price sequence will be $\{V_t\}$. Let $\{P_t\}$ be the asset price sequence from the Brock model. We can get a direct comparison of these two price sequences by expanding equation (3) and using equation (2) and the rational expectations equilibrium (REE) conditions:

$$P_t = E_t[\Gamma_{t+1}(P_{t+1} + \pi_{t+1})],$$

$$= E_t[\Gamma_{t+1}(P_{t+1} + y_{t+1} - r_{t+1}x_t)],$$

$$= E_t[\Gamma_{t+1}(P_{t+1} + x_{t+1} + c_{t+1})] - E_t[\Gamma_{t+1}r_{t+1}x_t],$$

$$= E_t[\Gamma_{t+1}(P_{t+1} + x_{t+1} + c_{t+1})] - x_t.$$

Rewrite this equation as

$$(P_t + x_t) = E_t[\Gamma_{t+1}(P_{t+1} + x_{t+1} + c_{t+1})]. \tag{10}$$

Now subtract equation (10) from equation (7):

$$(V_t - P_t - x_t) = E_t[\Gamma_{t+1}(V_{t+1} - P_{t+1} - x_{t+1})].$$

The transversality condition in this model is that if a random sequence, $\{Z_t\}$, satisfies

$$Z_t = E_t[\Gamma_{t+1}Z_{t+1}]$$

for all t, then $Z_t \equiv 0$. This follows from repeated substitutions:

$$Z_t = E_t[\Gamma_{t+1}\Gamma_{t+2}\cdots\Gamma_{t+s}Z_{t+s}]$$

and by taking the limit of this equation as $s \to \infty$. Therefore,

$$V_t - P_t - x_t \equiv 0$$

or

$$V_t = P_t + x_t. \tag{11}$$

This is the relationship between the asset prices in the two models.

Using equations (9) and (11), the difference between the returns on these two types of assets is

$$\Delta_t = E_t\left[\frac{P_{t+1} + \pi_{t+1}}{P_t} - \frac{P_{t+1} + x_{t+1} + c_{t+1}}{P_t + x_t}\right]$$

$$= E_t\left[\frac{P_{t+1} + y_{t+1} - f'(x_t, \xi_{t+1})x_t}{P_t} - \frac{P_{t+1} + y_{t+1}}{P_t + x_t}\right]. \tag{12}$$

We can use this equation to measure the difference between the equity premia in the two models. If we were to take the *utility weighted* difference between the equity premia, we would have

$$E_t \left[\Gamma_{t+1} \left(\frac{P_{t+1} + \pi_{t+1}}{P_t} - \frac{V_{t+1} + c_{t+1}}{V_t} \right) \right],$$

which, by the pricing equations (8) and (3), *is* zero. However, there is no reason (other than risk neutrality in which case $\Gamma_t = 1$) why this would imply that equation (9) is zero. In the next section we show with an example that this difference is *not necessarily* zero.

4. AN EXAMPLE

Take the case of a logarithmic utility function with a Cobb–Douglas production function that has a random elasticity of capital:

$$u(c) = \ln(c),$$
$$f(x, \alpha) = x^\alpha,$$

where $0 < \alpha \leq 1$ wp1. (For the production function f to be continuous in x at 0 wp1, we must rule out a mass point of α at 0.) Let $\bar{\alpha} = E[\alpha]$ and let $\{\alpha_t\}$ be iid. The solution of the dynamic programming problem is

$$x_t = \bar{\alpha}\beta y_t, \tag{13}$$

$$\pi_{t+1} = (1 - \alpha_{t+1})y_{t+1}, \tag{14}$$

$$P_t = \frac{\beta(1 - \bar{\alpha})}{1 - \beta} y_t. \tag{15}$$

(See the appendix for proofs of the equations in this section.) Now compute the difference in the asset returns from equation (12):

$$\Delta_t = E_t \left[\frac{P_{t+1} + (1 - \alpha_{t+1})y_{t+1}}{P_t} - \frac{P_{t+1} + y_{t+1}}{P_t + x_t} \right] \tag{16}$$

$$= \frac{(1 - \beta)}{\beta(1 - \bar{\alpha})y_t} E_t[(\bar{\alpha} - \alpha_{t+1})y_{t+1}] \tag{17}$$

The algebra of going from equation (16) to equation (17) is tedious and is in the appendix. Also in the appendix, we show that for this example, if $0 < y_0 < 1$ then $0 < y_t < 1$ for all t. In Theorem 1 of the appendix we show that this implies that $E_t[(\bar{\alpha} - \alpha_{t+1})y_{t+1}] > 0$ for all t. (The one exception occurs for the degenerate case that $\alpha_t = \bar{\alpha}$ for all t, in which case the equity premium is the same for the two models.) In particular, we show that α_{t+1} and $x_t^{\alpha_{t+1}}$ are negatively correlated for $0 < x_t < 1$.[2]

For this example, the sequence of outputs, $\{y_t\}$, is transient in the set $[1, \infty)$ and is eventually in $(0, 1)$ with probability 1 for any initial $y_0 > 0$. Once it has reached this set, $\Delta_t > 0$ and the equity premium on the model 1 asset prices is *strictly greater* than it is on the model 2 asset prices. In the transient set it is strictly less.

5. CONCLUSION

The model in Section 4 is not meant to be our view of the "correct" model that explains the equity premium. It is used *solely* because it is analytically tractable and it verifies the conclusion that there *can be* a difference between the equity premia in the production-based asset prices and the consumption-based asset prices. How much of the equity premium can be explained by this difference is an empirical issue that we do not address here.

This example should lay to rest the notion that production *in and of itself* cannot add anything to the analysis of the equity premium on asset prices. (Recall the quote of Mehra and Prescott in the introduction.) In this example, there is only one firm with 100% depreciation. In Akdeniz and Dechert (2007) it was shown that for a model with four firms, with partial depreciation and for which the technology parameters are state dependent, the equity premium can easily be twice the value of $\beta^{-1} - 1$. This is due, in part, to the fact that such a model also can include idiosyncratic risk, which is also a determinant of the equity premium.

NOTES

1. This claim is for reasonable parameterizations of the model, which they discuss in detail on pp. 154–155.
2. We thank the referee for this observation. The intuition behind this result is that α and x^α move in oposite directions when both x and α are in the interval $(0, 1)$.

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APPENDIX

Here are the proofs of the assertions in the text.

Equation (13). This was proved in Mirman and Zilcha (1975), Example A, p. 333.   ■

Equation (14). Since $f(x, \alpha) = x^\alpha$, then

$$x_t f'(x_t, \alpha_{t+1}) = \alpha_{t+1} x_t^{\alpha_{t+1}} = \alpha_{t+1} y_{t+1},$$

and thus $\pi_{t+1} = y_{t+1} - x_t f'(x_t, \alpha_{t+1}) = (1 - \alpha_{t+1}) y_{t+1}$.   ■

Equation (15). From equation (1) and using $u(c) = \ln(c)$ for the example, we have

$$\Gamma_t = \frac{\beta c_t}{c_{t+1}} = \beta \frac{y_t}{y_{t+1}}.$$

Insert this and the result above for profits into the asset pricing equation (3) to get

$$P_t = \mathsf{E}_t \left\{ \beta \frac{y_t}{y_{t+1}} [P_{t+1} + (1 - \alpha_{t+1}) y_{t+1}] \right\}.$$

Now assume that the pricing function is linear in output, $P_t = k y_t$, and see whether we get a consistent equation for k:

$$k y_t = \mathsf{E}_t \left\{ \beta \frac{y_t}{y_{t+1}} [k y_{t+1} + (1 - \alpha_{t+1}) y_{y+1}] \right\},$$

which reduces to

$$k = \mathsf{E}_t \{ \beta[k + (1 - \alpha_{t+1})] \} = \beta(k + 1 - \bar{\alpha}),$$

and we get $k = \beta(1 - \bar{\alpha})/(1 - \beta)$.   ■

Equation (17). The terms inside the expectation in equation (16) are

$$\left[\frac{k + (1 - \alpha_{t+1})}{k} - \frac{k + 1}{k + \bar{\alpha}\beta} \right] \frac{y_{t+1}}{y_t} = \left(1 + \frac{1 - \alpha_{t+1}}{k} - \frac{k + 1}{k + \bar{\alpha}\beta} \right) \frac{y_{t+1}}{y_t},$$

where k is defined above. Now,

$$k + 1 = \frac{\beta(1 - \bar{\alpha})}{1 - \beta} + 1 = \frac{1 - \bar{\alpha}\beta}{1 - \beta}$$

and

$$k + \bar{\alpha}\beta = \frac{\beta(1 - \bar{\alpha})}{1 - \beta} + \bar{\alpha}\beta = \frac{\beta(1 - \bar{\alpha}\beta)}{1 - \beta},$$

and so

$$\frac{k + 1}{k + \bar{\alpha}\beta} = \frac{1 - \bar{\alpha}\beta}{1 - \beta} \frac{1 - \beta}{\beta(1 - \bar{\alpha}\beta)} = \frac{1}{\beta}.$$

Putting these together we get

$$1 + \frac{1 - \alpha_{t+1}}{k} - \frac{k+1}{k + \bar{\alpha}\beta} = 1 + (1 - \alpha_{t+1})\frac{(1-\beta)}{\beta(1-\bar{\alpha})} - \frac{1}{\beta}$$

$$= \frac{(1 - \alpha_{t+1})(1-\beta)}{\beta(1-\bar{\alpha})} - \frac{1-\beta}{\beta}$$

$$= \frac{1-\beta}{\beta}\left(\frac{1 - \alpha_{t+1}}{1-\alpha} - 1\right)$$

$$= \frac{1-\beta}{\beta}\left(\frac{\bar{\alpha} - \alpha_{t+1}}{1-\bar{\alpha}}\right).$$

From this we conclude that

$$\Delta_t = \mathsf{E}_t\left[\frac{1-\beta}{\beta}\left(\frac{\bar{\alpha} - \alpha_{t+1}}{1-\bar{\alpha}}\right)\frac{y_{t+1}}{y_t}\right] = \frac{1-\beta}{\beta(1-\bar{\alpha})y_t}\mathsf{E}_t[(\bar{\alpha} - \alpha_{t+1})y_{t+1}],$$

which was to be shown. ∎

Notice that if $0 < y_t < 1$ then $0 < y_{t+1} < 1$ wp1. This follows from $y_{t+1} = x_t^{\alpha_{t+1}} = (\bar{\alpha}\beta y_t)^{\alpha_{t+1}}$, and the terms in parentheses are all between 0 and 1.

THEOREM 1. *Assume that the sequence* $\{\alpha_t\}$ *is iid. Then for the example in Section 4,* $\mathsf{E}_t[(\bar{\alpha} - \alpha_{t+1})y_{t+1}] > 0$ *for* $y_t \in (0, 1)$.

Proof. Note that $y_{t+1} = e^{\alpha_{t+1}\ln x_t}$ and that $\ln x_t < 0$ for $x_t \in (0, 1)$. Define the auxiliary function

$$\phi(c) = \mathsf{E}[(\bar{\alpha} - \alpha)e^{c\alpha}]$$

for a random variable α that has the same distribution as the iid sequence $\{\alpha_t\}$. The function ϕ is differentiable:

$$\phi'(c) = \mathsf{E}[(\bar{\alpha} - \alpha)\alpha e^{c\alpha}]$$

$$= \mathsf{E}[(\bar{\alpha} - \alpha)(\alpha - \bar{\alpha} + \bar{\alpha})e^{c\alpha}]$$

$$= -\mathsf{E}[(\bar{\alpha} - \alpha)^2 e^{c\alpha}] + \bar{\alpha}\mathsf{E}[(\bar{\alpha} - \alpha)e^{c\alpha}]$$

$$= -\mathsf{E}[(\bar{\alpha} - \alpha)^2 e^{c\alpha}] + \bar{\alpha}\phi(c).$$

Define another auxiliary function,

$$\psi(c) = \mathsf{E}[(\bar{\alpha} - \alpha)^2 e^{c\alpha}],$$

and note that $\psi(c) > 0$ for all c. The function ϕ satisfies the differential equation

$$\phi'(c) = \bar{\alpha}\phi(c) - \psi(c),$$

which has a general solution of

$$\phi(c) = e^{c\bar{\alpha}}\left[\phi(0) + \int_c^0 e^{-z\bar{\alpha}}\psi(z)dz\right].$$

This can be verified by differentiating the right-hand side with respect to c. From its definition, $\phi(0) = 0$, so the particular solution for ϕ is

$$\phi(c) = e^{c\bar{\alpha}} \int_c^0 e^{-z\bar{\alpha}} \psi(z) dz.$$

Since $\psi > 0$, it follows that $\phi(c) > 0$ for $c < 0$. Therefore,

$$\mathsf{E}_t[(\bar{\alpha} - \alpha_{t+1})e^{c\alpha_{t+1}}] > 0$$

for $c < 0$. Let $c = \ln x_t$ and the conclusion follows for $x_t \in (0, 1)$.   ∎

COROLLARY 2. $\Delta_t < 0$ *for* $x_t > 1$.

Proof. When $x_t > 1$, $c = \ln(x_t) > 0$ and so $\phi(c) < 0$.   ∎