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Integrable discrete systems on $\mathbb{R}$ and related dispersionless systems

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A general framework for integrable discrete systems on $\mathbb{R}$, in particular, containing lattice soliton systems and their q-deformed analogs, is presented. The concept of regular grain structures on $\mathbb{R}$, generated by discrete one-parameter groups of diffeomorphisms, in terms of which one can define algebra of shift operators is introduced. Two integrable hierarchies of discrete chains together with bi-Hamiltonian structures and their continuous limits are constructed. The inverse problem based on the deformation quantization scheme is considered. © *2008 American Institute of Physics.* [DOI: 10.1063/1.2948962]

I. INTRODUCTION

Recently, the so-called integrable q-analogs of KP- and Toda-type hierarchies together with related Hamiltonian structures, W-algebras, and τ-functions have become of increasing interest (see Refs. 1–9 and references therein). The q-deformed KP hierarchy (q-KP) with the reductions of q-KdV soliton-type systems are obtained by means of pseudodifferential operators defined in terms of the q-derivative ∂_q instead of the usual derivative ∂ used for ordinary KP and KdV hierarchies

$$\partial u(x) = \frac{\partial u(x)}{\partial x} \quad \rightarrow \quad \partial_q u(x) = \frac{u(qx) - u(x)}{(q-1)x}.$$

Analogously, the q-deformed Toda hierarchies can be constructed by means of the q-shift operators

$$Eu(x) = u(x+1) \quad \rightarrow \quad E_q u(x) = u(qx).$$

The scheme of the construction of integrable q-deformed systems is based on the classical R-matrix formalism that proved very fruitful for the systematic construction of field and lattice soliton systems[10–15] as well as dispersionless integrable field systems.[16–19] Moreover, the R-matrix approach allows a construction of Hamiltonian structures and conserved quantities. By an integrable system, we mean such a system which has infinite hierarchy of symmetries and conserved quantities.

Having all the above classes of integrable systems, with parallel schemes of construction, it is interesting how to embed them into a more general unifying framework. One of the possible approaches is to construct integrable systems on time scales.[20,21] A time scale $\mathbb{T}$ is an arbitrary

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nonempty closed subset of real numbers. It was introduced to unify all possible intervals on the real line $\mathbb{R}$, such as continuous (whole) $\mathbb{R}$, discrete $\mathbb{Z}$, and q-discrete $\mathbb{K}_q$ intervals. On a given time scale it is possible to construct Δ-derivative (being simultaneously a generalization of the ordinary derivative and the q-derivative) by forward $\sigma(x)$ and backward $\rho(x)$ jump operators, where $x \in \mathbb{T}$ (for all precise definitions see Refs. 20 and 21). Assuming the regularity property of $\mathbb{T}$, i.e., $\rho(\sigma(x))=x$, one can define an algebra of the Laurent series of Δ-operators

$$\Delta u(x) = \frac{u(\sigma(x)) - u(x)}{\mu(x)}, \quad \mu(x) \equiv \sigma(x) - x, \quad x \in \mathbb{T}$$

or shift operators as $Eu(x)=u(\sigma(x))$, leading to the construction of integrable systems on time scales.[21,22] Defining suitable inner products in this algebra, additionally one can construct conservation laws. In such a formulation, dynamical fields $u:\mathbb{T}\to\mathbb{R}$ are the mappings from a time scale to real numbers.

The main goal of this work is the formulation of a general unifying framework of integrable discrete systems, in such a way that the domain of dynamical fields u is always $\mathbb{R}$. We also consider the continuous limit and the inverse procedure. In Sec. II we introduce the concept of a regular grain structure on $\mathbb{R}$ defined by discrete one-parameter groups of diffeomorphisms $\sigma_{m\hbar}(x)$. Then, the shift operator can be constructed in terms of formal jump operator $\sigma(x)=\sigma_\hbar(x)$. In this section, elements of geometric scheme are defined as appropriate functionals, duality maps, adjoint operators, etc. A class of discrete systems is chosen in such a way that the limit $\hbar\to 0$ is dispersionless. In Sec. III, using the formalism of classical R-matrices, we construct two integrable hierarchies of discrete chains being counterparts of the original infinite-field Toda and modified Toda chains. Additionally bi-Hamiltonian structures are constructed. In Sec. IV the concept of the continuous limit, which in our case becomes the dispersionless limit, is explained. Further, in Sec. V, the theory of dispersionless chains, being dispersionless limits of discrete chains together with bi-Hamiltonian structures, is presented. In Sec. VI the inverse problem to the dispersionless limit is considered. It is based on the scheme of the deformation quantization formalism introduced in Ref. 15. As a result, we show that there is a class of gauge equivalent integrable discrete systems, being dispersive counterparts of dispersionless systems considered earlier. We end the paper with some final comments.

II. ONE-PARAMETER REGULAR GRAIN STRUCTURES ON $\mathbb{R}$

The main aim of this article is to present a general theory of integrable discrete systems on $\mathbb{R}$ that contains lattice soliton systems as well as q-discrete systems as particular cases. This theory is illustrated by integrable discrete chains that are infinite-field systems.

The maps $\sigma:\mathbb{R}\to\mathbb{R}$ and $\rho:\mathbb{R}\to\mathbb{R}$ are called the *forward* and *backward jump operators*, respectively. In $n\in\mathbb{Z}_+$ forward steps, a point $x\in\mathbb{R}$ is mapped to a point $\sigma^n(x)$, where σ^n is the n-times composition of forward jump operator σ. In n backward steps, x is mapped to a point $\rho^n(x)$. Then, the range of possible points to which we can map x by forward and backward steps (including x) is introduced by

$$\mathbb{G}_x := \{\rho^n(x):n \in \mathbb{Z}_+\} \cup \{x\} \cup \{\sigma^n(x):n \in \mathbb{Z}_+\}. \tag{2.1}$$

Hence, to each point x of $\mathbb{R}$ a set $\mathbb{G}_x$ is associated. The union of all $\mathbb{G}_x$ is given by $\mathbb{G}:=\cup_{x\in\mathbb{R}}\mathbb{G}_x$.

Definition 2.1: *We say that $\mathbb{G}$ defines the grain structure on $\mathbb{R}$. $\mathbb{G}$ is called as the regular grain structure, if there exist inverse maps σ^{-1} and ρ^{-1}, such that $\sigma(x)=\rho^{-1}(x)$ and $\rho(x)=\sigma^{-1}(x)$ for all $x\in\mathbb{R}$.*

So, in order to define the regular grain structure on $\mathbb{R}$, it is enough to use the forward jump operator σ, being bijection, since the backward jump operator can be written in terms of σ, i.e., $\rho=\sigma^{-1}$. Then, Definition 2.1 turns out to be $\mathbb{G}_x=\{\sigma^n(x):n\in\mathbb{Z}\}$, where we assume that $\sigma^0\equiv id_{\mathbb{R}}$. Besides, bijective σ defines a discrete one-parameter group of bijections on $\mathbb{R}$: $\mathbb{Z}\ni m\mapsto\{\sigma_m:\mathbb{R}\to\mathbb{R}\}$, such that $\sigma_m:=\sigma^m$, and vice versa each one-parameter group of bijections on $\mathbb{R}$ defines the regular grain structure on $\mathbb{R}$ with the forward jump operator defined by $\sigma := \sigma_1$. Note that the regular grain structure introduces equivalence classes between points of $\mathbb{R}$, such that $x \sim y$ if $\mathbb{G}_x = \mathbb{G}_y$ $(x, y \in \mathbb{R})$, i.e., there exists $k \in \mathbb{Z}$ such that $y = \sigma^k(x)$.

Further, we introduce a regular grain structure $\mathbb{G}$ on $\mathbb{R}$ by one-parameter group of diffeomorphisms instead of bijections, which is necessary as we deal with differential geometry of infinite-dimensional systems with smooth dynamical fields. Let $\mathbb{Z} \ni m \mapsto \sigma_{m\hbar}$ be a discrete one-parameter group of diffeomorphisms on $\mathbb{R}$: $\sigma_{m\hbar}: \mathbb{R} \to \mathbb{R}$, i.e.,

$$\sigma_0(x) = x \quad \text{and} \quad \sigma_{m\hbar}(\sigma_{n\hbar}(x)) = \sigma_{(m+n)\hbar}(x), \quad m,n \in \mathbb{Z},$$

where $\hbar > 0$ is some deformation parameter. It follows that $(\sigma_{n\hbar})^{-1}(x) = \sigma_{-n\hbar}(x)$. The continuous one-parameter group of diffeomorphisms $(\mathbb{R} \ni t \mapsto \sigma_t)$ can be completely determined by its infinitesimal generator $\mathcal{X}(x)\partial_x$ being a vector field on $\mathbb{R}$. We assume that the component $\mathcal{X}(x)$ is defined on $\mathbb{R}$ except at most at a finite number of points. Then,

$$\mathcal{X}(x) = \left.\frac{d\sigma_t(x)}{dt}\right|_{t=0} \quad \Leftrightarrow \quad \frac{d\sigma_t(x)}{dt} = \mathcal{X}(\sigma_t(x)), \tag{2.2}$$

where $t \in \mathbb{R}$. Arbitrary $\mathcal{X}\partial_x$ generates a continuous one-parameter group of diffeomorphisms only when it is a *complete* vector field, for which maximal integrals are defined on the whole $\mathbb{R}$, i.e., $\mathbb{R}$ is a domain of the mapping $t \mapsto \sigma_t$. In such a case the above discrete one-parameter group is well defined as it is enough to consider subgroup $\mathbb{Z}$ of $\mathbb{R}$. Incomplete $\mathcal{X}\partial_x$ might still well define a discrete group of diffeomorphisms, if $\hbar$ is properly chosen.

Lemma 2.2: *Let $\sigma_t(x)$ be a one-parameter group of diffeomorphisms generated by $\mathcal{X}(x)\partial_x$. Then, the following relation is valid:*

$$\mathcal{X}(x)\frac{d\sigma_t(x)}{dx} = \mathcal{X}(\sigma_t(x)). \tag{2.3}$$

Proof: From (2.2) one observes that $\mathcal{X}(\sigma_{s+t}(x)) = d\sigma_{s+t}(x)/ds$. By acting σ_s on both sides of (2.3), we have the following relation:

$$\mathcal{X}(\sigma_s(x))\frac{d\sigma_{s+t}(x)}{d\sigma_s(x)} = \mathcal{X}(\sigma_{s+t}(x)),$$

which completes the proof. $\square$

Now, we establish a phase space related to discrete systems. Let

$$\mathbf{u} := (u_0(x), u_1(x), u_2(x), \ldots)^T$$

be an infinite tuple of smooth functions $u_i: \mathbb{R} \to \mathbb{K}$, $x \mapsto u_i(x)$ with values in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Additionally we assume that u_i's depend on an appropriate set of evolution parameters, i.e., u_i's are dynamical fields. Let $\mathcal{U}$ be a linear topological space, with local independent coordinates $\mathbf{u}(\sigma_{m\hbar}(x))$ for all $m \in \mathbb{Z}$, which defines infinite-dimensional phase space. We use the following notation:

$$E^m u(x) := (E^m u)(x) = u(\sigma_{m\hbar}(x)), \quad m \in \mathbb{Z},$$

where $u(x)$ is some field. Let $\mathcal{C}$ be the algebra over $\mathbb{K}$ of functions on $\mathcal{U}$ of the form

$$f[\mathbf{u}] := \sum_{m \geq 0} \sum_{i_1,\ldots,i_m \geq 0} \sum_{s_1,\ldots,s_m \in \mathbb{Z}} a^{i_1 i_2 \ldots i_m}_{s_1 s_2 \ldots s_m} (E^{s_1} u_{i_1})(E^{s_2} u_{i_2}) \cdots (E^{s_m} u_{i_m}) \tag{2.4}$$

that are polynomials in $\mathbf{u}(\sigma_{m\hbar}(x))$ of finite order, with coefficients $a^{i_1 i_2 \ldots i_m}_{s_1 s_2 \ldots s_m} \in \mathbb{K}$. This algebra can be extended into operator algebra $\mathcal{C}[E, E^{-1}]$ ($\mathcal{C}[x, y, \ldots]$ stands for the linear space of polynomials in $x, y, \ldots$ with coefficients from $\mathcal{C}$), where the shift operator E is compatible with the grain structure defined by $\sigma_\hbar(x)$. Since $\sigma_\hbar(x)$ is an element of one-parameter group of diffeomorphisms, the equivalence

$$\sigma_\hbar(x) = e^{\hbar \mathcal{X}(x)\partial_x} x \quad \Leftrightarrow \quad e^{\hbar \mathcal{X}(x)\partial_x} u(x) = u(e^{\hbar \mathcal{X}(x)\partial_x} x), \tag{2.5}$$

where $u(x)$ is a smooth function. Formula (2.5) is valid on the whole real line if $\mathcal{X}(x)\partial_x$ is complete or where a discrete one-parameter group of diffeomorphisms is well defined. Thus, the shift operator E can be identified with $e^{\hbar \mathcal{X}(x)\partial_x}$, i.e.,

$$E^m \equiv e^{m\hbar \mathcal{X}(x)\partial_x}. \tag{2.6}$$

Example 2.3: *Consider vector fields of the form* $\mathcal{X}(x)\partial_x = x^{1-n}\partial_x$ *on* $\mathbb{R}$, *for* $n \in \mathbb{Z}$. *For* $n=0$, *integrating* (2.2) *one finds that*

$$\sigma_t(x) = e^t x \quad \Rightarrow \quad \sigma_{m\hbar}(x) = e^{m\hbar} x = q^m x \quad q \equiv e^\hbar,$$

which is defined for all $t \in \mathbb{R}$ *and so* $\mathcal{X}\partial_x = x\partial_x$ *is a complete vector field. When* $n=0$, *we deal with systems of "q-discrete" type. When* $n \neq 0$, *in general,* $\sigma_t(x)$ *is of the following implicit form:*

$$(\sigma_t(x))^n = x^n + nt.$$

For $n=1$, *we have*

$$\sigma_t(x) = x + t \quad \Rightarrow \quad \sigma_{m\hbar}(x) = x + m\hbar,$$

and $\mathcal{X}\partial_x = \partial_x$ *is obviously complete. In this case we deal with systems of "lattice" type. For* $n=-1$ *the related vector field* $\mathcal{X}\partial_x = x^2\partial_x$ *is incomplete as* $t \neq \frac{1}{x}$,

$$\sigma_t(x) = \frac{x}{1-tx} \quad \Rightarrow \quad \sigma_{m\hbar}(x) = \frac{x}{1-m\hbar x}.$$

However, if $x \neq 1/m\hbar$, *the related discrete one-parameter group of diffeomorphisms is well defined. When* n *is odd, we can always define a discrete one-parameter group of diffeomorphisms generated by* $\mathcal{X}\partial_x = x^{1-n}\partial_x$.

A space $\mathcal{F} = \{F:\mathcal{U} \to \mathbb{K}\}$ of functions on $\mathcal{U}$ is defined through linear functionals

$$\int (\cdot) d_\hbar x : \mathcal{C} \to \mathbb{K}, \quad f[\mathbf{u}] \mapsto F(\mathbf{u}) \coloneqq \int f[\mathbf{u}] d_\hbar x, \tag{2.7}$$

such that the following property is fulfilled:

$$\int Ef[\mathbf{u}] d_\hbar x = \int f[\mathbf{u}] d_\hbar x. \tag{2.8}$$

Here $\int d_\hbar x$ is a formal integration symbol. Property (2.8) entails the form of adjoint with respect to the duality map that will be defined in a moment.

Definition 2.4: *The explicit form of appropriate functionals can be introduced in two ways.*

(i) *A discrete representation is defined as*

$$F(\mathbf{u}) = \int f[\mathbf{u}] d_\hbar x \coloneqq \hbar \sum_{n \in \mathbb{Z}} f[\mathbf{u}(\sigma_{n\hbar}(x))]. \tag{2.9}$$

(ii) *A continuous representation is given as*

$$F(\mathbf{u}) = \int f[\mathbf{u}] d_\hbar x \coloneqq \int_{-\infty}^{\infty} f[\mathbf{u}(x)] \frac{dx}{\mathcal{X}(x)}, \tag{2.10}$$

where we assume that $u_i(x)$ *vanishes as* $|x| \to \infty$ *[if* $\mathcal{X}(x) \to 0$ *for* $|x| \to \infty$, *then* $u_i(x)$ *must* *vanish faster than $\mathcal{X}(x)$ does]. The above integral is in general improper, so additionally we assume that $u_i(x)$ behave properly as x tends to critical points x_c of $\mathcal{X}(x)$ ($\mathcal{X}(x_c)=0$). Then, evaluating the integral we take its principal value.*

When it is not necessary to differentiate between the above representations, we use only the formal integration symbol $\int d_\hbar x$. We have explicitly defined the functionals in two ways reflecting two different approaches developed for the lattice soliton systems. The first one is with the domain of dynamical fields $\mathbb{Z}$,[12,13] and the second one with $\mathbb{R}$.[15,23] So, functionals (2.9) and (2.10) are appropriate generalizations of these two approaches.

Proposition 2.5: *Both functionals from Definition 2.4 are well defined and satisfy* (2.8).

Proof: Both functionals are trivially linear. The discrete functional satisfies (2.8) since one can freely change the boundaries of the sum over the whole $\mathbb{Z}$. For the continuous functional we have

$$\int Ef[\mathbf{u}]d_\hbar x = \int_{-\infty}^{\infty} f[\mathbf{u}(\sigma_\hbar(x))]\frac{dx}{\mathcal{X}(x)} = \int_{-\infty}^{\infty} f[\mathbf{u}(x)]\frac{d\sigma_{-\hbar}(x)}{dx}\frac{dx}{\mathcal{X}(\sigma_{-\hbar}(x))}$$

$$= \int_{-\infty}^{\infty} f[\mathbf{u}(x)]\frac{dx}{\mathcal{X}(x)} = \int f[\mathbf{u}]d_\hbar x,$$

where the second equality is obtained by the change of variables $x \mapsto \sigma_\hbar(x)$, while the next one follows from Lemma 2.2. □

A vector field on $\mathcal{U}$ is given by a system of differential-difference equations. Here the difference calculus is performed with respect to the grain structure defined by $\sigma_\hbar$ and the first order differential calculus is with respect to the evolution parameter t,

$$\mathbf{u}_t = \mathbf{K}(\mathbf{u}), \tag{2.11}$$

where $u_t := \partial u/\partial t$ and $\mathbf{K}(\mathbf{u}) := (1/\hbar)(K_1[\mathbf{u}], K_2[\mathbf{u}], \ldots)^T$ with $K_i[\mathbf{u}] \in \mathcal{C}$. The class of the discrete systems is chosen in such a way that in the continuous limit $\hbar \to 0$, we obtain systems of hydrodynamic type (see Sec. IV). This assumption explains the appearance of the factor $\hbar$ in $\mathbf{K}$.

Let $\mathcal{V}$ be a linear space over $\mathbb{K}$, of all such vector fields on $\mathcal{U}$. Then the dual space $\mathcal{V}^*$ is a space of all linear maps $\eta : \mathcal{V} \to \mathbb{K}$. The action of $\eta \in \mathcal{V}^*$ on $\mathbf{K} \in \mathcal{V}$ can be defined through a duality map (bilinear functional) $\langle \cdot , \cdot \rangle : \mathcal{V}^* \times \mathcal{V} \to \mathbb{K}$ given by functional (2.7) as

$$\langle \eta, \mathbf{K} \rangle = \int \sum_{i=0}^{\infty} \eta_i K_i d_\hbar x = \int \eta^T \cdot \mathbf{K} d_\hbar x, \tag{2.12}$$

where the components of $\eta := (\eta_1, \eta_2, \ldots)^T$ belong to $\mathcal{C}$. With respect to the duality map (2.12) one finds that the adjoint of E^m is equal to E^{-m}, i.e., $(E^m)^\dagger = E^{-m}$.

Proposition 2.6: *The differential*

$$dF(\mathbf{u}) = \left(\frac{\delta F}{\delta u_0}, \frac{\delta F}{\delta u_1}, \ldots\right)^T \in \mathcal{V}^*$$

of a functional $F(\mathbf{u}) = \int f[\mathbf{u}]d_\hbar x$, such that its pairing with $\mathbf{K} \in \mathcal{V}$ assumes the usual Euclidean form

$$F'[\mathbf{K}] = \langle dF, \mathbf{K} \rangle = \int \sum_{i=0}^{\infty} \frac{\delta F}{\delta u_i}(u_i)_t d_\hbar x, \tag{2.13}$$

where $F'[\mathbf{K}]$ is the directional derivative, is defined by variational derivatives of the form

$$\frac{\delta F}{\delta u_i} := \sum_{m \in Z} E^{-m} \frac{\partial f[\mathbf{u}]}{\partial u_i(\sigma_{m\hbar}(x))}.$$

Proof: Let $\mathbf{u}_t = \mathbf{K}(\mathbf{u})$, then

$$F'(\mathbf{u})[\mathbf{u}_t] \equiv \frac{dF(\mathbf{u})}{dt} = \int \sum_{i=0}^{\infty} \sum_{m\in\mathbb{Z}} \frac{\partial f[\mathbf{u}]}{\partial u_i(\sigma_{m\hbar}(x))} \frac{du_i(\sigma_{m\hbar}(x))}{dt} d_\hbar x = \int \sum_{i=0}^{\infty} \frac{\delta F}{\delta u_i}(u_i)_t d_\hbar x,$$

where the last equality follows from (2.8). □

Furthermore, we are interested in bivector fields on $\mathcal{U}$ defined through linear operators $\pi:\mathcal{V}^* \to \mathcal{V}$, which are matrices with coefficients from $\mathcal{C}[E,E^{-1}]$ multiplied by $1/\hbar$ in a local representation. An operator π is a Poisson operator (tensor) if the bilinear bracket

$$\{H,F\}_\pi = \langle dF, \pi dH\rangle, \quad F,H \in \mathcal{F}$$

is a Poisson bracket.

Remark 2.7: *It is important to mention that the particular choice of the algebra $\mathcal{C}$, and consequently the algebra $\mathcal{C}[E,E^{-1}]$, determines the class of discrete systems considered, which tends to differential systems of first order, i.e., dispersionless ones, as $\hbar \to 0$. Alternative approach for the construction of discrete systems on $\mathbb{R}$ with the grain structure $\mathbb{G}$ is based on the use of Δ-derivative, instead of the shift operator, given by*

$$\Delta u(x) := \frac{(E-1)u(x)}{(E-1)x} = \frac{u(\sigma_\hbar(x)) - u(x)}{\mu_\hbar(x)}, \quad \mu_\hbar(x) \equiv \sigma_\hbar(x) - x.$$

In this case, the algebra $\mathcal{C}$ is composed of polynomials in $\Delta^m u$ $(m=0,1,\ldots)$ and the operator algebra is given by $\mathcal{C}[\Delta]$. Consequently the restriction (2.8) on the functional is replaced by

$$\int' \Delta f[u] d_\hbar x = 0, \tag{2.14}$$

which entails that $\Delta^\dagger = -\Delta E^{-1}$ with respect to the duality map generated by this functional. Prime in $\int'$ is used to differentiate the functional satisfying property (2.14) from the functional satisfying property (2.8). Nevertheless, both functionals are interrelated by the relation

$$\int' (\cdot) d_\hbar x = \int (\cdot) \mu_\hbar(x) d_\hbar x,$$

which is a consequence of the restrictions imposed on them. Contrary to the previous case, the continuous limit of discrete systems from the alternative approach with Δ-operator gives dynamical field systems with dispersion and is not considered in this article.

III. *R*-MATRIX APPROACH TO INTEGRABLE DISCRETE SYSTEMS ON $\mathbb{R}$

The construction of integrable discrete systems following from the scheme of classical R-matrix formalism is parallel to the one used in the case of lattice soliton systems.[12,14,15]

On $\mathbb{R}$ with the grain structure $\mathbb{G}$ defined by some diffeomorphism $\sigma_\hbar$, we introduce the algebra of shift operators with finite highest order,

$$\mathfrak{g} = \mathfrak{g}_{\geq k-1} \oplus \mathfrak{g}_{<k-1} = \left\{ \sum_{i\geq k-1}^{N} u_i(x)\mathcal{E}^i \right\} \oplus \left\{ \sum_{i<k-1} u_i(x)\mathcal{E}^i \right\}, \tag{3.1}$$

where

$$\mathcal{E}^m u(x) = (E^m u)(x)\mathcal{E}^m \equiv u(\sigma_{m\hbar}(x))\mathcal{E}^m, \quad \sigma_{m\hbar} := \sigma_\hbar^m, \quad m \in \mathbb{Z}, \tag{3.2}$$

and $u_i(x)$ are smooth dynamical fields.

Proposition 3.1: *The multiplication operation on $\mathfrak{g}$ defined by (3.2) is noncommutative and associative.*

Proof: Noncommutativity is obvious. Associativity follows from straightforward calculation and from the fact that $\sigma_{m\hbar}$ is a one-parameter group of diffeomorphisms. □

The Lie structure on $\mathfrak{g}$ is introduced by the commutator

$$[A,B] = \frac{1}{\hbar}(AB - BA), \quad A,B \in \mathfrak{g}.$$

Subsets $\mathfrak{g}_{\geq k-1}$ and $\mathfrak{g}_{<k-1}$ of $\mathfrak{g}$ are Lie subalgebras only if $k=1$ and $k=2$. As a result, we define the classical R-matrices $R=P_{\geq k-1}-\frac{1}{2}$, by appropriate projections, and related Lax hierarchies,

$$L_{t_n} = [(L^n)_{\geq k-1}, L] = \pi_0 dH_n = \pi_1 dH_{n-1}, \quad n \in \mathbb{Z}_+, \quad k = 1,2, \tag{3.3}$$

of infinitely many mutually commuting systems. The evolution equations from (3.3) are generated by powers of appropriate Lax operators $L \in \mathfrak{g}$ of the form

$$k=1: \quad L = \mathcal{E} + u_0 + u_1\mathcal{E}^{-1} + u_2\mathcal{E}^{-2} + \cdots = \mathcal{E} + \sum_{i\geq 0} u_i \mathcal{E}^{-i}, \tag{3.4}$$

$$k=2: \quad L = u_0\mathcal{E} + u_1 + u_2\mathcal{E}^{-1} + u_3\mathcal{E}^{-2} + \cdots = \sum_{i\geq 0} u_i \mathcal{E}^{1-i}. \tag{3.5}$$

Then, the first chains from (3.3) are

$$(u_i)_{t_1} = \frac{1}{\hbar}[(E-1)u_{i+1} + u_i(1-E^{-i})u_0], \tag{3.6}$$

$$(u_i)_{t_2} = \frac{1}{\hbar}[(E^2-1)u_{i+2} + Eu_{i+1}(E+1)u_0 - u_{i+1}(E^{-i}+E^{-i-1})u_0 + u_i(1-E^{-i})u_0^2$$
$$+ u_i(E+1)(1-E^{-i})u_1]$$

$$\vdots$$

for $k=1$, and

$$(u_i)_{t_1} = \frac{1}{\hbar}[u_0 E u_{i+1} - u_{i+1}E^{-i}u_0],$$

$$(u_i)_{t_2} = \frac{1}{\hbar}[u_0 E u_0 E^2 u_{i+2} - u_{i+2}E^{-i-1}u_0E^{-i}u_0 + u_0(E+1)u_1Eu_{i+1} - u_{i+1}E^{-i}u_0(E^{1-i}+E^{-i})u_1]$$

$$\vdots$$

for $k=2$. Throughout this work, the shift operators E^m in the evolution equations and conserved quantities act only on the nearest field to the right and in Poisson operators act on everything to the right of the symbol E^m.

Example 3.2: *The lattice case:* $\mathcal{X}=1$. *Let* $\hbar=1$. *The first chains of the evolution equations from* (3.3) *have the forms*

$$k=1: \quad u_i(x)_{t_1} = u_{i+1}(x+1) - u_{i+1}(x) + u_i(x)(u_0(x) - u_0(x-i)),$$

$$k=2: \quad u_i(x)_{t_1} = u_0(x)u_{i+1}(x+1) - u_0(x-i)u_{i+1}(x).$$

These are Toda and modified Toda chains, respectively.

Example 3.3: *The q-discrete case:* $\mathcal{X}=x$ $(q\equiv e^{\hbar})$. *In this case the same evolution equations are*

$$k=1: \quad u_i(x)_{t_1} = u_{i+1}(qx) - u_{i+1}(x) + u_i(x)(u_0(x) - u_0(q^{-i}x)),$$

$$k=2: \quad u_i(x)_{t_1} = u_0(x)u_{i+1}(qx) - u_0(q^{-i}x)u_{i+1}(x),$$

where the constant factor $\hbar$ is absorbed into the evolution parameter t_1 through simple rescaling. These are q-deformed analogs of the chains from the previous example.

In this work we do not consider finite-field reductions of (3.3) as the procedure immediately follows from Refs. 12 and 15. To construct Hamiltonian structures for (3.3), one has to define an appropriate inner product on $\mathfrak{g}$.

Definition 3.4: *Let* $\mathrm{Tr}: \mathfrak{g} \to \mathbb{K}$ *be a trace form, being a linear map, such that*

$$\mathrm{Tr}(A) := \int \mathrm{res}(A\mathcal{E}^{-1}) d_\hbar x,$$

where $\mathrm{res}(A\mathcal{E}^{-1}) := a_0$ *for* $A = \Sigma_i a_i \mathcal{E}^i$. *Then, the bilinear map* $(\cdot,\cdot): \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ *defined as*

$$(A,B) := \mathrm{Tr}(AB) \tag{3.7}$$

is an inner product on $\mathfrak{g}$.

Proposition 3.5: *The inner product (3.7) is nondegenerate, symmetric, and* ad-*invariant, i.e.,*

$$([A,B],C) = (A,[B,C]), \quad A,B,C \in \mathfrak{g}.$$

Proof: The nondegeneracy of (3.7) is obvious. The symmetricity follows from (2.8). The ad-invariance is a consequence of the associativity of multiplication operation in $\mathfrak{g}$. □

Next, the differentials $dH(L)$ of functionals $H(L) \in \mathcal{F}(\mathfrak{g})$ for (3.4) and (3.5) have the forms

$$k=1: \quad dH = \sum_{i\geq 0} \mathcal{E}^i \frac{\delta H}{\delta u_i},$$

$$k=2: \quad dH = \sum_{i\geq 0} \mathcal{E}^{i-1} \frac{\delta H}{\delta u_i},$$

which follow from the assumption that the inner product on $\mathfrak{g}$ is compatible with (2.13), i.e.,

$$(dH, L_t) = \int \sum_{i=0}^{\infty} \frac{\delta H}{\delta u_i} (u_i)_t d_\hbar x.$$

Then, the bi-Hamiltonian structure of the Lax hierarchies (3.3) is defined by the compatible (for fixed k) Poisson tensors given by

$$k=1,2: \quad \pi_0 : dH \mapsto [L, (dH)_{<k-1}] + ([dH, L])_{<2-k}$$

and

$$k=1: \quad \pi_1 : dH \mapsto \frac{1}{2}([L, (L\,dH + dH\,L)_{<0}] + L([dH,L])_{<1} + ([dH,L])_{<1}L)$$
$$+ \hbar[(E+1)(E-1)^{-1} \,\mathrm{res}([dH,L]\mathcal{E}^{-1}), L],$$

$$k=2: \quad \pi_1 : dH \mapsto \frac{1}{2}([L, (L\,dH + dH\,L)_{<1}] + L([dH,L])_{<0} + ([dH,L])_{<0}L),$$

where the operation $(E-1)^{-1}$ is the formal inverse of $(E-1)$ and one can show that $(E+1)(E-1)^{-1} = \Sigma_{i=1}^{\infty}(E^{-i} - E^i)$. The appropriate Hamiltonians (conserved quantities) are

$$H_n(L) = \frac{1}{n+1}\mathrm{Tr}(L^{n+1}), \quad dH_n(L) = L^n,$$

and the explicit bi-Hamiltonian structure of (3.3) is given by

$$(u_i)_{t_n} = \sum_{j\geq 0} \pi_0^{ij}\frac{\delta H_n}{\delta u_j} = \sum_{j\geq 0} \pi_1^{ij}\frac{\delta H_{n-1}}{\delta u_j}, \quad i \geq 0.$$

The Poisson tensors for $k=1$ are

$$\pi_0^{ij} = \frac{1}{\hbar}[E^j u_{i+j} - u_{i+j}E^{-i}],$$

$$\pi_1^{ij} = \frac{1}{\hbar}\left[\sum_{k=0}^{i}(u_k E^{j-k}u_{i+j-k} - u_{i+j-k}E^{k-i}u_k + u_i(E^{j-k} - E^{-k})u_j) + u_i(1 - E^{j-i})u_j + E^{j+1}u_{i+j+1} - u_{i+j+1}E^{-i-1}\right],$$

together with the hierarchy of Hamiltonians in the forms

$$H_0 = \int u_0 d_\hbar x,$$

$$H_1 = \int \left(u_1 + \frac{1}{2}u_0^2\right)d_\hbar x,$$

$$H_2 = \int \left(u_2 + u_0(E+1)u_1 + \frac{1}{3}u_0^3\right)d_\hbar x,$$

$$\vdots$$

For $k=2$ the first Poisson tensor has the following form:

$$\pi_0^{10} = \frac{1}{\hbar}(1 - E^{-1})u_0, \quad \pi_0^{01} = \frac{1}{\hbar}u_0(E-1),$$

$$\pi_0^{ij} = \frac{1}{\hbar}[E^{j-1}u_{i+j-1} - u_{i+j-1}E^{1-i}], \quad i,j \geq 2,$$

with all remaining π_0^{ij} equal to zero, the second one is

$$\pi_1^{ij} = \frac{1}{\hbar}\left[\sum_{k=0}^{i-1}(u_k E^{j-k}u_{i+j-k} - u_{i+j-k}E^{k-i}u_k) + \frac{1}{2}u_i(E^{1-i} - 1)(E^{j-1} + 1)u_j\right],$$

and the first Hamiltonians are

$$H_0 = \int u_1 d_\hbar x,$$

$$H_1 = \int \left(\frac{1}{2}u_1^2 + u_0 E u_2\right)d_\hbar x,$$

$$H_2 = \int \left(\frac{1}{3}u_1^3 + u_0 E u_0 E^2 u_3 + u_0 u_1 E u_2 + u_0 E u_1 E u_2 \right) d_\hbar x,$$

$$\vdots \, .$$

IV. THE CONTINUOUS LIMIT

The aim of this section is to consider the limit of discrete systems (2.11) as $\hbar$ tends to 0. The class of discrete systems is determined by the choice of the algebra $\mathcal{C}$. Assume that the dynamical fields from $\mathcal{C}$ depend on $\hbar$ in such a way that the expansion, with respect to $\hbar$ near zero, is of the form

$$u_i(x) = u_i^{(0)}(x) + u_i^{(1)}(x)\hbar + O(\hbar^2),$$

i.e., u_i tends to $u_i^{(0)}$ as $\hbar \to 0$. In further considerations we use u_i instead of $u_i^{(0)}$. In the continuous limit $\mathcal{C}$ turns out to be the algebra of polynomial functions in $u_i(x)$, denoted by $\mathcal{C}_0$,

$$\mathcal{C}_0 \ni f(\mathbf{u}) \coloneqq \sum_{m \geq 0} \sum_{i_1,\ldots,i_m \geq 0} a^{i_1 i_2 \ldots i_m} u_{i_1}(x) u_{i_2}(x) \cdots u_{i_m}(x).$$

In general, the limit of discrete systems (2.11) does not have to exist. For the limit procedure, one should first expand the coefficients of $\mathbf{K}(\mathbf{u})$ into a Taylor series with respect to $\hbar$ near 0, i.e.,

$$E^m u = e^{m\hbar \mathcal{X}\partial_x} u = u + m\hbar \mathcal{X} u_x + \frac{m^2}{2}\hbar^2 (\mathcal{X}\mathcal{X}_x u_x + \mathcal{X}^2 u_{2x}) + O(\hbar^3).$$

Thus, the continuous limit of (2.11) exists only if zero order terms in $\hbar$ will mutually cancel in the above expansion. In this case, as $\hbar \to 0$, the discrete systems (2.11) tend to the systems of hydrodynamic type given in the following form:

$$\mathbf{u}_t = \mathcal{X}\mathbf{A}(\mathbf{u})\mathbf{u}_x, \tag{4.1}$$

where $\mathbf{A}(\mathbf{u})$ is the matrix with coefficients from $\mathcal{C}_0$, and the continuous limit is indeed the dispersionless limit.

Proposition 4.1: *Assume that the fields $u_i(x)$ vanish as $|x| \to \infty$ in the continuous limit. Then the functionals from Definition 2.4 are given by*

$$\int (\cdot) d_0 x : \mathcal{C}_0 \to \mathbb{K}, \quad f[\mathbf{u}] \mapsto F(\mathbf{u}) = \int f(\mathbf{u}) d_0 x = \int_{-\infty}^{\infty} f(\mathbf{u}(x)) \frac{dx}{\mathcal{X}(x)}. \tag{4.2}$$

Proof: For the continuous case (2.10) the proof is straightforward. In the case of discrete functionals (2.9), by the concept of Riemann integral construction, we have

$$\int f[\mathbf{u}] d_0 x \equiv \lim_{\hbar \to 0} \int f[\mathbf{u}] d_\hbar x = \lim_{\hbar \to 0} \sum_{n \in \mathbb{Z}} \hbar f[\mathbf{u}(\sigma_{n\hbar}(x))]$$
$$= \lim_{\hbar \to 0} \sum_{n \in \mathbb{Z}} f[\mathbf{u}(\sigma_{n\hbar}(x))] \left(\frac{\mu_\hbar(x)}{\hbar} \right)^{-1} \mu_\hbar(x) = \int_{-\infty}^{\infty} f(\mathbf{u}(x)) \frac{dx}{\mathcal{X}(x)}.$$

□

Thus, bivectors π are matrices with coefficients of the operator form $a\mathcal{X}\partial_x b$, where $a, b \in \mathcal{C}_0$. With respect to the duality map defined by the "dispersionless" functional (4.2), the adjoint of the operator ∂_x is given as

$$(\partial_x)^\dagger = \frac{\mathcal{X}_x}{\mathcal{X}} - \partial_x. \tag{4.3}$$

Consequently, the variational derivatives of functionals $F=\int f d_0 x=\int_{-\infty}^{\infty} f(dx/\mathcal{X})$ are given by the derivatives of densities f with respect to the fields u_i, i.e.,

$$\frac{\delta F}{\delta u_i} = \frac{\partial f}{\partial u_i}.$$

Example 4.2: *The dispersionless limit of the system* (3.6) *together with its Hamiltonian structure with respect to the first Poisson tensor is given by*

$$(u_i)_{t_1} = \mathcal{X}[(u_{i+1})_x + iu_i(u_0)_x] = \pi_0^{ij}\frac{\delta H_1}{\delta u_j}, \tag{4.4}$$

where

$$\pi_0^{ij} = j\mathcal{X}\partial_x u_{i+j} + iu_{i+j}\mathcal{X}\partial_x \quad \textit{and} \quad H_1 = \int \left(u_1 + \frac{1}{2}u_0^2\right) d_0 x.$$

The Hamiltonian representation of the systems (4.1) with the functional (4.2) follows directly from the continuous limit and leads to the nonstandard form with the adjoint operator of the differential operator given by (4.3). A more natural representation is the one with the components $\mathcal{X}(x)$ included in the densities of functionals given in the standard form

$$F(\mathbf{u}) = \int_{-\infty}^{\infty} \mathcal{X}(x)^{-1} f(\mathbf{u}(x))dx \equiv \int_{-\infty}^{\infty} \varphi(\mathbf{u}(x))dx,$$

for which the variational derivatives preserve the form $\delta F/\delta u_i = \partial\varphi/\partial u_i$. As a consequence, bivectors π from the previous representation must be multiplied on the right-hand side by $\mathcal{X}$. Now, the adjoint of the operator ∂_x takes the standard form $(\partial_x)^\dagger = -\partial_x$. Therefore, in what follows we use only the natural Hamiltonian representation of dispersionless systems (4.1).

Example 4.3: *The natural Hamiltonian structure of* (4.4) *is given by*

$$\pi_0^{ij} = j\mathcal{X}\partial_x\mathcal{X}u_{i+j} + iu_{i+j}\mathcal{X}\partial_x\mathcal{X} \quad \textit{and} \quad H_1 = \int_{-\infty}^{\infty} \mathcal{X}^{-1}\left(u_1 + \frac{1}{2}u_0^2\right)dx.$$

In the next section we consider the R-matrix formalism of the dispersionless systems (4.1) that can be considered as the continuous limit of the formalism presented in Sec. III.

V. *R*-MATRIX APPROACH TO INTEGRABLE DISPERSIONLESS SYSTEMS ON $\mathbb{R}$

The theory of classical R-matrices on commutative algebras, with the multi-Hamiltonian formalism, was given in Ref. 17. Here we follow the particular scheme of R-matrix parallel to the one developed in Refs. 18 and 19.

Let us consider the algebra of polynomials in p with the finite highest order,

$$\mathcal{A} = \mathcal{A}_{\geq k-1} \oplus \mathcal{A}_{<k-1} = \left\{\sum_{i\geq k-1}^{N} u_i(x)p^i\right\} \oplus \left\{\sum_{i<k-1} u_i(x)p^i\right\}, \tag{5.1}$$

equipped with the Lie structure induced by the Poisson bracket in the form

$$\{f,g\} := p\mathcal{X}(x)\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}\right), \quad f,g \in \mathcal{A}. \tag{5.2}$$

Subsets $\mathcal{A}_{\geq k-1}$ and $\mathcal{A}_{<k-1}$ of $\mathcal{A}$ are Lie subalgebras only if $k=1$ and $k=2$. Thus, the classical R-matrices $R=P_{\geq k-1}-\frac{1}{2}$ determine the Lax hierarchies,

$$L_{t_n} = \{(L^n)_{\geq k-1}, L\} = \pi_0 dH_n = \pi_1 dH_{n-1}, \quad n \in \mathbb{Z}_+, \quad k=1,2, \tag{5.3}$$

that are generated by powers of the Lax functions $L \in \mathcal{A}$ given in the forms

$$k=1: \quad L = p + u_0 + u_1 p^{-1} + u_2 p^{-2} + \cdots = p + \sum_{i\geq 0} u_i p^{-i}, \tag{5.4}$$

$$k=2: \quad L = u_0 p + u_1 + u_2 p^{-1} + u_3 p^{-2} + \cdots = \sum_{i\geq 0} u_i p^{1-i}. \tag{5.5}$$

The first dispersionless chains from (5.3) take the following form for $k=1$:

$$\begin{gathered}(u_i)_{t_1} = \mathcal{X}[(u_{i+1})_x + iu_i(u_0)_x],\\ (u_i)_{t_2} = 2\mathcal{X}[(u_{i+2})_x + u_0(u_{i+1})_x + (i+1)u_{i+1}(u_0)_x + iu_iu_0(u_0)_x + iu_i(u_1)_x],\\ \vdots\end{gathered} \tag{5.6}$$

and for $k=2$,

$$\begin{gathered}(u_i)_{t_1} = \mathcal{X}[u_0(u_{i+1})_x + iu_{i+1}(u_0)_x],\\ (u_i)_{t_2} = 2\mathcal{X}[u_0^2(u_{i+2})_x + (i+1)u_0u_{i+2}(u_0)_x + u_0u_1(u_{i+1})_x + iu_{i+1}(u_0u_1)_x],\\ \vdots .\end{gathered} \tag{5.7}$$

Example 5.1: *For $\mathcal{X}=1$ chains (5.6) and (5.7) are dispersionless Toda and modified Toda chains, respectively, while for $\mathcal{X}=x$ chains (5.6) and (5.7) are dispersionless limits of the q-analogs of Toda and modified Toda.*

The appropriate trace form is defined as

$$\mathrm{Tr}(A) := \int_{-\infty}^{\infty} \mathcal{X}^{-1}\mathrm{res}(Ap^{-1})dx,$$

where $\mathrm{res}(A):=a_{-1}$ for $A=\Sigma_i a_i p^i$, and the inner product on $\mathcal{A}$ is given by

$$(A,B) := \mathrm{Tr}(AB).$$

Proposition 5.2: *The above inner product is nondegenerate, symmetric, and* ad*-invariant with respect to the Poisson bracket, i.e.,*

$$(\{A,B\},C) = (A,\{B,C\}), \quad A,B,C \in \mathcal{A}.$$

Proof: The nondegeneracy and symmetricity is obvious. The ad-invariance is a consequence of the following equality: $\mathrm{Tr}\{A,B\}=0$, which is valid for arbitrary $A,B \in \mathcal{A}$. □

Then, the differentials $dH(L)$ of functionals $H(L) \in \mathcal{F}(\mathcal{A})$ related to the Lax functions (5.4) and (5.5) have the forms

$$k=1: \quad dH = \mathcal{X}\sum_{i\geq 0}\frac{\delta H}{\delta u_i}p^i,$$

$$k=2: \quad dH = \mathcal{X}\sum_{i\geq 0}\frac{\delta H}{\delta u_i}p^{i-1}.$$

The bi-Hamiltonian structure of the Lax hierarchies (3.3) is defined through the compatible (for fixed k) Poisson tensors

$$k=1,2: \quad \pi_0{:}dH \mapsto \{L,(dH)_{<k-1}\} + (\{dH,L\})_{<2-k},$$

and

$$k=1: \quad \pi_1{:}dH \mapsto \{L,(dHL)_{<0}\} + L(\{dH,L\})_{<1} + \{\partial_x^{-1}\mathrm{res}(\mathcal{X}^{-1}p^{-1}\{dH,L\}),L\},$$

$$k=2: \quad \pi_1{:}dH \mapsto \{L,(dHL)_{<1}\} + L(\{dH,L\})_{<0}.$$

Then, for Hamiltonians

$$H_n(L) = \frac{1}{n+1}\mathrm{Tr}(L^{n+1}), \quad dH_n(L) = L^n,$$

the explicit bi-Hamiltonian structure of (3.3) is given by

$$(u_i)_{t_n} = \sum_{j\geq 0}\pi_0^{ij}\frac{\delta H_n}{\delta u_j} = \sum_{j\geq 0}\pi_1^{ij}\frac{\delta H_{n-1}}{\delta u_j}, \quad i\geq 0.$$

So, the Poisson tensors for $k=1$ are given by

$$\pi_0^{ij} = \mathcal{X}[j\partial_x u_{i+j} + iu_{i+j}\partial_x]\mathcal{X},$$

$$\pi_1^{ij} = \mathcal{X}\Bigg[\sum_{k=0}^{i}((j-k)u_k\partial_x u_{i+j-k} + (i-k)u_{i+j-k}\partial_x u_k) + i(j+1)u_i\partial_x u_j + (j+1)\partial_x u_{i+j+1} + (i+1)u_{i+j+1}\partial_x\Bigg]\mathcal{X},$$

where the related Hamiltonians are

$$H_0 = \int_{-\infty}^{\infty}\mathcal{X}^{-1}u_0dx,$$

$$H_1 = \int_{-\infty}^{\infty}\mathcal{X}^{-1}\left(u_1 + \frac{1}{2}u_0^2\right)dx,$$

$$H_2 = \int_{-\infty}^{\infty}\mathcal{X}^{-1}\left(u_2 + 2u_0u_1 + \frac{1}{3}u_0^3\right)dx,$$

$$\vdots.$$

For $k=2$ we have the first Poisson tensor

$$\pi_0^{10} = \mathcal{X}\partial_x\mathcal{X}u_0, \quad \pi_0^{01} = u_0\mathcal{X}\partial_x\mathcal{X},$$

$$\pi_0^{ij} = \mathcal{X}[(j-1)\partial_x u_{i+j-1} + (i-1)u_{i+j-1}\partial_x]\mathcal{X}, \quad i,j \geq 2,$$

where all remaining π_0^{ij} are equal to zero, and the second one is as follows:

$$\pi_1^{ij} = \mathcal{X}\left[\sum_{k=0}^{i-1}((j-k)u_k\partial_x u_{i+j-k} + (i-k)u_{i+j-k}\partial_x u_k) + (1-i)u_i\partial_x u_j\right]\mathcal{X}.$$

Finally the related Hamiltonians are

$$H_0 = \int_{-\infty}^{\infty} \mathcal{X}^{-1}u_1 dx,$$

$$H_1 = \int_{-\infty}^{\infty} \mathcal{X}^{-1}\left(\frac{1}{2}u_1^2 + u_0u_2\right)dx,$$

$$H_2 = \int_{-\infty}^{\infty} \mathcal{X}^{-1}\left(\frac{1}{3}u_1^3 + u_0^2u_3 + 2u_0u_1u_2\right)dx,$$

$$\vdots$$

One can observe that the chains, together with the bi-Hamiltonian structures, constructed in this section are dispersionless limits of the discrete chains considered in Sec. III.

VI. DEFORMATION QUANTIZATION PROCEDURE

The aim of this section is to formulate the inverse procedure of the dispersionless limit considered earlier. The quantization deformation formalism (for the references see Ref. 15) which is the unified approach to the lattice and field soliton systems was presented in Ref. 15. Here we follow the scheme from that article.

The Poisson bracket (5.2) can be written in the form

$$\{f,g\} \coloneqq f(p\partial_p \wedge \mathcal{X}(x)\partial_x)g \quad f,g \in \mathcal{A},$$

where the derivations $p\partial_p$ and $\mathcal{X}(x)\partial_x$ commute. Hence, it can be quantized in infinitely many ways via $\star$-products being deformed multiplications

$$f\star^{\alpha}g = f\exp\left[\frac{\hbar}{2}((\alpha+1)p\partial_p \otimes \mathcal{X}(x)\partial_x + (\alpha-1)\mathcal{X}(x)\partial_x \otimes p\partial_p)\right]g. \tag{6.1}$$

This $\star$-product for $\alpha=0$ and $\alpha=1$ is the generalization of the Moyal and Kuperschmidt–Manin products, respectively. Expanding (6.1) one finds that

$$f\star^{\alpha}g = \sum_{k=0}^{\infty}\frac{\hbar^k}{2^k k!}\sum_{j=0}^{k}(\alpha+1)^{k-j}(\alpha-1)^j[(p\partial_p)^{k-j}(\mathcal{X}\partial_x)^j f]\cdot[(\mathcal{X}\partial_x)^{k-j}(p\partial_p)^j g]. \tag{6.2}$$

Algebra $\mathcal{A}$ (5.1) with the multiplication defined by (6.1), with a fixed α, is an associative, but not commutative, algebra with the following Lie bracket, being a deformed Poisson bracket:

$$\{f,g\}_{\star^\alpha} = \frac{1}{\hbar}(f\star^{\alpha}g - g\star^{\alpha}f). \tag{6.3}$$

Then, as $\hbar \to 0$, we have

$$\lim_{\hbar\to 0} f\star^{\alpha} g = fg,$$

$$\lim_{\hbar\to 0}\{f,g\}_{\star^{\alpha}} = \{f,g\}.$$

Algebra $\mathcal{A}$ with $\star^{\alpha}$-product will be denoted as $\mathcal{A}_{\alpha}$.

The associativity property of $\star^{\alpha}$-products is a purely algebraic consequence of the construction. For the simple proof, see Ref. 15. Moreover, we could treat these products only formally not requiring a convergence of the sum in (6.2). In order to make the $\star^{\alpha}$-products consistent with the introduced formalism of grain structures, we assume that vector fields $\mathcal{X}\partial_x$ are such that formula (2.5) is valid, i.e., $\mathcal{X}\partial_x$ is complete or it generates well defined discrete one-parameter group of diffeomorphisms. From the simple observation

$$(p\partial_p)^k p^m = m^k p^m,$$

one finds that

$$p^m \star^{\alpha} u(x) = \sum_{k=0}^{\infty} \frac{\hbar^k}{2^k k!}(\alpha+1)^k m^k (\mathcal{X}\partial_x)^k u(x) p^m = e^{m(\alpha+1)(\hbar/2)\mathcal{X}\partial_x} u(x) p^m = E^{m[(\alpha+1)/2]} u(x) p^m,$$

$$u(x) \star^{\alpha} p^m = \sum_{k=0}^{\infty} \frac{\hbar^k}{2^k k!}(\alpha-1)^k m^k (\mathcal{X}\partial_x)^k u(x) p^m = e^{m(\alpha-1)(\hbar/2)\mathcal{X}\partial_x} u(x) p^m = E^{m[(\alpha-1)/2]} u(x) p^m,$$

where the last equalities follow from (2.6).

Note that the decomposition of (5.1) into Lie subalgebras is still preserved after deformation quantization and they are Lie subalgebras with respect to the Lie bracket (6.3). Hence, we have Lax hierarchies

$$L_{t_n} = \{(L^n)_{\geq k-1}, L\}_{\star^{\alpha}}, \quad n \in \mathbb{Z}_+, \quad k = 1,2, \tag{6.4}$$

which are well defined for Lax functions in the form of (5.4) and (5.5). Notice that the Lax hierarchies are generated by powers with respect to $\star^{\alpha}$-products, i.e., $L^n = L\star^{\alpha}\cdots\star^{\alpha}L$. The first chains from Lax hierarchies (6.4) are

$$k=1: \quad (u_i)_{t_1} = \frac{1}{\hbar}[(E-1)E^{(\alpha-1)/2}u_{i+1} + u_i(1-E^{-i})E^{i[(1-\alpha)/2]}u_0],$$

$$k=2: \quad (u_i)_{t_1} = \frac{1}{\hbar}[E^{i[(1-\alpha)/2]}u_0 E^{(\alpha+1)/2}u_{i+1} - E^{(\alpha-1)/2}u_{i+1}E^{-i[(\alpha+1)/2]}u_0].$$

One can observe that they coincide with the respective discrete systems for $\alpha=1$.

Nevertheless, all algebras $\mathcal{A}_{\alpha}$ are gauge equivalent under the isomorphism

$$D^{\alpha'-\alpha}: \mathcal{A}_{\alpha} \to \mathcal{A}_{\alpha'}, \quad D^{\alpha'-\alpha} = \exp\left[(\alpha-\alpha')\frac{\hbar}{2}\mathcal{X}(x)\partial_x p\partial_p\right],$$

such that

$$f\star^{\alpha'} g = D^{\alpha'-\alpha}[D^{\alpha-\alpha'} f \star^{\alpha} D^{\alpha-\alpha'} g],$$

$$\{f,g\}_{\star^{\alpha'}} = D^{\alpha'-\alpha}\{D^{\alpha-\alpha'}f, D^{\alpha-\alpha'}g\}_{\star^{\alpha}}.$$

It is also straightforward to prove that under the above isomorphism, the Lax hierarchy structure is preserved. Let $L_{\alpha}=\Sigma_i u_i p^i \in \mathcal{A}_{\alpha}$ and $L_{\alpha'}=\Sigma_i u_i' p^i \in \mathcal{A}_{\alpha'}$. Then, the transformation between fields is as follows:

$$L_{\alpha'} = D^{\alpha'-\alpha}L_{\alpha} \quad \Rightarrow \quad u_i' = E^{i\alpha-\alpha'/2}u_i.$$

On the other hand, (6.1) implies the following commutation rules:

$$u \star v = uv,$$

$$p^m \star p^n = p^{m+n},$$

$$p^m \star u = (e^{m\hbar\mathcal{X}\partial_x}u) \star p^m = E^m u \star p^m,$$

$$u \star p^m = p^m \star (e^{-m\hbar\mathcal{X}\partial_x}u) = p^m \star E^{-m}u,$$

which are independent of the choice of $\star^{\alpha}$-product. Therefore, we skip the related index. Hence, we can quantize algebra $\mathcal{A}$ to the following algebra separately:[24]

$$\mathfrak{a} = \left\{\sum_i u_i \star p^i\right\},$$

which is obviously associative under the above commutation rules. Notice that algebra $\mathfrak{a}$ differs from algebras $\mathfrak{g}_{\alpha}$ as in $\mathfrak{a}$ we also deform the polynomial functions, i.e., we are not using the standard multiplication anymore. Notice that algebra $\mathfrak{a}$ is trivially equivalent to algebra $\mathcal{A}_1$ as $u\star^1 p^m = up^m$ and $p^m \star^1 u = E^m u p^m$. Also, it is straightforward to see that $\mathfrak{a}$ is isomorphic to the algebra of shift operators $\mathfrak{g}$ (3.1) defined on the grain structure by some discrete one-parameter group of diffeomorphisms on $\mathbb{R}$. Hence, it is clear that algebra (5.1) with Poisson bracket (5.2) is the limit of algebra (3.1) of shift operators with the Lie structure defined by the commutator as $\hbar \to 0$.

VII. CONCLUSIONS

In the present article, we have introduced a general framework of integrable discrete systems on $\mathbb{R}$. This formalism is based on the construction of shift operators by means of discrete one-parameter groups of diffeomorphisms on $\mathbb{R}$, which are determined by infinitesimal generators $\mathcal{X}\partial_x$. Particularly, if $\mathcal{X}=1$ or $\mathcal{X}=x$ the related discrete systems are of lattice Toda or q-deformed Toda type, respectively. All integrable discrete systems defined by different vector fields $\mathcal{X}(x)\partial_x$ are not equivalent in the sense that these vector fields are not globally equivalent. Nevertheless, one can find a local transformation relating respective vector fields.

Consider the vector fields from Example 2.3. Let $\mathcal{X}(x)=x^{1-n}$ for odd $n\neq 0$ and $\mathcal{X}'(x')=1$ (the lattice case). Then one finds that $x'=(1/n)x^n$ is a bijection on $\mathbb{R}\backslash\{0\}$. Hence, all discrete systems generated by $\mathcal{X}\partial_x=x^{1-n}\partial_x$, with odd n, can be reduced to the original lattice Toda-type systems, excluding the point $x=0$. For $n=0$, $\mathcal{X}(x)=x$ (the q-discrete case) and let $\mathcal{X}'(x')=1$. Then we have $x=e^{x'}$, which is not a bijection. However, if the domain of dynamical fields of q-discrete systems is restricted to $x\in\mathbb{R}_+$, then the above map is a bijection and q-discrete systems on $\mathbb{R}_+$ became equivalent to the lattice systems on $\mathbb{R}$.

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