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Joint inventory and constant price decisions for a continuous review system

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Abstract

Purpose – The purpose of this paper is to study joint inventory and pricing strategy for a continuous inventory review system. While dynamic pricing decisions are often studied in the literature along with inventory management, the authors' aim in this study is to obtain a single long-run optimal price; also to gain insight about how to obtain the optimal price and inventory control variables simultaneously and then the benefits of joint optimization of the inventory and pricing decisions over the sequential optimization policy often followed in practice.

Design/methodology/approach – A general (*R;Q*) policy system with fixed cost of ordering is modelled and then the case where unsatisfied demand is lost is studied. General forms of both the additive and multiplicative demand models are used to obtain structural results.

Findings – By showing optimality conditions on the price and inventory decision variables, two algorithms on how to obtain optimal decision variables, one for additive and another for multiplicative demand-price model are provided. Through extensive numerical analyses, the potential profit increases are reported if the price and inventory problem are solved simultaneously instead of sequentially. In addition, the sensitivities of optimal decision variables to system parameters are revealed.

Practical implications – Although there are several studies in the literature investigating emergency price change models, they use arbitrary exogenous prices menus. However, the value of a price change can be better appreciated if the long-run price is optimal for the system.

Originality/value – Very few researchers have investigated constant price and inventory optimization, and while there are several past studies demonstrating the benefits of dynamic pricing over a static one, there still are not many findings on the benefit of joint price and inventory optimization.

Keywords Inventory control, Pricing, Demand model, Optimal pricing, *R;Q* policy, Lost sales, Sequential optimization

Paper type Research paper

1. Introduction

Retail replenishment is a high value activity and as such according to the US Commerce Department (2004), 1.1 trillion US dollars in inventory supports 3.2 trillion US dollars in annual US retail sales. This inventory is spread out across the value chain, with 400 billion US dollars at retail locations. Firms are now sourcing up to 75% of the value of goods and services globally and uncertainty of demand creates longer safety lead times resulting in high inventory value. Demand conditions are such that it is difficult to meet supply chain expectations as either some supply chain member will be required to expedite shipments (high cost) or hold high levels of inventory

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(Farris and Hutchison, 2002; Hoffman, 2009). High levels of stock adversely affect profitability (Bhatnagar and Teo, 2009).

To coordinate demand requests, transportation and inventory management utilize the benefits of strategic supply chain tools such as information technology to assist in keeping inventory lower and ordering more efficient (Auramo *et al.*, 2005). This real-time information in regard to inventory levels throughout the supply chain assists in lowering the costs of back orders, lost orders, and obsolescence (Yao *et al.*, 2007; Lumsden and Mirzabeiki, 2008). However, inventories at retailers actually have not decreased in time (Chen *et al.*, 2007). Also, Gaur *et al.* (2005) use firm-level data and conclude that inventory turnover has a downward sloping trend between 1987-2000 for US retailers. With such a large stockpile of inventory, it should be expected that stockouts at the retail level should be very low, but research suggests that it is not the case. Global studies have shown that stockouts occur at 8.3% of all retail sales worldwide (Gruen *et al.*, 2002). Research studying 71,000 customers worldwide concluded that customers lose patience with stockouts. Only 15% of the customers will delay the purchase to another time until the item is back in stock. Even after recouping some of the loss with sales of alternative products, retailers will still suffer about 4% of sales due to stockouts (Gruen *et al.*, 2002). Thistakes an enormous toll on retail margins. Rapidly increasing product variety (Hoole, 2006) with long lead times due to sourcing from overseas to assure low cost (Quint and Shorten, 2005) enhance the difficulty of aligning the supply with the demand.

To cope with the increasing uncertainty both on demand and supply sides, it becomes a crucial requirement to make demand and supply decisions through the cooperation of marketing and operations managers. Traditionally, demand management is theresponsibility of marketing managers who estimate demand determinants such as pricing, promotion, and advertising. To anticipate pricing, demand forecasts have become sophisticated, though still flawed due to the sheer number of products in a retailer's store (the US typical grocery store carries up to 31,000 items (Kahn and McDonough, 1997))(Srinivasan *et al.*, 2008). Pricing appropriately for retailers is so complicated that often they do not adapt prices based on demand conditions which lead to past price dependency and lower margins (Nijs *et al.*, 2007).

On the other side, supply management (which includes supplier selection, contracting, and quality and inventory control) is the responsibility of operations managers. Due to global supply chains, inventory interactions often involve many differentfirms with long product replenishment times and inventory imbalances (Bhatnagar and Teo, 2009). The supply chain focus in today's marketplace is increasingly important, as for example, the US imported 1.48 trillion in 2004 (CIA, 2004) and Wal-Mart would be China's eight largest trading partner if it was a nation (Jiang, 2004).

The combination of both Marketing and Operations management (demand and supply) are by two different departments in a firm and the difficulty of coordinated decision making and possible benefits of such collaboration is of interest to researchers. When coordinated effectively these interdepartmental relations will contribute greatly to supply chain effectiveness (Kim *et al.*, 2006). Various marketing and operations conflicts have been studied in literature including quality versusprice (Balasubramanian and Bhardwaj, 2004), lead time versus price (Pekgün *et al.*, 2008), product variety versus production flexibility (De Groote, 1994), advertising versus inventory control (Khouja and Robbins, 2003), and price versus inventory control decision making (Li and Atkins, 2002; Yin and Rajaram, 2007), which is main question area of this study. For a review on marketing and operations collaboration issues, see Eliashberg and Steinberg (1993) and Tang (2010).

The incompatibility between pricing and inventory decisions can lead to important profit losses, while a good match has promising profit improvements. For example, in new market entry, the simple cost-plus strategy (which does not take inventory into consideration) has accounted for many business failures. Following the earthquake on March 11, 2011 in Japan, for certain models of Japanese cars, US dealers realized shortages by the beginning of the spring 2011 and expected even more in summer months with the significant losses in production capacities. To keep up with the decreases in supply in upcoming months, dealers aimed "to boost prices on Japanese cars and trucks by an average of about $400 a vehicle", which may decrease demand, but would increase profits (Boudette, 2011).

Research conducted among more than 11,000 euro-area companies, suggested half of the companies do not use an information set that includes future expectations and costs in their price-setting and rather base their decisions on rule of thumb or past experiences (Fabiani *et al.*, 2006). Moreover, firms in the survey change their price once a year, which indicates the use of constant prices rather than dynamic pricing. Firms' stickiness to constant prices is explained mainly by the implicit and explicit contracts with the customers. Thus, every company should first incorporate future operational concerns such as inventory planning into its price-setting process to obtain its optimal constant market-price, then any complex price strategies can be considered building on these information.

In this study, joint inventory and pricing decisions are considered in a continuous review inventory replenishment system, where the orders are triggered whenever the inventory level drops below a certain level. Continuous review policy models are widely used in practice. Their popularity is supported by the existence of supplier-buyer contracts as a supplier would prefer to supply a fixed quantity for each order instead of arbitrary amounts (Urban, 2000). Orders arriveafter a significant lead time and most unsatisfied demand is lost. Customers arrive according to a general price dependent demand function, where a higher price leads to lower demand arrival rate. Our first interest is to gain insights about how to obtain optimal inventory and constant price variables simultaneously.

Our second purpose is to investigate the benefits that can be obtained by making the price decision along with the inventory control decisions. While there are several past studies demonstrating the benefits of dynamic pricing over a static one, there still are not many findings on the benefit of joint price and inventory optimization.

Obtaining the optimal constant price jointly with inventory policy is also important to evaluate the benefit of dynamic pricing policies. To reveal the correct benefit of a price change decision, the base price considered should be the optimal constant price of the system. So any improvement in system payoff after price change can be devoted to the price change policy due to the changing needs of the system. Otherwise, if the base price is not already optimal for the base system setting, a price change may improve system payoff not because of being a better fit for changing system conditions, but the new price can be more closer to the optimal base price. For example, studies investigating the profit increases of a supplier due to the price discounts offered (Cheung, 1998; Klastorin *et al.*, 2002) would provide better results when the price before the discount would be taken as the optimal price for the system.

There are a quite large number of studies on joint pricing and inventory decisions in the literature. The earliest work on integrating inventory control was with the endogenous demand model by Whitin (1955). This study and a number of later research focus on non-perishable and replenishable products, which does not have have narrow life span like perishable goods and unsold units can be carried between consecutive replenishment cycles. Some further research incorporated pricing decisions also with the use of limited perishable inventories termed revenue management (Gallego and van Ryzin (1994). Our study focuses on the stream of literature in regard to replenishable inventory.

Very few constant price and inventory optimization studies have followed Whitin (1955) (Chan *et al.*, 2004). Petruzzi and Dada (1999) study the constant price and inventory decisions in a newsboy setting. Kunreuther and Schrage (1973), Gilbert (1999) and Gilbert (2000) consider demand as a deterministic function of single price, which is to be determined along with ordering quantities in multiple periods, with additive demand, multiplicative demand, and multiplicative demand with multiple products sharing a common capacity, respectively.

Previous studies on inventory-pricing coordination have mainly focused on joint inventory control and dynamic pricing, where pricing decisions are made either after every customer arrival or at every replenishment epochs. Some of these dynamic pricing problems are studied with periodic review systems (Federgruen and Heching, 1999; Chen and Simchi-Levi, 2004) and some with continuous review systems (Chen and Simchi-Levi, 2006; Gayon *et al.*, 2009). For complete reviews on joint dynamic pricing and inventory control problems, see Chan *et al.* (2004), Yano and Gilbert (2005), Gimpl-Heersink *et al.* (2008) and Chen and Simchi-Levi (2010).

Although the research on joint control of inventories and dynamic pricing has promising benefits, several studies show that only a few good prices are enough to capture most of these benefits. Chen *et al.* (2010) and Gayon *et al.* (2009) show that most of the benefit of multiple pricing is reached by using only two prices. The optimal constant price decision along with the optimal inventory decisions should be defined initially. Although optimal single price might be obtained asa special case of dynamic pricing in previous studies, to our knowledge, there is no model in the literature that is similar to our study. For example, Chen and Simchi-Levi (2006) and Feng and Chen (2003) state the challenges in modeling positive lead time, so they do not include it.

There are several past studies that can be more closely related to our work than others. Guan and Zhao (2011) study joint constant price and inventory decisions of multiple retailers under both the centralized and decentralized settingswith Poisson demand and backlogged model. Chao and Zhou (2006) investigate joint dynamic pricing and inventory strategy for a continuous review system. They assume that the lead time is zero, unsatisfied demand can be backlogged, and demand has Poisson distribution. Chen *et al.* (2010) show the benefits of dynamic pricing in a continuous review system where replenishments can be done instantaneously when the inventory level drops to zero. So no stocking-out is allowed.

This paper is organized as follows. In §2, the demand function is first introduced and then the objective function is developed. In §3 and §4, respectively, the optimality properties of systems with additive and multiplicative demands are obtained. In §5, numerical studies on the expected performance of the joint price and inventory optimization over sequential optimization and also sensitivity of optimal decision variables to system parameters are reported. §6 concludes the paper by summarizing the results and further research areas.

2. Model framework

This study's focus is a continuously reviewed inventory replenishment model. The replenishment policy has the form such as whenever the inventory level hits R, an order of Q units is placed. After the placement of a regular order, it takes L time units to receive the order. This is the well-known (R, Q) policy.

The time of a replenishment decision is called a reorder point and the arrival of an order is called a regeneration point. Referring to renewal theory, the system goes through one replenishment cycle between the placement of two successive orders where the inventory level regenerates itself from the inventory level R to R during a cycle. Unlike the case for deterministic demands, the system might not exactly repeat itself within each cycle. Even the length of the cycle is now a random variable. However, the system does repeat itself in the sense that the inventory level returns back to same point after a random cycle time.

A price sensitive, stochastic demand model is used, where the demand at any point depends on price p charged. Let $D(p, t)$ be the demand during any time interval t, when price p is charged and the demands realized in consecutive time units are independent. Randomness in demand is defined to be price independent and the total demand is composed of a price dependent deterministic portion and the random demand portion. Then the demand per unit time $D(p, 1)$ is a function of the deterministic portion $y(p)$ and random factor ε, which is a non-negative, continuous random variable defined on the range $[A, B]$ and distributed with the probability density function $f(\cdot)$, the cumulative function $F(\cdot)$, expected value $E(\varepsilon) = \mu$, and the standard deviation σ. The mean of $D(p,1)$ is denoted by $v(p)$. The random and deterministic demand effects can combined in additive or multiplicative form, which are detailed in §3 and S 4, respectively.

The objective is to maximize the long-run average profit of the firm, which is composed of the revenue obtained from sales, fixed cost K charged per ordering, variable cost c paid per unit ordered, holding cost h incurred per unit kept in stock per unit time, and the cost of unsatisfied, which is lost in this study, sales b incurred per unit lost. Although, a profit maximization model implicitly incorporates the cost of a lost sales as the profit lost from the available sales opportunity, the cost of a lost sales may be greater than the per unit profit. It may include the goodwill loss, as well as the negative effects on future sales. Thus, the profit maximization studies that explicitly define a lost sales cost such as Petruzzi and Dada (1999), Chen *et al.* (2006) and Serel (2009) are followed. The important notation used throughout the paper is summarized in Table I.

A cost and revenue calculation approach is used, which is similar to the seminal work of Hadley and Whitin (1963) for the simple (R, Q) model. It is a well-known inventory estimation approach where stockout probability is assumed to be sufficiently small. Low stockout probability is clearly supported by our model, where pricing decision aims to prevent stockouts. The exact computation can be done fairly more easily for Poisson demand with backorders, but very complicated for other distributions and lost sales case. It is easier to work with backorder model, because when demands arrive in single units, the inventory position is uniformly distributed between R and $R + Q$. When the unsatisfied demand can be backordered, the inventory position change between two states is independent of the current inventory state. On the other hand, when unsatisfied

Parameters	
K	Fixed ordering cost per order
C	Purchasing cost per unit
h	Inventory holding cost per unit per time unit
B	Shortage cost per unit demand lost
L	Lead time for an order
$D(p,t)$	Demand during a time interval t, when price p is charged
$\nu(p)$	Mean demand during a unit time interval
$y(p)$	Price dependent deterministic portion of the demand during unit time interval
ε	Random portion of the demand per unit time interval distributed with density $f(\cdot)$ cumulative function $F(\cdot)$, mean μ, and standard deviation σ
$\bar{F}(\cdot)$	Complement of cumulative distribution function $F(\cdot)$, i.e. $\bar{F}(\cdot) = 1 - F(\cdot)$
Variables	
Q	Order quantity
R	Reorder level
p	A constant selling price per unit

Table I. Notation for the formulations

demand is lost, between two demand arrivals, the inventory position may not change, if the system is already stocked-out (Sezen, 2006). Then the change in inventory position after a demand arrival depends on whether the inventory level is positive or zero as shown by Figure 1. Therefore, lost sales models are more scarce in the literature and using the approximate payoff calculationof Hadley and Whitin (1963) is quite reasonable for this model. For a detailed discussion on lost sales model for a continuous review inventory system, see Bijvank and Vis (2011). Hadley and Whitin (1963)'s approximate average profit treatment for a continuous review inventory model is highly common in the literature (Moinzadeh and Nahmias, 1988; Cheung, 1998; Johansen and Thorstenson, 1998; Tekin *et al.*, 2001; Durán *et al.*, 2004). Also, it is shown by Lau and Lau (2002) that Hadley-Whitin's average inventory computing method is quite robust and often more accurate than alternatives suggested in the literature.

At the outset, the assumption is that there is only a single order outstanding at any time. When the unmet demand is lost, the number of outstanding orders is the largest integer less than or equal to $(Q + R)/Q = 1 + R/Q$ (Hadley and Whitin, 1963).

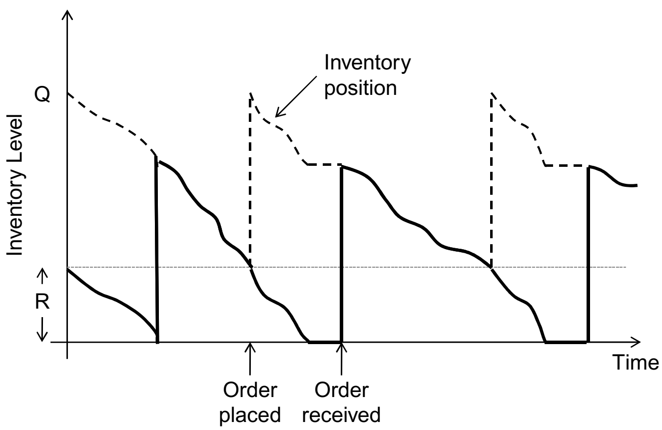

Figure 1. The change in inventory position and level in time in a continuous review system with lost sales

Then, if itis assumed that $R < Q$, it is always true that there is only a single order outstanding. Single outstanding order assumption is extensively utilized in the literature especially with the lost sales model because of the difficulties in model tractability detailed above; see Hadley and Whitin (1963), Archibald (1981), Tekin *et al.* (2001) and Hill and Johansen (2006).

The objective of the firm is to maximize the long-run average profit with the decision variables (Q, R, p). The expected revenue of a cycle is pQ, as the total units sold between two replenishment epochs is Q. The expected amount of average inventory held during a cycle includes first the average order level, which is $Q/2$. There is also the safety stock that is equal to $R - E[D(p, L)]$, where $E[D(p, L)]$ is the expected demand during lead time. Note that the safety stock is not restricted to be non-negative, which can be the case when backordering is possible. However, the inventory level is always non-negative when unmet demand is lost. Therefore, consistent with Hadley and Whitin (1963) for their lost sales treatment, the expected holding cost is adjusted by adding the expected lost sales to the average inventory level to include the increased holding costs when the unmet demand is lost instead of a backorder model. The resulting expected average inventory level is $Q/2 + R - E[D(p, L)] + E[S]$, where $E[S]$ is the expected lost sales during the lead time that is also the expected lost sales in a cycle. The explicit lost sales cost is $bE[S]$ per cycle.

The expected length of a cycle in a backorder model is $Q/v(p)$, where Q is the total demand received and $v(p)$ is the expected demand per unit time. When the unmet demand is lost, the expected cycle time can be extended by an amount $E[S]/v(p)$, as the total demand in a cycle is now $Q + E[S]$. However, when the lost sales is small enough, the lost sales effect can be neglected and the expected cycle time is approximated by $Q/v(p)$, which is also done in Hadley and Whitin (1963).

The resulting expected long-run average profit, which is denoted by $\Pi(Q, R, p)$ is:

$$\Pi(Q,R,p) = (p-c)\nu(p) - K\frac{\nu(p)}{Q} - h\left(\frac{Q}{2} + R - E[D(p,L)]\right) - E[S] \times \left(\frac{b\nu(p)}{Q} + h\right), \tag{1}$$

where $E[S]$, the expected lost sales per cycle can be obtained as follows:

$$S = \begin{cases} 0, & D(p,L) \leq R \\ D(p,L) - R, & D(p,L) > R. \end{cases}$$

3. Additive demand model

The firm needs to find the best price p with the continuous review ordering policy parameters Q and R. For this purpose, the demand function needs to be defined in more detail. The price dependent random demand has been defined in the literature mainly by two methods, additive and multiplicative forms of deterministic and random parts. In additive form, demand during a unit time period is $D(p, 1) = y(p) + \varepsilon$, where $y(p)$ is a decreasing function of price p. Additive demand models are very common in the inventory pricing literature (Petruzzi and Dada, 1999; Ray *et al.*, 2005; Gimpl-Heersink *et al.*, 2008). Specifically, $y(p) = \alpha - \beta p$, where $\alpha > 0$ and $\beta > 0$. The demand is not allowed to take negative values, thus the price is restricted to the range $[c, \alpha/\beta]$ and $A > -\alpha$, where A is the lower bound of the range of ε. Then the mean demand during a unit time interval is $\nu(p) = y(p) + \mu$. $D(p, t)$ has two parts: a certain demand part $ty(p)$ and a random portion that has a t-fold convolution of the distribution $f(\cdot)$ and denoted by $f_t(\cdot)$ with mean $t\mu$ and standard deviation $\sigma\sqrt{t}$. So $D(p, t)$ has the mean $tv(p)$ and the standard deviation $\sigma\sqrt{t}$.

The total mean demand during t is denoted by $v_0 t$, when the lowest possible price c is charged, $\nu_0 = y(c) + \mu$. It is assumed that $y(c) > \mu$, i.e. highest certain demand rate is greater than the mean uncertain demand rate.

Demand during period L is $D(p, L) = Ly(p) + \varepsilon_L$, where ε_L is distributed with $f_L(\cdot)$. By making a variable transformation similar to the one made by Petruzzi and Dada (1999), $z = R - Ly(p)$, which denotes the stock intended to satisfy the random demand part. By using the variable z, the lost sales can be rewritten as:

$$S = \begin{cases} 0, & \varepsilon_L \leq z \\ D(p, L) - R, & \varepsilon_L > z. \end{cases}$$

The expected lost sales per cycle $E[S]$ can now be denoted as a function of z such as:

$$E[S] = S(z) = \int_z^{BL} (x - z) f_L(x) dx.$$

As B is defined as the upper limit on the random demand per unit time, BL is the maximum value ε_L can take. Accordingly, the objective function of the problem in equation (1) can be reformulated as a function of Q, z, and p as follows:

$$\max_{Q,z,p} \ \Pi(Q,z,p) = (p - c)\nu(p) - K\frac{\nu(p)}{Q} - h\left(\frac{Q}{2} + z - L\mu\right) - S(z)\left(\frac{b\nu(p)}{Q} + h\right). \quad (2)$$

To analyze the existence of optimal pricing and inventory policy, the first and second partial derivatives of $\Pi(Q, z, p)$ with respect to Q, z, and p are:

$$\frac{\partial \Pi(Q,z,p)}{\partial Q} = \frac{\nu(p)}{Q^2}(K + bS(z)) - \frac{h}{2},$$

$$\frac{\partial^2 \Pi(Q,z,p)}{\partial Q^2} = -\frac{2\nu(p)}{Q^3}(K + bS(z)) < 0, \quad (3)$$

$$\frac{\partial \Pi(Q,z,p)}{\partial z} = -h + \bar{F}_L(z)\left(\frac{b\nu(p)}{Q} + h\right)$$

$$\frac{\partial^2 \Pi(Q,z,p)}{\partial z^2} = -f_L(z)\left(\frac{b\nu(p)}{Q} + h\right) < 0, \quad (4)$$

$$\frac{\partial \Pi(Q,z,p)}{\partial p} = \nu(p) - \beta\left(p - c - \frac{K}{Q} - S(z)\frac{b}{Q}\right) \quad (5)$$

$$\frac{\partial^2 \Pi(Q,z,p)}{\partial p^2} = -2\beta < 0.$$

As the second partial derivatives are negative, $\Pi(Q, z, p)$ is concave in each of three decision variables separately. Then an optimal decision variable when the other two variables are fixed can be extracted from the first order conditions.

From equation (5):

$$p^*(Q,z) = \frac{\nu_0 + 2c\beta}{2\beta} + \frac{K + bS(z)}{2Q}.$$

In fact $(\nu_0 + 2c\beta)/2\beta$ is the base price p^0 that maximizes the expected revenue minus the purchase cost $\pi(p) = (p - c)*\nu(p)$, such that $p^0 = (\nu_0 + 2c\beta)/2\beta$. Thus, by using p^0, the optimal price given (Q, z) is $p^*(Q,z) = p^0 + ((K + bS(z))/2Q)$. Since $(K + bS(z))/2Q$ is non-negative, $p^* \geq p^0$. This result is the opposite of what has been found for a single cycle newsboy problem. Studies by Mills (1959) and Petruzzi and Dada (1999) show that when the newsboy problem is optimized jointly over ordering quantity and single price with additive demand, the resulting best price is not greater than the base price that is optimal for the expected marginal revenue. Our finding shows that the result is different in a multiperiod continuous review model than the newsboy problem.

By definition the price is limited to the region $[c, \alpha/\beta]$. It can be easily shown that given $y(c) = \alpha - \beta c \geq 0$, $p^*(Q,z) \geq c$ for all z and Q. Then $p^*(Q, z)$ is limited by its upper limit α/β. Lemma 1 demonstrates our findings.

Lemma 1. (i) For a fixed z and p, $\Pi(Q, z, p)$ is concave in Q and optimal $Q^*(z, p)$ can be uniquely defined as:

$$Q^*(z,p) = \sqrt{\frac{2\nu(p)(K + S(z)b)}{h}}. \tag{6}$$

(ii) For a fixed Q and p, $\Pi(Q, z, p)$ is concave in z and optimal $z^*(Q, p)$ can be uniquely defined as:

$$F_L(z^*(Q,p)) = \frac{b\nu(p)}{hQ + b\nu(p)}. \tag{7}$$

(iii) For a fixed Q and z, $\Pi(Q, z, p)$ is concave in p and optimal $p^*(Q, z)$ can be uniquely defined as:

$$p^*(Q,z) = \begin{cases} p^0 + ((K + bS(z))/2Q), & (K + bS(z))/Q < (\nu_0 - 2\mu)/\beta \\ \alpha/\beta, & (K + bS(z))/Q \geq (\nu_0 - 2\mu)/\beta. \end{cases}$$

By using the optimality conditions stated in Lemma 1, after fixing one of the decision variables, the effect of a change in the second decision variable on the optimal value of the third one can be seen. For example, from Lemma 1i, for a fixed z, Q decreases as the price increases and so the demand rate. Similar findings are summarized in Table II.

Results indicate that in general, decision variables behave as substitutes such that an increase in one of them leads to a decrease in the optimal value of the other. For example, for a fixed price p, if the reorder level is increased, then the order quantity should be decreased as otherwise the average inventory level would increase that would cause high holding cost. Similarly, for a fixed reorder level, if the order quantity is increased, then the price should be decreased so that a higher demandrate can be generated to sell the increased orders.

Next, the behavior of $\Pi(Q, z, p)$ in two decision variables is analyzed when the third variable is kept fixed. For a fixed price p, the joint concavity of $\Pi(Q, z, p)$ in (Q, z) is shown by Brooks and Lu (1969) under a restriction on the distribution of random demand. Zipkin (1986) shows the joint concavity without any restriction for the model where backorder costs are charged per unit per time and with the same constraint on demand distribution when backorder cost is charged per item. This finding is reproduced here and demonstrated in Lemma 2.

Lemma 2. For a fixed price p, $\Pi(Q, z, p)$ is jointly concave in (Q, z) for $z \geq \mu L$ given that $f_L(z)$ is non-increasing for $z \geq \mu L$. Thus, there is a unique optimal ordering quantity $Q^*(z^*, p)$ and unique optimal $z^*(Q^*, p)$ such thatthe optimal reorder point is $R^*(Q^*, p) = Ly(p) + z^*$.

The constraint is not very restrictive, as most of the well-known distributions such as Normal and Poisson are non-increasing for the values greater than the mean.

For a fixed z, $\Pi(Q, z, p)$ can be reduced to a single variable of Q by replacing p with the optimal price $p^*(Q, z)$ from Lemma 1iii. Then $\Pi(Q, z, p(Q, z))$ becomes a function of only Q and it has the properties stated in Lemma 3.

Lemma 3. For a fixed z, optimal ordering quantity is $Q^*(z, p^*)$ to sell at the optimal price $p^*(Q^*, z)$ as specified in Lemma 1iii such that:

$$(a) \quad 2\mu(\nu_0 - 2\mu)^2 \leq h\beta^2(K + bS(z))$$

$$(b) \quad \nu_0 > 6\mu$$

(i) if both parameter relations (a) and (b) hold simultaneously, then Q^* is either the smallest or the largest of three Q's satisfying $\partial\Pi(Q, z, p(Q, z))/\partial Q = 0$,

(ii) if at least one of the parameter inequalities (a) and (b) fails, then Q^* is the unique Q satisfying $\partial\Pi(Q, z, p(Q, z))/\partial Q = 0$.

For a fixed Q, $\Pi(Q, z, p)$ can be reduced to a single variable of z by replacing p with the optimal price $p^*(Q, z)$ from Lemma 1iii. Then, how to obtain the optimal z follows from Lemma.

Lemma 4. For a fixed Q, optimal reordering point is $R^*(Q, p^*) = Ly(p) + z^*(Q, p^*)$ and the optimal selling price is $p^*(Q, z^*)$ as specified in Lemma 1iii. z^* is obtained such as if the distribution of uncertain demand portion satisfies $r'_L(z) + 2(r_L(z))^2 > 0$ for $z \in [AL, BL]$, where $r(\cdot) = f(\cdot)/(1 - F(\cdot))$ is the failure rate, then z^* is equal to the single root or either the smallest or the largest of three roots of $\partial\Pi(Q, z, p(Q, z))/\partial Q = 0$. Otherwise z^* can be obtained by an extensive search over different values of z.

Fixed	Effect	Cause
z	p ↗	Q ↘
z	Q ↗	p ↘
Q	p ↗	z ↘
Q	z ↗	p ↘
p	z ↗	Q ↘
p	Q ↗	z ↘

Table II. Interaction of optimal decision variables with additive demand form

The condition $r'_L(z) + 2(r_L(z))^2 > 0$ is always satisfied when the failure rate $r_L(\cdot)$ of uncertain demand function $f_L(\cdot)$ is nondecreasing. Many commonly used theoretical distributions in the literature such as Normal and Exponential have nondecreasing failure rates. A similar conditioning on the failure rate of the demand function is also needed in the pricing-inventory analysis for a single period problem as in Petruzzi and Dada (1999) and Chen *et al.* (2006), which also provide a discussion on the generality of nondecreasing failure rates.

Given the structure of the long-run profit function over two variables through Lemmas 2-4, it can be easily conjectured that the simultaneous analysis of the long-run profit $\Pi(Q, z, p)$ over all three decision variables Q, z, and p is hard to develop. In fact, $\Pi(Q, z, p)$ is not jointly concave in (Q, z, p). Therefore, the optimal solution of the problem can be found by an algorithm building on Lemmas 1-4 based on the assumption that the failure rate $r_L(\cdot)$ of $f_L(\cdot)$ satisfies $r'_L(z) + 2(r_L(z))^2 > 0$. The algorithm is given in Table III.

4. Multiplicative demand model

The demand can be also defined in multiplicative form of deterministic and random demand portions such that unit time demand is $D(p, 1) = y(p)\varepsilon$, where $y(p) = \alpha p^{-\beta}$, $\alpha > 0$ and $\beta > 1$. Petruzzi and Dada (1999) also workwith a similar demand model. As demand should be always positive, the lower limit of range of ε should satisfy $A > 0$. The demand per unit time has a mean $\nu(p) = y(p)\mu$, standard deviation σ, a density function $f(\cdot)$, and cumulative function $F(\cdot)$. The demand during t time periods has a mean $y(p)\mu t$, standard deviation $\sigma\sqrt{t}$, density function $f_t(\cdot)$, and cumulative function $F_t(\cdot)$.

To customize the expected profit function (1) for the multiplicative demand, let $z = R/y(p)$ and substitute R with z. The expected lost sales per cycle can be rewritten as:

$$S = \begin{cases} 0, & \varepsilon_L \leq z \\ D(p, L) - R, & \varepsilon_L > z. \end{cases}$$

The expected lost sales per cycle $E[S]$ can now be denoted as a function of z such as:

$$\begin{aligned} E[S] &= y(p)\int_z^{BL} (x - z) f_L(x) dx \\ &= y(p)S(z). \end{aligned}$$

Initialize	j = 1. Set $z = AL$, where A is the lower limit on ε and $Best\ \pi = -M$, where M is a large number
Step 1.	Compute $S(z)$
Step 2.	If $2\mu(\nu_0 - 2\mu)^2 \leq h\beta^2(K + bS(z))$ and $\nu_0 > 6\mu$, $[Q_1, Q_2, Q_3] = roots[\partial\pi(Q, z, p(Q, z))/\partial Q]$ $Q^* = arg\ \max_{Q_1, Q_3}(\pi(Q_1, z, p(Q_1, z)),\ \pi(Q_3, z, p(Q_3, z)))$. Otherwise $Q^* = root[\partial\pi(Q, z, p(Q, z))/\partial Q]$
Step 3.	Compute $p^* = (\nu_0 + 2c\beta)/(2\beta) + (K + bS(z))/(2Q^*)$ and $\pi(Q^*, z, p^*)$
Step 4.	If $\pi(Q^*, z, p^*) > Best\ \pi$, $Best\ \pi = \pi(Q^*, z, p^*)$, $Best\ z = z$, $Best\ p = p^*$, $Best\ R = z + y(p^*)$, $j = j + 1$, $z = z + 1$, and go to Step 1 if $z \leq BL$
Step 5.	Stop

Table III.
Optimal constant price and inventory policy algorithm for additive demand form

Then the expected profit function becomes:

$$\pi(Q,z,p) = \max_{Q,z,p} \quad \Pi(Q,z,p) = (p-c)\nu(p) - K\frac{\nu(p)}{Q} - h\left(\frac{Q}{2} + y(p)(z-L\mu)\right) - y(p)S(z)\left(\frac{b\nu(p)}{Q} + h\right). \tag{8}$$

To analyze the existence of optimal pricing and inventory policy, the first and second partial derivatives of $\Pi(Q, z, p)$ with respect to Q and z are considered:

$$\frac{\partial \Pi(Q,z,p)}{\partial Q} = \frac{\nu(p)}{Q^2}(K + by(p)S(z)) - \frac{h}{2},$$

$$\frac{\partial^2 \Pi(Q,z,p)}{\partial Q^2} = -\frac{2\nu(p)}{Q^3}(K + by(p)S(z)) < 0,$$

$$\frac{\partial \Pi(Q,z,p)}{\partial z} = -hy(p) + y(p)\bar{F}_L(z)\left(\frac{b\nu(p)}{Q} + h\right)$$

$$\frac{\partial^2 \Pi(Q,z,p)}{\partial z^2} = -y(p)f_L(z)\left(\frac{b\nu(p)}{Q} + h\right) < 0,$$

As the second partial derivatives are negative, $\Pi(Q, z, p)$ is concave in Q for fixed (z, p) and in z for fixed (Q, p). Then an optimal decision variable when the other two variables are fixed can be extracted from the first order conditions.

For fixed (Q, z), the pattern of $\pi(Q, z, p)$ in p is more complex:

$$\frac{\partial \Pi(Q,z,p)}{\partial p} = \mu(\beta-1) \times \frac{y(p)}{p}\left[-p + \frac{\beta}{\beta-1}\left(c + \frac{K}{Q} + \frac{h}{\mu}(z-\mu L + S(z))\right) + \frac{2b\beta y(p)S(z)}{(\beta-1)Q}\right], \tag{9}$$

In equation (9), $\mu(\beta-1)(y(p)/p)$ is always positive as $\beta > 1$ and $y(p) > 0$ by definition. Regarding the part in parenthesis $[\cdot]$, as p increases $y(p)$ decreases given that $y(p) = \alpha p^{-\beta}$. $z - \mu L + S(z)$ isnon-negative as:

$$\begin{aligned}
\int_{AL}^{BL}(x-z)f_L(x)dx &= \int_{AL}^{z}(x-z)f_L(x)dx + \int_{z}^{BL}(x-z)f_L(x)dx,\\
\mu L - z &= \int_{AL}^{z}(x-z)f_L(x)dx + S(z),\\
z - \mu L + S(z) &= -\int_{AL}^{z}(x-z)f_L(x)dx,
\end{aligned} \tag{10}$$

where the right hand side of equation (10) is non-negative. Thus, $c + K/Q + h(z - \mu L + S(z))/\mu > 0$. When the p is small the value of $[\cdot]$ in equation (9) can be small. As p increases, $y(p)$ decreases, so the value in $[\cdot]$ decreases.

So, as p increases, the value of $[\cdot]$ can be equal to zero only once at the global maxima. Therefore, for a given (Q, z), there exits a single price that maximizes the expected profit function. These results are summarized in Lemma 5.

Lemma 5. (i) For a fixed z and p, $\Pi(Q, z, p)$ is concave in Q and optimal $Q^*(z, p)$ can be uniquely defined as:

$$Q^*(z,p) = \sqrt{\frac{2\nu(p)(K + by(p)S(z))}{h}}. \tag{11}$$

(ii) For a fixed Q and p, $\Pi(Q, z, p)$ is concave in z and optimal $z^*(Q, p)$ can be uniquely defined as:

$$F_L(z^*(Q,p)) = \frac{b\nu(p)}{hQ + b\nu(p)}. \tag{12}$$

(iii) For a fixed Q and z, there exits a single $p^*(Q, z)$ that maximizes $\Pi(Q, z, p)$ such that:

$$p^*(Q,z) - y(p^*(Q,z))\frac{2b\beta S(z)}{\beta - 1} = \frac{\beta}{\beta - 1}\left(c + \frac{K}{Q} + \frac{h}{\mu}(z - \mu L + S(z))\right). \tag{13}$$

When the demand has a multiplicative form, the base price p^0 that maximizes the expected revenue minus the purchase cost $\pi(p) = (p - c)*\nu(p)$ is $p^0 = c\beta/(\beta - 1)$. Thus, by using p^0, the optimal price given (Q, z) is $p^*(Q,z) = p^0 + (\beta/(\beta - 1))(K/Q + h(z - \mu L + S(z))/\mu + 2by(p^*(Q,z))S(z)$. From the discussion preceding the Lemma 5, it is known that the part added to p^0 is always positive. So, $p^* \geq p^0$. This result is similar to that is shown by Petruzzi and Dada (1999) for the multiplicative demand in a newsboy setting. Our result shows that this finding is also confirmed in a multiperiod continuous review model.

The effect of a change in a decision variable on the other one is studied by using the optimality conditions stated in Lemma 5. For example, from Lemma 5i, for a fixed z, Q decreases as the price increases as the demand rate decreases. Similar findings are summarized in Table IV.

As in the case of additive demand form summarized by Table II, according to Table table:effects-multiplicative, decision variables work as substitutes, in general, to minimize costs. For example, for a fixed price order level, if the order quantity happens to be below its optimal value as a result of quality issue or a supplier delivery problem, then the optimal price to charge should be increased to sell the available inventory at a slower rate but more profitably.

Table IV. Interaction of optimal decision variables with multiplicative demand form

Fixed	Effect	Cause
z	p ↗	Q ↘
z	Q ↗	p ↘
Q	p ↗	z ↘
Q	z ↗	p ↘ ↗
p	z ↗	Q ↘
p	Q ↗	z ↘

As a closed-form solution for $p^*(Q, z)$ cannot be obtained, it is difficult to investigate the joint optimality of $\pi(Q, z, p)$ for more than one variable for the multiplicative demand form. Still, the results (11), (12), and (13) can be used to simplify the algorithm to search for the optimal set of (Q, z, p) as given in Table V.

5. Numerical experiments

After getting some information on optimality conditions of a joint constant price and inventory model, through numerical exercises, insights on the (i) sensitivity of optimal variables to the system parameters, (ii) value gained by joint optimization overa sequential price and inventory optimization, and (ii) sensitivity of these gains to system parameters are obtained.

During the numerical analyses, it is assumed that random demand portion of the demand has Poisson arrivals with mean μL during the lead time L. To define the base problem setting, it is benefited from the parameter values used in Chen *et al.* (2006) and Chen and Simchi-Levi (2006). It would be preferable to test our results on exact parameter settings that are already used in other studies, but any study with a price dependent demand, fixed ordering cost, non-zero lead time, and lost salescosts could not be found in the literature. The base problem setting has $K = 45$, $c = 3$, $L = 3$, $h = 0.2$, and $b = 3.5$. The similar tests are done first with additive demand model and then for multiplicative demand. For the additive demand form, the base demand settings are $\alpha = 27$, $\beta = 3.5$, and $\mu = 4.5$.

The algorithm defined in Table III is run first for the 29 problem settings defined in Table VI by changing each system parameter at a time by keeping the others fixed. In each problem setting, only the parameter that is changed wrt the base setting is reported for clarity of the table. As the random demand distribution is defined as Poisson, running the optimal solution algorithm over only integer values of z is reasonable, because z is the amountof stock kept for the random demand portion. Although the optimal z^* in Table VI is always integer, the optimal order quantity and reorder level can be non-integer as $R = z^* + y(p^*)$ and $y(p^*)$ can be non-integer at optimal p^*.

The results indicate that each parameter may have different effects on decision variables. For example, as the mean of the random demand μ or lead time L increases, the main effect is on the optimal reorder level R, while the order quantity and the optimal price change slightly or not at all. On the other hand, the optimal ordering quantity Q changes radically with the fixed cost of ordering K or the holding cost per unit h, while the reorder level and the price change relatively less. The demand-price relationship factor β and the unit purchase price c affect all variables at a great extent. The main

Initialize	j = 1. Set $z = AL$, where A is the lower limit on ε and $Best\ \ \pi = -M$, where M is a large number. Compute $p^0 = c\beta/(\beta-1)$ and $EOQ = \sqrt{2\nu(p^0)K/h}$
Step 1.	Compute $S(z)$.
Step 2.	For $0 \leq Q \leq EOQ$, i. Compute $p^* = root[p - y(p)2bS(z)\beta/(\beta-1) = \beta/(\beta-1)(c + K/Q + h(z - \mu L + S(z))/\mu)]$ ii. Compute $\pi(Q, z, p*)$ iii. If $\pi(Q, z, p^*) > Best\ \ \pi$, $Best\ \ \pi = \pi(Q, z, p^*)$, $Best\ \ z = z$, $Best\ \ R = zy(p^*)$, and $Best\ \ p = p^*$
Step 3.	Set $j = j + 1$, $z = z + 1$, and go to Step 1 if $z \leq BL$
Step 4.	Stop

Table V. Optimal constant price and inventory policy algorithm for multiplicative demand form

Table VI. Optimal constant price and inventory control variables with additive demand $D(p,1) = \alpha - \beta p + \varepsilon$

	Parameter setting								Optimal solution				
	α	β	μ	K	c	L	h	b	Q^*	R^*	p^*	z^*	π^*
P0	27	3.5	4.5	45	3	3	0.2	3.5	66.6	29.2	6.4	15	17.3
P1		2.5							72.5	36.1	8.1	16	42.2
P2		3.0							69.3	33.2	7.1	16	27.4
P3		4.0							62.8	26.2	5.8	15	10.1
P4		4.5							58.6	23.0	5.4	15	5.1
P5			3.5						64.3	27.6	6.2	12	14.8
P6			4.0						65.1	28.9	6.3	14	16.0
P7			5.0						67.3	30.5	6.4	17	18.6
P8			5.5						68.1	31.8	6.5	19	19.9
P9				35					59.0	30.7	6.3	16	18.7
P10				40					62.6	30.5	6.3	16	18.0
P11				50					69.7	29.0	6.4	15	16.6
P12				55					72.7	28.8	6.4	15	15.9
P13					2.0				72.3	35.8	5.8	16	27.4
P14					2.5				69.2	33.1	6.1	16	22.1
P15					3.5				63.1	26.4	6.6	15	12.9
P16					4.0				59.3	23.5	6.9	15	9.0
P17						1			65.4	10.8	6.4	6	17.6
P18						2			65.7	20.5	6.4	11	17.4
P19						4			66.7	38.9	6.4	20	17.1
P20						5			66.7	48.7	6.4	25	17.0
P21							0.10		95.2	31.4	6.2	16	21.5
P22							0.15		76.9	30.8	6.3	16	19.2
P23							0.25		59.0	28.7	6.4	15	15.6
P24							0.30		53.4	28.3	6.4	15	14.1
P25								2.5	66.0	29.3	6.4	15	17.4
P26								3.0	66.3	29.2	6.4	15	17.3
P27								4.0	66.1	30.2	6.4	16	17.2
P28								4.5	66.3	30.2	6.4	16	17.2

reason is that both β and c are two parameters that have important effects onthe optimal price. As the price changes, then the demand rate changes, so do the ordering quantity and the reorder level. One interesting result that can deducted from Table VI is that the lost sales cost rate b has a very slight effect on optimality results. The intuition behind this result is that the profit maximization model implicity incorporates the cost of a lost sales as the profit margin. So the additional stockout cost does not effect the optimal decision variables, unless it is very significant wrt the profit margin.

According to the traditional business definitions in a firm, marketing determines the sales price of a product after evaluating the competitor and customer preference effects, which in fact shapes the customer demand. The supply and operations management does the routine inventory decisions given the demand structure. So a sequential price and inventory control mechanism is in effect. The joint price and inventory control optimization offered in this study aims to eliminate possible inefficiencies in sequential optimization. For this purpose, through numerical analyses, the gap between the joint and sequential solutions of the price and inventory control problems is measured.

In a sequential solution, first the price is optimized by considering the demand function and only the purchasing cost, as holding and stockout costs are results of inventory decisions. In joint pricing and inventory literature, this optimal price is called the base price denoted by p^0. The base price for additive and multiplicative demand models are shown in §3 and §4, respectively. Given a base price, which also determines the structure of the demand function, the inventory problem in our setting reduces to a regular (R, Q) problem.

The percent increase in the expected profit of a firm by using a joint price and inventory control is denoted by $\Delta\pi$, so $\Delta\pi = (\pi(Q,z,p) - \pi(Q,z|p^0))/\pi(Q,z|p^0)*100$, where $\pi(Q,z|p^0)$ is the expected profit under sequential optimization. Also the percent change in optimal decision variables by joint optimization over sequential optimization are measured where ΔQ, ΔR, Δp, and Δz denote, respectively, the percent change in the optimal ordering quantity, reorder level, price, and the stock indented for random demand portion during the lead time. The gap between joint and sequential optimizations is evaluated for 29 problem scenarios defined in Table VI. The results are reported in Table VII.

According to Table VII, as expected, the optimal price in the joint optimization is larger than the base price of sequential optimization, i.e. $p^* \geq p^0$. The gap in optimal price increases especially when β or h increases. The sequential optimization leads to lower price, so higher demand rate, which results in higher optimal ordering quantity and reorder levels than those in joint optimization. The differences in optimal ordering quantity and reorder level areamplified as β, c, and h increase. The change in the lead time L and b seem not to affect the gap significantly. Except the base problem P0, the optimal stock for the random demand portion z is the same in both joint and sequential optimization. Thus, the gap can be mainly devoted to the difference in optimal price and the resulting effect in demand rate.

The average increase in expected profit by joint optimization of price and inventory variables is over 3% in Table VII. This profit improvement can be as much as 15% when β is high. The improvement by the joint optimization is also measured for 1,000 problem instances with randomly chosen parameter values for each problem. Each parameter is randomly generated using a uniform distribution from the ranges given in Table VIII. Our results demonstrate average profit improvements of 4.97% over sequential optimization provided by the joint price and inventory control for the tested problems with a standard deviation of 5.75. These results indicate that by using our joint optimization algorithm, significant improvements can be provided over the sequential decision making, which is common in practice.

Numerical tests are also repeated with the multiplicative demand. The algorithm defined in Table V is run for 29 problem settings defined in Table IX by changing each system parameter at a time by keeping the others fixed. In each problem setting, only the parameter that is changed wrt the base setting, which is named as P0, is reported for clarity of the table.

The sensitivity of optimal decision variables to system variables are mainly similar to those reported for additive demand in Table VI. However, there are two exceptional results that is worth pointing. With multiplicative demand, as the mean of the random demand μ increases, the optimal price decreases, which is the opposite with additive demand. As the price decreases, the demand increases. So the increases in ordering quantity and reorder level with the increase in μ are greater than those with the additive demand.

The second difference between the additive and multiplicative demand cases is the sensitivity of the optimal price to holding cost. While the holding cost does not

Problem	$\Delta\pi$	ΔQ	ΔR	Δp	Δz
P0	2.51	−5.2	−14.1	6.0	−6.3
Increasing β					
P1	0.60	−3.4	−6.3	4.2	0
P2	1.21	−4.6	−8.4	5.0	0
P3	5.55	−8.2	−14.9	7.0	0
P4	15.23	−10.8	−19.4	8.2	0
Increasing μ					
P5	3.09	−6.7	−12.3	6.3	0
P6	2.76	−6.4	−11.6	6.1	0
P7	2.27	−5.9	−10.9	5.8	0
P8	2.05	−5.7	−10.3	5.7	0
Increasing K					
P9	1.77	−5.4	−9.7	5.2	0
P10	2.11	−5.8	−10.4	5.6	0
P11	2.93	−6.5	−12.1	6.3	0
P12	3.36	−6.9	−12.7	6.7	0
Increasing c					
P13	1.30	−4.8	−8.7	5.9	0
P14	1.76	−5.4	−9.7	5.9	0
P15	3.78	−7.2	−13.1	6.1	0
P16	6.18	−8.4	−15.3	6.2	0
Increasing L					
P17	2.37	−6.1	−10.3	5.9	0
P18	2.43	−6.1	−10.8	5.9	0
P19	2.56	−6.2	−11.5	6.0	0
P20	2.58	−6.2	−11.5	6.0	0
Increasing h					
P21	0.96	−4.2	−7.6	4.1	0
P22	1.64	−5.2	−9.4	5.1	0
P23	3.56	−7.0	−12.9	6.8	0
P24	4.83	−7.8	−14.3	7.5	0
Increasing b					
P25	2.46	−6.1	−11.4	5.9	0
P26	2.49	−6.2	−11.4	6.0	0
P27	2.49	−6.1	−11.0	6.0	0
P28	2.52	−6.2	−11.1	6.0	0
Average $\Delta\pi$: 3.08 percent					

Table VII.
The performance of joint optimization over sequential optimization for additive demand as % change in optimal variables

Table VIII.
Distributions of random parameters used for additive demand model

α	27	β	U(2.5, 4.5)	μ	U(3, 6)	K	U(35, 60)
c	U(2, 4)	L	U(1, 4)	h	U(0.1, 0.4)	b	U(2.5, 5)

have a direct effect on optimal price with additive demand from Lemma 1iii, it has also a direct increasing effect on optimal price with multiplicative demand from Lemma 5. As a result, p^* increases with h in Table IX.

The performance of the joint optimization over sequential solution is also evaluated with multiplicative demand. The gap between joint and sequential optimizations is evaluated for 29 problem scenarios defined in Table IX. The results

	Parameter setting								Optimal solution				
	α	β	μ	K	c	L	h	b	Q^*	R^*	p^*	z^*	π^*
P0	27	2	35	45	3	3	0.2	3.5	93.0	58.9	7.2	113.0	57.0
P1		1.6							112.0	85.4	9.4	114	143.8
P2		1.8							103.5	72.3	8.0	113	90.0
P3		2.2							82.5	46.0	6.7	112	36.1
P4		2.4							72.0	34.8	6.4	111	22.5
P5			25						76.5	39.4	7.4	80	38.3
P6			30						85.0	49.1	7.3	97	47.6
P7			40						101.0	69.1	7.1	129	66.6
P8			45						108.5	79.9	7.0	145	76.2
P9				35					84	60.5	7.1	113	59.1
P10				40					89	60.5	7.1	113	58.0
P11				50					98	58.3	7.2	112	56.1
P12				55					101	56.7	7.3	112	55.2
P13					2.0				138.5	129.3	4.9	115	84.0
P14					2.5				112.0	85.5	6.0	114	67.9
P15					3.5				80.5	43.9	8.3	112	49.1
P16					4.0				70.0	33.2	9.5	111	43.1
P17						1			93.5	20.9	7.1	39	57.6
P18						2			95.0	40.7	7.1	76	57.3
P19						4			93.5	77.6	7.2	149	56.8
P20						5			93.5	96.4	7.2	185	56.6
P21							0.10		138.5	67.1	6.8	115	63.2
P22							0.15		110.0	62.8	7.0	114	59.8
P23							0.25		81.0	55.2	7.4	112	54.6
P24							0.30		73.5	53.3	7.5	111	52.5
P25								2.5	93.0	57.8	7.2	111	57.2
P26								3.0	93.0	58.3	7.2	112	57.1
P27								4.0	93.5	58.9	7.2	113	56.9
P28								4.5	93.0	59.4	7.2	114	56.9

Table IX. Optimal constant price and inventory control variables with multiplicative demand $D(p,1) = \alpha p^{-\beta}\varepsilon$

are reported in Table X. The range of gap between the optimality results of joint and sequential optimization are different for multiplicative and additive demands because of the change in ranges of β and μ. However, the sensitivity of the gap is mainly similar. The only exception is the effect of purchasing cost c. While the relative performance of joint optimization improves with c in additive demand model, it deteriorates with multiplicative demand.

6. Conclusion

Firm performance increases when supply chain members work together in cooperation, as customer demands are often unpredictable (Richey *et al.*, 2009). Contrarily, research suggests that the links between pricing, inventory control, supply chain management and firm performance have not been empirically substantiated (Frohlich and Westbrook, 2001). However, some conclusions have been suggested: reducing inventory holding costs through improved inventory management improves profitability (Stapleton *et al.*, 2002).

While much of the supply chain management literature has concentrated on inter-firm conflicts and collaboration, there is still a need for research within each firm;

Problem	$\Delta\pi$	ΔQ	ΔR	Δp	Δz
P0	4.01	−17.1	−31.2	20.0	−0.9
Increasing β					
P1	1.32	−12.9	−22.7	17.5	0.0
P2	2.33	−14.6	−27.0	18.5	−0.9
P3	6.92	−20.0	−35.8	21.8	−0.9
P4	12.30	−23.3	−41.9	24.4	−1.8
Increasing μ					
P5	5.96	−19.7	−35.1	23.3	−1.2
P6	4.80	−18.1	−33.1	21.7	−1.0
P7	3.45	−15.8	−29.1	18.3	−0.8
P8	3.02	−14.7	−27.0	16.7	−0.7
Increasing K					
P9	3.18	−15.8	−29.2	18.3	−0.9
P10	3.59	−16.1	−29.2	18.3	−0.9
P11	4.44	−17.3	−31.2	20.0	−0.9
P12	4.88	−18.5	−33.0	21.7	−0.9
Increasing c					
P13	5.11	−19.1	−33.9	22.5	−0.9
P14	4.42	−17.3	−31.2	20.0	−0.9
P15	3.74	−15.9	−29.5	18.6	−0.9
P16	3.55	−16.2	−29.7	18.8	−0.9
Increasing L					
P17	3.40	−15.7	−30.4	18.3	−2.5
P18	3.74	−15.0	−29.5	18.3	−1.3
P19	4.25	−17.2	−31.0	20.0	−0.7
P20	4.46	−17.6	−30.9	20.0	−0.5
Increasing h					
P21	1.65	−11.9	−22.8	13.3	−0.9
P22	2.75	−14.6	−27.2	16.7	−0.9
P23	5.43	−19.7	−34.8	23.3	−0.9
P24	7.01	−20.7	−36.6	25.0	−0.9
Increasing b					
P25	3.86	−17.1	−31.2	20	−0.9
P26	3.94	−17.1	−31.2	20	−0.9
P27	4.07	−16.5	−31.8	20	−1.7
P28	4.12	−17.3	−31.2	20	−0.9
Average $\Delta\pi$: 4.33 percent					

Table X.
The performance of joint optimization over sequential optimization for multiplicative demand as % change in optimal variables

that of intra-firm departmental collaboration. Our research hoped to illustrate that intra-firm conflicts between marketing and operations management of supply and demand can be successfully modeled and coordinated. Demand management is the marketing department's focus which is known to be a function of price, advertising, and product variety; manipulating these variables to achieve the greatest revenue. Supply management then coordinates supply with the given demand at lowest cost. For example, recent research into efficient consumer response (for retailers and suppliers to work closely together to reduce costs and add customer value) focus on both demand management as well as cost control such as: store assortment, efficient replenishment, efficient promotion and product introduction (Martens and Dooley, 2010).

The frequent mismatches between supply and demand such as unexpectedly short lasting promotion periods due to out-of-stocks or piled-up stocks due to overpriced products, have directed researchers and practitioners to rethink the hierarchical decision making between marketing and operations. From just such a need in the research stream in collaborated decision-making our study integrated the joint price and inventory management. We chose to work with a continuous review inventory system, non-zero lead time, fixed ordering cost, and lost sales model in case of a stockout. While continuous inventory review systems are highly practical because of constant ordering quantities, lead time, fixed cost; lost sales are unavoidable in reality and difficult to include in analytical studies. Our aim by this study is to gain knowledge on the relationship between price and inventory optimization by incorporating as many aspects of reality as possible.

The issue of inventory management within the supply chain has become exacerbated on a global scale to such an extent that firms are developing unique concepts to combat the high costs of insufficient customer focus and stock out. An example of an innovate technique to combat this issue are that firms are shipping large containers of goods (called floating docks) without demand requirements, towards the customers drawing upon both transportation and inventory control issues. This floating dock concept maylead to less storage costs and a shorter order time, but the amount of inventory may actually be increased, but in different locations of the supply chain (Dekker *et al.*, 2009).

Although the joint inventory and pricing research has been improving in recent years, most of the research's focus has been in obtaining dynamic pricing strategies, where price can be updated after every demand arrival or at each replenishment opportunity. While dynamic pricing is easier to implement with improving price tagging and point-of-sales technologies and transition to online sales, still many sellers follow a static price policy especially for functional products. Without dealing with complex dynamic price policies, a seller should know how to determine the best long-run prices for his products along with the best inventory replenishment decisions. The supply chain complexity, although sophisticated models are continuously assisting practitioners, still cannot ever provide full information as the level of coordination will vary due to organizational culture, size, technology, etc. (Choi and Krause, 2006; Jonsson *et al.*, 2007).

Therefore, in this study, our focus is on joint optimization of inventory replenishment and the constant selling price. By showing optimality conditions on the price and inventory decision variables, two algorithms on how to obtain optimal decision variables, one for additive and another for multiplicative demand-price model, are provided. Through extensive numerical analyses, the potential profit increases are reported if price and inventory problem is solved simultaneously instead of sequentially. In addition, the sensitivities of optimal decision variables to system parameters are revealed.

Our results mainly support the practical intuition that marketing wants to sell more with lower prices, while operations is more concerned about inventory levels such that if price and inventory decisions are made together, the optimal price is higher andstocking levels are lower than those in sequential decision making. Moreover, the benefit of joint decision making increases significantly as the fixed ordering and/or holding costs increase. This indicates the increasing importance of coordination for expensive to order and stock products.

The model used in this study incorporates many aspects of reality such as replenishment fixed cost and lead time, both multiplicative and additive forms of price dependent demand function, and lost sales in case of a stockout. The analytical optimality results provided would help a practitioner to obtain the optimal decision variables in a reasonable time period. Among the inputs that are required to implement the model, the most time consuming to obtain might be the demand function and the shortage cost per unit lost demand, while the rest of parameters are relatively straightforward to obtain. The price dependent demand function can be obtained by analyzing the past data, which can be even more useful if enough price changes are made in the past. The lost sales cost can be approximately obtained by considering the follow-up service revenues from a sale as in auto-dealer industry and/or repetitive purchases of customers that is measured by customer loyalty.

The study in this paper can be further extended. A price change model in the case of low inventory levels where a price discount or increase can be exercised can be studied by first utilizing the methodology described in this study to obtain the optimal constant price. Although there are several studies in the literature investigating emergency price change models, they use arbitrary exogenous prices. However, the value of a price change can be better appreciated if the long-run price is optimal for the system. Another follow-up study can be a backordering model in a similar problem context. Any difference in optimal decision variables can be investigated with the lost sales model.

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Appendix

Proof of Lemma 2. For a fixed price p, to show the joint concavity in (Q, z), the Hessian matrix of the profit function $\pi(Q, z)$ needs to be checked. The first principal minors should negative and the second principal minor, which is also the determinant, should be positive. The first principal minors $\partial^2\Pi(Q,z)/\partial Q^2$ and $\partial^2\Pi(Q,z)/\partial z^2$ are negative from equations (3) and (4), respectively. To calculate the determinant we also need:

$$\frac{\partial^2\Pi(Q,z)}{\partial Q\partial z} = \frac{\partial^2\Pi(Q,z)}{\partial z\partial Q} = -\frac{b\bar{F}_L(z)\nu(p)}{Q^2}.$$

Then the determinant is:

$$\begin{aligned}|H| &= \frac{\partial^2\Pi(Q,z)}{\partial Q^2}\frac{\partial^2\Pi(Q,z)}{\partial z^2} - \frac{\partial^2\Pi(Q,z)}{\partial Q\partial z}\frac{\partial^2\Pi(Q,z)}{\partial z\partial Q} \\ &= \frac{\nu(p)}{Q^4}(2(K + bS(z))f_L(z)(b\nu(p) + hQ) - b^2\bar{F}_L(z)^2\nu(p)). \qquad \text{(A1)}\end{aligned}$$

To show that the determinant $|H|$ is positive, first the limit according to z is checked, which is:

$$\lim_{z\to\infty}|H| = 0.$$

Then, check how $|H|$ approaches the limit in z:

$$\frac{\partial|H|}{\partial z} = \frac{2\nu(p)}{Q^4}\left(-bhQ\bar{F}_L(z)f_L(z) + (b\nu(p) + hQ)(K + bS(z))\frac{df_L(z)}{dz}\right), \qquad \text{(A2)}$$

which is negative if $df_L(z)/dz \le 0$. Remember that $z = R - Ly(p)$ and $f_L(\cdot)$ has a mean μL. So, if $f_L(\cdot)$ is nondecreasing for all values greater than equal to $L\mu$ and $z \ge \mu L$ then $|H| > 0$ as it has a limit zero and is non-increasing while approaching the limit. This concludes the proof that $\pi(Q, z)$ is jointly concave in (Q, z) if $f_L(x)$ is non-increasing for $x \ge \mu L$. □

Proof of Lemma 3. Given $p^*(Q, z)$, the optimal price for fixed Q and z from Lemma 3iii and assuming that z is fixed, the profit function as a function of only Q is obtained as follows:

$$\pi^a(Q,z,p(Q,z)) = \frac{1}{\beta}\left(\frac{\nu_0}{2} - \beta\left(\frac{K+bS(z)}{2Q}\right)^2\right) - h\left(\frac{q}{2} + z - \mu L + S(z)\right), \tag{A3}$$

for $(K + bS(z))/Q < (\nu_0 - 2\mu)/\beta$. The superscript "a" is added to denote this condition. Otherwise:

$$\pi^b(Q,z,p(Q,z)) = \mu\left(\frac{\nu_0-\mu}{\beta} - \frac{K+bS(z)}{Q}\right) - h\left(\frac{Q}{2} + z - \mu L + S(z)\right), \tag{A4}$$

where the superscript "b" is added to denote the second possible case.

First take the derivative of $\pi(Q, z, p(Q, z))$ wrt Q and check, whether it is a continuous function of Q. The derivative of equation (A3) is as follows:

$$R^a(Q) = \frac{\partial \pi^a(Q,z,p(Q,z))}{\partial Q} = \frac{K+bS(z)}{2Q^3}(\nu_0 Q - \beta(K+bS(z))) - \frac{h}{2}. \tag{A5}$$

Then the derivative of equation (A4) is:

$$R^b(Q) = \frac{\partial \pi^b(Q,z,p(Q,z))}{\partial Q} = \frac{K+bS(z)}{2Q^3}(\nu_0 Q - \beta(K+bS(z))) - \frac{h}{2}. \tag{A6}$$

As the z is taken as fixed and p is a function of Q and z, $R(Q) = (\partial\pi(Q,z,p(Q,z)))/\partial Q$ for ease of notation.

To check whether $R(Q)$ is continuous on Q, first find the Q^0, where the optimal price is at its upper limit $p^*(Q^0, z) = \alpha/\beta$. By using the definition of $p^*(Q, z)$:

$$Q^0 = \frac{\beta(K+bS(z))}{\nu_0 - 2\mu}.$$

By definition, $\nu_0 = y(c) + \mu$ and $y(c) > \mu$, so $\nu_0 - 2\mu > 0$. Thus, $Q^0 > 0$, for $K > 0$. It is easy to confirm that $R^a(Q) = R^b(Q)$ at $Q = Q^0$. So, $R(Q)$ is continuous on Q.

It is known from the limit condition of the price, while $\pi^b(Q,z,p(Q,z))$ is valid for $Q \in [0, Q^0)$, $\pi^a(Q,z,p(Q,z))$ is defined on $Q \in [Q^0, \infty)$. So let us first check the behavior of $R(Q)$ for $Q \in [0, Q^0)$. For $Q = 0$, $R^b(Q) = \infty$, from equation (A6). Between 0 and Q^0:

$$\frac{dR^b(Q)}{dQ} = -\frac{2\mu(K+bS(z))}{Q^3} < 0. \tag{A7}$$

So, $R(Q)$ is decreasing from infinity on $[0, Q^0)$. Next, check the behavior of $R(Q)$ on $[Q^0, \infty)$:

$$\frac{dR^a(Q)}{dQ} = \frac{K+bS(z)}{2Q^3}\left(-2\nu_0 + \frac{3\beta}{Q}(K+bS(z))\right), \tag{A8}$$

$$\frac{d^2R^a(Q)}{dQ^2}\Big|_{dR(Q/dQ=0} = \frac{16\nu_0^5}{18\beta^4(K+bS(z))^3} < 0. \tag{A9}$$

From equation (A9), it can be concluded that $R^a(z)$ is unimodal, as it first increases and then decreases. Combining all obtained information, $R(Q)$ might have one of the three possible paths shown in Figure A1.

If $R(Q)$ is positive at Q^0, then $R(Q) = 0$ can be satisfied only at a single Q, around which $R(Q)$ turns from positive to negative. Thus, this single root of $R(Q)$ is the single and the global maximum for $\pi(Q,z,p(Q,z))$, denoted by Q^* in the figure.

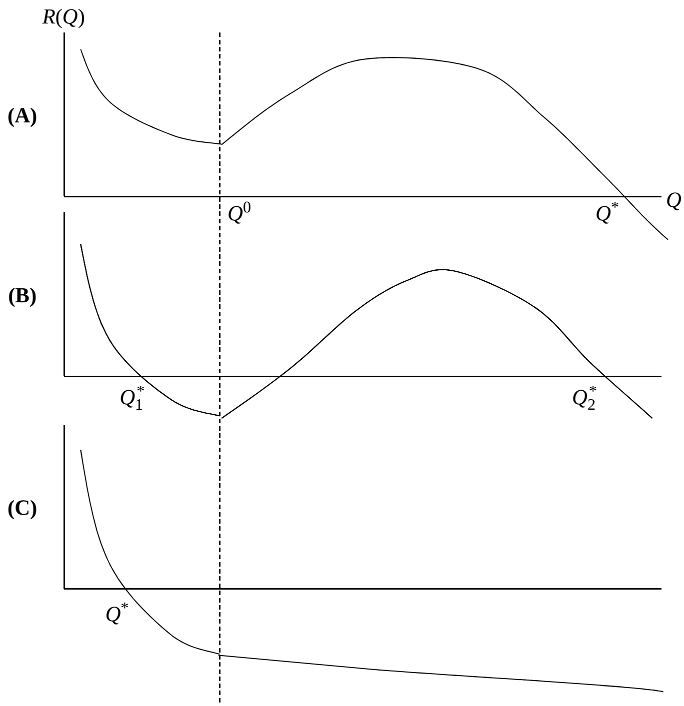

Figure A1.
For a fixed z, in search for the optimal Q

If $R(Q)$ is non-positive at Q^0, this indicates that $R^b(Q) = Q$ at a single point $Q \in (0, Q^0)$. In the region $[Q^0, \infty)$, as $R^a(Q)$ is first increasing, and then decreasing, $R^a(Q) = 0$ can be satisfied at two points or none. These two possibilities are illustrated by cases (B) and (C), respectively, in Figure A1. In case (B), there can be two maxima Q_1^* and Q_2^* that satisfy $R(Q) = 0$ and around which $R(Q)$ changes from positive to negative. The global maxima can be obtained by comparing $\pi(Q_1^*, z, p(Q_1^*, z))$ and $\pi(Q_2^*, z, p(Q_2^*, z))$. In case (C), the single and global maximum is attained at $R^b(Q) = 0$.

To get more information on which case described above apply for any given problem setting, check whether $R(Q^0)$ is positive or non-positive. By using the definition of Q^0, and equation (A5) or (A6):

$$R(Q^0) = \frac{\mu(\nu_0 - 2\mu)^2}{\beta^2(K + bS(z))} - \frac{h}{2}. \tag{A10}$$

Equation (A10) is positive, if $2\mu(\nu_0 - 2\mu)^2 > h\beta^2(K + bS(z))$. So, case (A) in Figure (A1) applies and there is a single Q that maximizes $\pi(Q, z, p(Q, z))$.

If Equation (A10) is non-positive, but $R(Q)$ is non-increasing for $Q \geq Q^0$, there is still a single root maxima of $R(Q)$. For this case to hold, equation (A8) should be non-positive for $Q \geq Q^0$, which requires:

$$\frac{3\beta(K + bS(z))}{2\mu} \leq Q^0 = \frac{\beta(K + bS(z))}{\nu_0 - 2\mu},$$

$$\nu_0 \leq 6\mu.$$

If $2\mu(\nu_0 - 2\mu)^2 \leq h\beta^2(K + bS(z))$ and $\nu_0 \leq 6\mu$, then case (C) in Figure (A1) applies and there is a single maxima of $\pi(Q, z, p(Q, z))$.

If neither of these conditions are satisfied, there can be at most three roots of $R(Q)$, where the smallest and largest ones can be local maxima, while the second root is a local minima. This completes the proof of the lemma. □

Proof of Lemma 4. Using $p^*(Q, z)$ from Lemma 3iii and assuming that Q is fixed, the profit function as a function of Q, z can be obtained as in equations (A3) and (A4). Similar to the proof of Lemma 4, first check whether the derivative of $\pi(Q, z, p(Q, z))$ is continuous on z. $p^*(Q, z)$ reaches its upper limit α/β at z^0 that satisfies the following equality:

$$S(z^0) = \frac{Q(\nu_0 - 2\mu) - K\beta}{b\beta}.$$

It is known from the limit condition of the price that while $\pi^b(Q, z, p(Q, z))$ is valid for $z \in [AL, z^0)$, $\pi^a(Q, z, p(Q, z))$ is defined on $z \in [z^0, BL]$, where A and B are the lower and upper limits of the range of the random demand per unit time, respectively.

The derivative of equation (A3) wrt to z is as follows:

$$R^a(z) = \frac{\partial \pi^a(Q, z, p(Q, z))}{\partial z} = \frac{b\bar{F}_L(z)}{Q}\left(\frac{\nu_0}{2} - \frac{\beta}{2Q}(K + bS(z))\right) - hF_L(z). \tag{A11}$$

Then the derivative of equation (A4) wrt to z is:

$$R^b(z) = \frac{\partial \pi^b(Q, z, p(Q, z))}{\partial z} = \bar{F}_L(z)\left(\frac{b\mu}{Q} + h\right) - h. \tag{A12}$$

As the Q is taken as fixed and p is a function of Q and z, $R(z) = (\partial \pi(Q, z, p(Q, z)))/\partial z$ for ease of notation.

At z^0, it is easy to show that $R^a(z) = R^b(z)$. So $R(z)$ is continuous on z. Then let us first check the behavior of $R(z)$ for $z \in [AL, z^0)$. For $z = AL$, $R^b(z) = b\mu/Q$, from equation (A12). Between AL and z^0:

$$\frac{dR^b(Qz)}{dz} = -f_L(z)\left(\frac{b\mu}{Q} + h\right) < 0. \tag{A13}$$

So, $R(z)$ is decreasing from $b\mu/Q$ on $[AL, z^0)$. Next, check the behavior of $R(z)$ on $[z^0, BL]$:

$$\frac{dR^a(z)}{dQz} = -f_L(z)\left(h + \frac{\nu_0 b}{2Q} - \frac{b\beta}{2Q^2}(K + bS(z))\right) + \bar{F}_L(z)^2\frac{b^2\beta}{2Q^2}, \tag{A14}$$

$$\begin{aligned}\frac{d^2R^a(z)}{dz^2} &= -\frac{df_L(z)}{dz}\left(h + \frac{\nu_0 b}{2Q} - \frac{b\beta}{2Q^2}(K + bS(z))\right) - \bar{F}_L(z)f_L(z)\frac{3b^2\beta}{2Q^2}, \\ \frac{d^2R^a(z)}{dz^2}\Big|_{dR(z)/dz=0} &= -\frac{\bar{F}_L(z)b^2\beta}{2f_L(z)Q^2}\left(\frac{df_L(z)}{dz}\bar{F}_L(z) + 3f_L(z)^2\right).\end{aligned} \tag{A15}$$

Equation (A15) is negative if $df_L(z)/dz{*}\bar{F}_L(z) + 3f_L(z)^2$ is positive. Let $r(\cdot)$ denote a failure rate function such that $r(\cdot) = f(\cdot)/\bar{F}(\cdot)$. Then $2r_L(z)^2 + dr_L(z)/dz = (df_L(z)/dz\bar{F}_L(z) + 3f_L(z)^2)/\bar{F}_L(z)^2$. Thus, if $2r_L(z)^2 + dr_L(z)/dz > 0$, then $df_L(z)/dz{*}\bar{F}_L(z) + 3f_L(z)^2 > 0$, which makes sure that equation (A15) is less than zero. The negativity of equation (A15) indicates that $R^a(z)$ is unimodal over $[z^0, BL]$, first increasing and then decreasing.

Combining all obtained information, it is known that $R(z)$ is first decreasing on z until $z = z^0$ and then unimodal beyond z^0. Similar to the pattern of $R(Q)$ in Lemma 4, $R(z)$ can have either one root at the point where $R^b(z) = 0$, or three roots, one at $R^b(z) = 0$ and two for $R^a(z) = 0$. If there is single root at $R^b(z) = 0$, then this is the single global maximum of $\pi(Q, z, p(Q, z))$ for the fixed Q. If $R^a(z) = 0$ at two values of z, then the global maximum is either at $R^a(z) = 0$ or the larger root of $R^b(z) = 0$. This completes the proof of the lemma. □

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