can write

$$\sum_{k=1}^{n}\gamma_f(k)\epsilon_k S_{k-1} = o\left(\sum_{k=2}^{n}\gamma_f^2(k)S_{k-1}^2\right) + O(1)$$
$$= o\left(\sum_{k=2}^{n}f(k)S_{k-1}^2\right) + O(1) \quad \text{a.s.}$$

where the last estimate was made by using condition ii). The third term is or order of 1 and regarding the fourth term, we know by the Law of Iterated Logarithm that $S_n^2 = o(n\log n)$ almost surely. Therefore, condition iii) implies

$$\gamma_f(n+1)S_n^2 = o\left(\sum_{k=1}^{n-1}\gamma_f(k)\right) \quad \text{a.s.}$$

Overall, we have the following:

$$\sum_{k=1}^{n}f(k)S_k^2 + o\left(\sum_{k=1}^{n}f(k)S_k^2\right) = \sigma^2\sum_{k=1}^{n-1}\gamma_f(k) + o\left(\sum_{k=1}^{n-1}\gamma_f(k)\right) + O(1) \quad \text{a.s.}$$

which together with condition i) imply (23). □

Let us now consider some applications of Lemma 2 which were used in this paper:

1) For $\sum_{k=1}^{n}\bar{\epsilon}_k^2 = \sum_{k=1}^{n}k^{-2}S_k^2$, $f(k) = 1/k^2$ and $\gamma_f(n) \sim \int_n^\infty dx/x^2 = 1/n$. Clearly, all the three conditions of Lemma 2 are satisfied. Hence, by (23),

$$\sum_{k=1}^{n}\bar{\epsilon}_k^2 \sim \sigma^2\sum_{k=1}^{n}\frac{1}{k} \sim \sigma^2\log n \quad \text{a.s.} \tag{24a}$$

2) For $\sum_{k=1}^{n}(\log k+1)^{-1}\bar{\epsilon}_k^2$, $f(k) = 1/(k^2(\log k+1))$ and $\gamma_f(n) \sim \int_n^\infty dx/(x^2\log x) \sim 1/(n\log n)$. Again, the three conditions can easily be verified. Thus

$$\sum_{k=1}^{n}(\log k+1)^{-1}\bar{\epsilon}_k^2 \sim \sigma^2\sum_{k=2}^{n}\frac{1}{k\log k} \sim \sigma^2\int_e^n\frac{dx}{x\log x} = \sigma^2\log\log n \quad \text{a.s.} \tag{24b}$$

3) For $\sum_{k=1}^{n}(\log k+1)^{-1/2}\bar{\epsilon}_k^2$, $f(k) = 1/(k^2\sqrt{\log k+1})$, $\gamma_f(n) \sim \int_n^\infty dx/(x^2\sqrt{\log x}) \sim 1/(n\sqrt{\log n})$ and

$$\sum_{k=1}^{n}(\log k+1)^{-1/2}\bar{\epsilon}_k^2 \sim \sigma^2\sum_{k=2}^{n}\frac{1}{k\sqrt{\log k}} \sim \sigma^2\int_1^n\frac{dx}{x\sqrt{\log x}} = 2\sigma^2\sqrt{\log n} \quad \text{a.s.} \tag{24c}$$

Acknowledgment

The authors wish to thank Prof. P. R. Kumar for his useful discussions and comments.

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On Stability of Interval Matrices

M. E. Sezer and D. D. Šiljak

***Abstract*—New sufficient, and sometimes necessary and sufficient conditions, are obtained for Schur- and Hurwitz-stability of interval matrices by relying on the concept of connective stability and M-matrices. The necessity part is broadened to include interval matrices with mixed signs of the off-diagonal elements, provided the sign patterns follow that of the Morishima matrix. The obtained results are extended to cover convex combinations of interval matrices.**

I. Introduction

Robust stability of dynamic systems with modeling uncertainties is oftenformulated as stability of a polytope of matrices in the parameter space. In this context, stability analysis has been centered on proving stability of an entire family of matrices by establishing stability of a number of test matrices, which usually are, or generated from, the vertices of the polytope (see, for example, [1]–[3], and the references therein). The main difficulty in this approach is that (unlike polytopes of polynomials) vertex or even edge stability does not, in general, imply stability of the whole polytope [4]. The only case for which this is known to be true is when the vertices are simultaneously stable [5], which is not an easy condition to test. The difficulty remains even in the special case of interval matrices.

As pointed out in [6], a straightforward solution of the stability problemfor certain classes of interval matrices is available in the framework of connective stability via M-matrix theory [7]. It is well known that the connective stability approach provides a simple sufficient condition for stability of an interval matrix in terms of stability of a test matrix with nonnegative off-diagonal coefficients. Alternatively, such a test matrix can be used to generate an interval matrix whose stability is equivalent to the stability of the test matrix itself. In this note we show that any Morishima matrix [7] serves the same purpose, and thus we obtain a more general characterization of interval matrices whose stability is equivalent to stability of one

Manuscript received July 6, 1992; revised December 22, 1992. This work was supported by the National Science Foundation under Grant ECS-9114872.

M. E. Sezer is with Bilkent University, 06533 Bilkent, Ankara, Turkey.

D. D. Siljak is with Santa Clara University, Santa Clara, CA 95053.

IEEE Log Number 9213956.

of its vertices. Although all known results on stability of interval matrices can be deduced from our characterization, it still inherits the conservatism (as well as generality) of the connective stability approach, indicating that available results on interval stability are still far from being complete.

Our second objective is to make use of the proportional domination conditions[8] to establish stability of a collection of special matrices, and hence, of the polytope formed by their convex combinations. When applied to interval matrices, this result allows for generating a larger stability region in the parameter space from their hyperrectangular stability regions.

II. Preliminaries

For two $n \times n$ matrices A and B, $A \geq B$ denotes element-by-element inequality. A family of interval matrices is defined as

$$\mathcal{A}(\underline{A}, \overline{A}) = \{A \in \boldsymbol{R}^{n\times n}\colon\ \underline{A} \leq A \leq \overline{A}\} \tag{1}$$

where $\underline{A} = (\underline{a}_{ij})$ and $\overline{A} = (\overline{a}_{ij})$ are fixed matrices. The family $\mathcal{A}$ is described geometrically as a hyperrectangle in the space $\boldsymbol{R}^{n\times n}$ of the coefficients a_{ij}. We say that a family $\mathcal{A}$ is stable (Schur- or Hurwitz-stable) if every $A \in \mathcal{A}$ is stable.

We recall several definitions and results of [7], which we want to apply tointerval matrices. An $n \times n$ matrix A is said to be positive quasi-dominant diagonal (PQDD) if there exist numbers $d_i > 0$ such that

$$a_{ii} > d_i^{-1}\sum_{j\neq i}^{n} d_j|a_{ij}|, \qquad i = 1, 2, \cdots, n. \tag{2}$$

If (2) holds when all $d_i = 1$, then A is said to be positive diagonal dominant (PDD). It follows that A is PQDD if and only if $D^{-1}AD$ is PDD for some diagonal matrix $D > 0$. If $a_{ij} \leq 0$, $i \neq j$, then A is PQDD if and only if all leading principal minors of A are positive, that is, A is an M-matrix.

A matrix A is called a Morishima matrix if by symmetric row and columnpermutations it can be transformed into the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{3}$$

where $A_{11} \geq 0$, $A_{22} \geq 0$ are square submatrices and $A_{12} \leq 0$, $A_{21} \leq 0$. It is easy to see that A is a Morishima matrix if and only if $SAS = |A|$ for some $S = \operatorname{diag}\{s_1, s_2, \cdots, s_n\}$ with $s_i = \pm 1$, $i = 1, 2, \cdots, n$, where $|A|$ denotes the matrix obtained from A by taking the absolute values of all elements.

Based on the above, we have the following stability results concerning somespecial matrices:

If $A \geq 0$, then A is *Schur-stable* if and only if $I - A$ is an$M$-matrix, or equivalently, PQDD. Note that this requires that $a_{ij} \in [0, 1)$, $i = 1, 2, \cdots, n$. If A is a Morishima matrix, then it is Schur-stable if and only if $|A|$ is, that is, $I - |A|$ is PQDD.

Let $A = A_D + A_C$, where $A_D = \operatorname{diag}\{a_{11}, a_{22}, \cdots, a_{nn}\}$. If$A_C \geq 0$, then A is *Hurwitz-stable* if and only if $-A$ is PQDD. If A_C is a Morishima matrix, then A is Hurwitz-stable if and only if $A_D + |A_C|$ is, that is, $-A$ is PQDD.

III. Schur-Stability of Interval Matrices

Our first result characterizes a class of interval matrices whose stabilityis equivalent to stability of a single test matrix.

Theorem 1: Let $\pm V$ be a Morishima matrix. Then, the following are equivalent:

i) V is Schur-stable.
ii) $|V|$ is Schur-stable.
iii) $\mathcal{A}(-|V|, |V|)$ is Schur-stable.

Proof: The implications i) $\Leftrightarrow$ ii) and iii) $\Rightarrow$ ii) areobvious. To prove that ii) $\Rightarrow$ iii), suppose $|V|$ is Schur-stable, that is, $I - |V|$ is PQDD. Then, $0 \leq |v_{ii}| < 1$, and for any $A \in \mathcal{A}(-|V|, |V|)$, $|a_{ij}| \leq |v_{ij}|$, so that

$$\begin{aligned} 1 - a_{ii} \geq 1 - |v_{ii}| &> d_i^{-1}\sum_{j\neq i}^{n} d_j|v_{ij}| \\ &\geq d_i^{-1}\sum_{j\neq i}^{n} d_j|a_{ij}|, \qquad i = 1, 2, \cdots, n \end{aligned} \tag{4}$$

for some $d_i > 0$. By Gershgorin's theorem [3], the eigenvalues of $D^{-1}AD$ all lie in the union of the discs $\mathcal{D}_{1-a_{ii}}(d_i^{-1}\sum d_j|a_{ij}|)$, which, by virtue of (4), are included in $\mathcal{D}_{1-a_{ii}}(1 - |v_{ii}| - \epsilon)$ for some $\epsilon > 0$, where

$$\mathcal{D}_c(r) = \{\gamma \in C\colon\ |\gamma - c| \leq r\}. \tag{5}$$

Therefore, the eigenvalues A, which are the same as those of $D^{-1}AD$, all lie in the union of $\mathcal{D}_{a_{ii}}(1 - |v_{ii}| - \epsilon)$ which are all contained in $\mathcal{D}_0(1 - \epsilon)$. Thus A is Schur-stable, and the proof is complete.

An immediate consequence of Theorem 1 is the following.

Corollary 1.1: Let $\underline{A} \geq 0$ (resp. $\overline{A} \leq 0$). Then, $\mathcal{A}(\underline{A}, \overline{A})$ isSchur-stable if and only if $\overline{A}$ (resp. $\underline{A}$) is Schur-stable.

Corollary 1.1 is the main result of [9], which we obtained without resortingto interval arithmetic.

Let us associate with a given family $\mathcal{A}(\underline{A}, \overline{A})$ a nonnegative testmatrix $\hat{A} = (\hat{a}_{ij})$, where

$$\hat{a}_{ij} = \max\{|\underline{a}_{ij}|, |\overline{a}_{ij}|\}. \tag{6}$$

From Theorem 1, the following result is automatic.

Corollary 1.2: $\mathcal{A}(\underline{A}, \overline{A})$ is Schur-stable if $\hat{A}$ is Schur-stable.

Corollary 1.2 provides a sufficient condition for Schur-stability of ageneral interval matrix having an arbitrary sign pattern of the interval bounds $\underline{a}_{ij}$ and $\overline{a}_{ij}$. The necessity is missing because $\hat{A}$ itself may not belong to the family $\mathcal{A}(\underline{A}, \overline{A})$. To recover necessity in Corollary 1.2, we denote by V_k, $k = 1, 2, \cdots, m$, those extreme vertices of $\mathcal{A}$ for which $|V_k| = \hat{A}$. Note that, in almost all cases, there is only one such vertex, which is farthest from the origin of the coefficient space.

Corollary 1.3: Suppose that $\pm V_k$ is a Morishima matrix corresponding to an extreme vertex of $\mathcal{A}(\underline{A}, \overline{A})$. Then, $\mathcal{A}$ is Schur-stable if and only if $\hat{A}$ is Schur-stable.

To illustrate the results above, consider an interval family $\mathcal{A}$ with

$$\underline{A} = \begin{bmatrix} 0.4 & -0.6 \\ -0.6 & 0.4 \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} 0.5 & -0.4 \\ -0.4 & 0.5 \end{bmatrix}.$$

For this particular example, Corollary 1.1 is inapplicable, and Corollary 1.2 is inconclusive as the nonnegative test matrix

$$\hat{A} = \begin{bmatrix} 0.5 & 0.6 \\ 0.6 & 0.5 \end{bmatrix} \tag{7}$$

is unstable. However, by noting that the unique extreme vertex

$$V = \begin{bmatrix} 0.5 & -0.6 \\ -0.6 & 0.5 \end{bmatrix}$$

of the family $\mathcal{A}$ is a Morishima matrix, we can use Corollary 1.3 to establish instability of $\mathcal{A}$.

We note that all the results above are essentially concerned with thestability of an uncertainty hyperrectangle symmetrically located about the origin. They all have a simple interpretation in the context of connective stability of multiparameter perturbed systems: A typical member A of the family $\mathcal{A}$ can be viewed as a perturbation of a nominal matrix $A_N = 0$ by additive perturbations $p_{ij}A_{ij}$, $i, j = 1, 2, \cdots, n$, where $A_{ij} = \hat{a}_{ij}e_ie_j^T$, with e_i representing the ith standard basis vector in $\boldsymbol{R}^n$, and $p_{ij} \in [-1, 1]$ are interconnection parameters. The usual connective stability approach of [7] via the use of a diagonal Liapunov function for the nominal matrix $A_N = 0$, produces directly the result of Corollary 1.2. However, since the uncertain family

$$\overline{\mathcal{A}} = \left\{ A \in \boldsymbol{R}^{n\times n}\colon\ A = A_N + \sum_{i,j}^{n} p_{ij}A_{ij}, \qquad p_{ij} \in [-1, 1] \right\} \tag{8}$$

represents the smallest hyperrectangle symmetrically located around the origin which includes $\mathcal{A}$, the result may be quite conservative. To reduce conservatism of this approach, it may be useful to choose the nominal matrix $A_N = (a_{ij}^N)$ at the center of the uncertainty hyperrectangle by defining

$$a_{ij}^N = \tfrac{1}{2}(\underline{a}_{ij} + \overline{a}_{ij}), \qquad i, j = 1, 2, \cdots, n. \tag{9}$$

Then, by modifying the perturbation matrices as $A_{ij} = \frac{1}{2}(\overline{a}_{ij} - \underline{a}_{ij})e_ie_j^T$, the uncertainty family $\tilde{\mathcal{A}}$ of (8) can be made to match $\mathcal{A}$ exactly. Stability of the family $\tilde{\mathcal{A}}$ can then be checked using the approach of [10]. A disadvantage of this approach is that the Liapunov matrix for A_N is no longer diagonal, and simplicity of the connective stability approach is lost.

To demonstrate the application of Liapunov approach to interval matrices, weconsider a family $\mathcal{A}$ with

$$\underline{A} = \begin{bmatrix} 0.5 & -0.6 \\ 0.4 & 0.5 \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} 0.5 & -0.4 \\ 0.6 & 0.5 \end{bmatrix}.$$

The unique extreme vertex

$$V = \begin{bmatrix} 0.5 & -0.6 \\ 0.6 & 0.5 \end{bmatrix}$$

of the family $\mathcal{A}$ corresponds to the same unstable test matrix $\hat{A}$ in (7). However, unlike the previous example, since $\pm V$ is not a Morishima matrix, Corollary 1.3 is not applicable. Nevertheless, the class $\mathcal{A}$ can be characterized as in (8) with

$$A_N = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad A_{12} = A_{21}^T = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix}$$

and $A_{11} = A_{22} = 0$. The Liapunov approach of [10] shows that $A_N + p_{12}A_{12} + p_{21}A_{21}$ is stable provided that

$$0.3|p_{12}| + 0.3|p_{21}| < 1. \tag{10}$$

Since (10) is satisfied for $|p_{12}|, |p_{21}| \le 1$, $\mathcal{A}$ is stable.

IV. Hurwitz–Stability of Interval Matrices

The results of the previous section can be restated for a Hurwitz-stability analysis of interval matrices. The following result is a continuous-time counterpart of Theorem 1, and can be proved in pretty much the same way as Theorem 1.

Theorem 2: Let $V = V_D + V_C$, where $V_D = \operatorname{diag}\{v_1, v_2, \cdots, v_n\}$ with $v_i < 0$ and $\pm V_C$ is a Morishima matrix. Then, the following are equivalent.

i) V is Hurwitz-stable,

ii) $V_D + |V_C|$ is Hurwitz-stable,

iii) $\mathcal{A}(A_D - |V_C|, V_D + |V_C|)$ is Hurwitz-stable for all $A_D \le V_D$.

Theorem 2 characterizes the largest family of matrices [i.e. $\mathcal{A}(A_D - |V_C|, V_D + |V_C|)$] whose stability can be inferred from stability of a single matrix with nonnegative off-diagonal elements (i.e., $V_D + |V_C|$). Like the class of Schur-stable matrices characterized by Theorem 1, the class of Hurwitz-stable matrices characterized by Theorem 2 corresponds to a symmetric hyperrectangle in the parameter space of A_C, but it is open toward $-\infty$ in the parameter space of A_D. For this reason, in the following, we characterize $\mathcal{A}$ by the bounds of A_C only; that is, we set $\underline{A}_D = \overline{A}_D = \hat{A}_D$, and let $\underline{A} = \hat{A}_D + \underline{A}_C$, $\overline{A} = \hat{A}_D + \overline{A}_C$.

The following continuous-time counterpart of Corollary 1.1 provides anecessary and sufficient condition for Hurwitz-stability of interval matrices with nonnegative (resp. nonpositive) off-diagonals.

Corollary 2.1: Let $\underline{A}_C \ge 0$ (resp. $\overline{A}_C \le 0$). Then $\mathcal{A}(\underline{A}, \overline{A})$ isHurwitz-stable if and only if $\overline{A}$ (resp. $\underline{A}$) is Hurwitz-stable.

To obtain a continuous-time counterpart of Corollary 1.2, we define a testmatrix $\hat{A} = \hat{A}_D + \hat{A}_C$ associated with the family $\mathcal{A}(\underline{A}, \overline{A})$, where $\hat{A}_C = (\hat{a}_{ij}^C)$ with

$$\hat{a}_{ij}^C = \max\{|\underline{a}_{ij}^C|, |\overline{a}_{ij}^C|\}, \qquad j \ne i. \tag{11}$$

Corollary 2.2: $\mathcal{A}(\underline{A}, \overline{A})$ is Hurwitz-stable if $\hat{A}$ is Hurwitz-stable.

As in the previous section, to recover necessity in Corollary 2.2, we need torestrict the signs of the off diagonal elements of the test matrix. For this purpose, we denote by $V_k = \hat{A}_D + V_k^C$, $k = 1, 2, \cdots$, those vertices of $\mathcal{A}(\underline{A}, \overline{A})$ for which $|V_k^C| = \hat{A}_C$, and state the following.

Corollary 2.3: Suppose that $\pm V_k^C$ is a Morishima matrix corresponding to anextreme vertex of $\mathcal{A}(\underline{A}, \overline{A})$. Then $\mathcal{A}(\underline{A}, \overline{A})$ is Hurwitz-stable if and only if $\hat{A}$ is Hurwitz-stable.

Finally, we note that when Corollary 2.3 is inapplicable to a given family$\mathcal{A}(\underline{A}, \overline{A})$ because none of the vertices V_k^C is a Morishima matrix, one can still try the Liapunov approach mentioned in the previous section by characterizing $\mathcal{A}$ as in (8).

V. Convex Combinations of Stable Matrices

Finally, we would like to discuss a related problem, namely, stability ofconvex combinations of stable matrices. The motivation for this study is to derive conditions which allow for generating from stable interval matrices other stable ones.

It is well known that convex combination of two stable matrices A and$B$ is not necessarily stable, even when stability is established by M-matrix conditions. The concept of *proportional domination* [8] provides stronger conditions needed.

Let A and B be two $n \times n$ matrices with positive diagonals andnonpositive off-diagonal elements. If

$$a_{ii}^{-1}a_{ij} \ge b_{ii}^{-1}b_{ij}, \qquad i, j, = 1, 2, \cdots, n \tag{12}$$

then A is said to proportionally dominate B rowwise. Columnwise proportional domination is defined similarly. It was shown in [8] that if A and B are both M-matrices with one proportionally dominating the other, then $\alpha A + \beta B$ is also an M-matrix for all $\alpha, \beta > 0$. Based on this we state the following:

Theorem 3: Let $A_0 \ge 0$ be an $n \times n$ Schur-stable matrix, and let $A_k \ge 0$be $n \times n$ matrices such that $I - A_k$ proportionally dominates $I - A_0$, $k = 1, 2, \cdots, N$, either all rowwise or all columnwise. Then a convex combination $\sum_{k=0}^{n} \alpha_k A_k$ with $\alpha_k \ge 0$, $k = 0, 1, \cdots, N$, $\sum_{k=0}^{n} \alpha_k = 1$, is also Schur-stable.

Proof: Assume, without loss of generality, that $I - A_k$ all proportionallydominate $I - A_0$ rowwise, so that $1 - a_{ii}^k > 0$, and

$$(1 - a_{ii}^k)^{-1}a_{ij}^k \le (1 - a_{ii}^0)^{-1}a_{ij}^0, \qquad i, j = 1, 2, \cdots, n; \quad k = 1, 2, \cdots, N. \tag{13}$$

Since A_0 is Schur-stable, $I - A_0$ is QPDD so that

$$1 - a_{ii}^0 > d_i^{-1}\sum_{j\ne i}^{n} d_j a_{ij}^0, \qquad i = 1, 2, \cdots, n \tag{14}$$

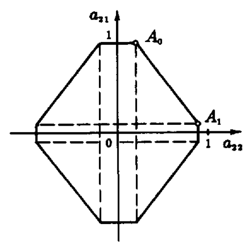

Fig. 1. Convex combination of stable interval matrices.

for some $d_i > 0$. By multiplying both sides of (13) by $(1 - a_{ii}^k)(1 - a_{ii}^0)d_j$, summing for $j \neq i$, and using (14), we get

$$1 - a_{ii}^k > d_i^{-1}\sum_{j\neq i}^{n} d_j a_{ij}^k, \qquad i = 1, 2, \cdots, n; \quad k = 0, 1, \cdots, N. \tag{15}$$

which implies that

$$1 - \left(\sum_{k=1}^{N}\alpha_k a_{ii}^k\right) > d_i^{-1}\sum_{j\neq i}^{n} d_j\left(\sum_{k=1}^{N}\alpha_k a_{ij}^k\right), \qquad i = 1, 2, \cdots, n. \tag{16}$$

Thus, $I - \sum_{k=0}^{N}\alpha_k A_k$ is PQDD, and the proof is complete.

Note that under the conditions of Theorem 3, the matrices$A_0, A_1, \cdots, A_N$ are simultaneously PQDD, as established by (15), and so are their convex combinations.

The following continuous-time version of Theorem 3 can be proved similarly.

Theorem 4: Let $A_0 = A_0^D + A_0^C$, with $A_0^C \geq 0$, be an $n \times n$Hurwitz-stable matrix, and $A_k = A_k^D + A_k^C$, with $A_k^C \geq 0$, be $n \times n$ matrices such that $-A_k$ proportionally dominates $-A_0$, $k = 1, 2, \cdots, N$, either all rowwise or all columnwise. Then, a convex combination $\sum_{k=0}^{N}\alpha_k A_k$, with $\alpha_k \geq 0$, $k = 0, 1, \cdots, N$, $\sum_{k=0}^{N}\alpha_k = 1$, is also Hurwitz-stable.

To illustrate the result of Theorem 3, we consider two positive matrices

$$A_0 = \begin{bmatrix} 0.4 & 0.4 \\ 1 & 0.2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.4 & 0.4 \\ 0.1 & 0.9 \end{bmatrix}.$$

both of which can easily be shown to be Schur-stable. Scaling the rows of $I - A_0$ and $I - A_1$ by the reciprocals of their diagonal elements we observe that $I - A_1$ proportionally dominates $I - A_0$. Then Theorem 3 implies that any convex combination

$$\alpha A_0 + (1-\alpha)A_1 = \begin{bmatrix} 0.4 & 0.4 \\ 0.1 + 0.9\alpha & 0.9 - 0.7\alpha \end{bmatrix}, \qquad \alpha \in [0, 1]$$

is also stable. The significance of this result can be seen by considering the stability regions in the (a_{21}, a_{22}) plane: While Theorem 1 produces two thin rectangular regions associated with A_0 and A_1, Theorem 3 combines these into a much larger one as shown in Fig. 1.

Notice that in this example, although $I - A_0$ and $I - A_1$ aresimultaneously PQDD, there exist no diagonal-type Liapunov function that would prove simultaneous stability [5] of A_0 and A_1; this function being a natural one in the context of M-matrices. This fact points out the significance of connective stability, which is diagonal-type simultaneous stability that can be established *via* simultaneous PQDD conditions [7].

VI. Conclusions

We have obtained necessary and sufficient conditions for stability ofMorishima-type interval matrices using the results available in the context of connective stability. The conditions have been further broadened to include convex combination of interval matrices. In future research, attempts shall be made to apply the results to establish connective stability of convex polytopes of nonlinear time varying matrix systems.

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Unified Formulation of Variable Structure Control Schemes for Robot Manipulators

Bin Yao, S. P. Chan, and Danwei Wang

***Abstract*—A general target model is proposed in the task space to represent motion trajectory, interaction force trajectory, and second-order function relating the motion errors and the interaction force errors. Using variable structure model reaching control (VSMRC) strategy, the model is achieved in the sliding mode with robust performance. Reaching transient can be eliminated or guaranteed with prescribed quality. By choosing a suitable model for the application, robust motion control, impedance control, hybrid position/force control, or constrained motion control are achieved, respectively.**

Manuscript received July 21, 1992; revised November 24, 1992.
B. Yao is with the Department of Mechanical Engineering, University of California, Berkeley, CA 94720.
S. P. Chan and D. Wang are with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 2263, Republic of Singapore.
IEEE Log Number 9213957.