and the functions

$$\begin{aligned} m''(s) &= m''(\alpha(t_i-)) + P''(\alpha(t_i-)) \\ &\quad \times \{I + k^2 G^T(t_i) H(t_i) G(t_i) P''(\alpha(t_i-))[s - \alpha(t_i-)]\}^{-1} \\ &\quad \times k^2 G^T(t_i) H(t_i) \{[y''(s) - y''(\alpha(t_i-))] \\ &\quad - G(t_i) m''(\alpha(t_i-))[s - \alpha(t_i-)]\}, \\ P''(s) &= P''(\alpha(t_i-))\{I + k^2 G^T(t_i) \\ &\quad \times H(t_i) G(t_i) P''(\alpha(t_i-))[s - \alpha(t_i-)]\}^{-1} \end{aligned}$$

are solutions of (10) for $s \in [\alpha(t-), \alpha(t+)]$.

Let us note that jumps $m''(\alpha(t_i+)) - m''(\alpha(t_i-)), P''(\alpha(t_i+)) - P''(\alpha(t_i-))$ of the variables $m''(\alpha(t))$, $P''(\alpha(t))$ coincide with jumps $m(t_i+) - m(t_i-)$, $P(t_i+) - P(t_i-)$ of the solutions $m(t)$, $P(t)$ of the equations with a measure (5). Thus, the optimal parameters $m''(\alpha(t))$, $P''(\alpha(t))$ are solutions of (5) everywhere in the considered time interval. This proves the theorem.

V. Conclusions

The minmax filtering equations over discrete-continuous observations follow from the minmax filtering equations over continuous observations given in [1] by virtue of replacing an absolutely continuous function $u(t)$ by a bounded variation one in accordance with an observation equation. No additional computation is needed.

The minmax filtering equations over discrete observations follow from the minmax filtering equations over continuous ones by virtue of transferring to discrete-continuous observations and assuming a bounded variation function $u(t)$ to be piecewise constant.

The definition of a vibrosolution ensures the stability of the optimal estimate with respect to small variations of a bounded variation function $u(t)$ and therefore observations.

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On the Stabilization and Stability Robustness Against Small Delays of Some Damped Wave Equations

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***Abstract*— In this note we consider a system which can be modeled by two different one-dimensional damped wave equations in a bounded domain, both parameterized by a nonnegative damping constant. We assume that the system is fixed at one end and is controlled by a boundary controller at the other end. We consider two problems, namely the stabilization and the stability robustness of the closed-loop system against arbitrary small time delays in the feedback loop. We propose a class of dynamic boundary controllers and show that these controllers solve the stabilization problem when the damping coefficient is nonnegative and stability robustness problem when the damping coefficient is strictly positive.**

I. Introduction

In recent years boundary control of flexible systems has become an active area of research. Most of the research in this area is concentrated on the problem of control and stabilization of conservative linear flexible systems (e.g., strings or beams without damping). Such systems have infinitely many eigenvalues on the imaginary axis and can be uniformly stabilized by using simple velocity feedback laws at their boundaries; see, e.g., [2] and [3]. It was shown, however, that these systems become unstable when arbitrary small time delays were introduced into the feedback laws; see, e.g., [5] and [6]. This lack of robustness and some other related results indicate that most of the conservative models in flexible structures are not well posed from the control theory point of view and possess potential limitations for the feedback design; see [8]. Recently in [7] it was argued that mathematical conservative models are never meant to represent physical systems for infinite time interval; hence any control theory based on these models should attempt to justify its conclusions by using an appropriately damped version of the corresponding conservative model.

In this paper we consider two different damped wave equations both parameterized by a damping coefficient $a \geq 0$. When $a = 0$, these models reduce to the standard conservative wave equation. To stabilize these systems, we propose a dynamic boundary control law. Following [7], we try to answer the following questions:

i) Does the proposed control law stabilize the conservative model and improve the stability of the damped models?

ii) Does the proposed control law robustly stabilize the damped models against small time delays in the feedback loop?

In the following section we propose a class of dynamic boundary controllers to solve these problems. It should be emphasized that these controllers do not robustly stabilize the conservative wave equation against small time delays in the feedback loop.

This note is organized as follows. In Section II, we give two examples of the damped wave equation used in this note and propose a class of dynamic controllers to solve the problems stated above. In Section III we give stability results [i.e., answer to problem i)], and in Section IV we give robustness results [i.e., answer to problem ii)]. Finally we give some concluding remarks.

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II. Damped Models

We first consider the following damped wave equation

$$w_{tt} = w_{xx} - 2aw_t - a^2 w, 0 < x < 1, t \geq 0, \tag{1}$$

$$w(0,t) = 0, w_x(1,t) = v(t), y(t) = w_t(1,t) \tag{2}$$

where $a \geq 0$ is a damping constant, $v(t)$ is the boundary control input, and $y(t)$ is the measured output. For simplicity, some coefficients are chosen to be unity. The system given by (1)-(2) is first introduced in [5] and later investigated in [1], [10], and [7]. For $a = 0$, the system given by (1) and (2) reduces to the standard conservative wave equation with a boundary controller.

To stabilize this system, the following simple controller and feedback law can be used

$$v(t) = ku(t) \tag{3}$$

$$u(t) = -y(t) \tag{4}$$

where $k > 0$. It is known that the closed-loop system given by (1)–(4) is exponentially stable; see [5]. When the feedback law in (4) is replaced by $u(t) = -y(t - h)$, where the constant $h > 0$ represents a small time delay, the stability of the closed-loop system depends on k and a. It is known that if

$$k > \frac{1 - e^{-2a}}{1 + e^{-2a}} \tag{5}$$

then the closed-loop system is unstable for arbitrary small time delays $h > 0$. On the other hand, if the inequality in (5) is reversed, then there exists an $h_0 > 0$ such that for any h, $0 \leq h \leq h_0$, the closed-loop system is L_2-stable. This result is obtained in [5] by calculating the eigenvalues directly and in [1] and [10] by using frequency domain techniques.

The second example of the damped wave equation that we consider is the following

$$w_{tt} = w_{xx} + aw_{xxt}, 0 < x < 1, t \geq 0, \tag{6}$$

$$w(0,t) = 0, w_x(1,t) + \alpha a w_{xt}(1,t) = v(t), y(t) = w_t(1,t) \tag{7}$$

where $a \geq 0$ is a damping constant, α is either zero or one. This type of damping is not unnatural and is similar to Kelvin–Voight damping for the Euler–Bernoulli beam. The system (6) and (7), with $\alpha = 0$, is first introduced and investigated in [7]. It can be shown that the closed-loop system given by (3), (4), (6), and (7) is exponentially stable (see Theorem 2 in Section III). It was shown in [7], however, by direct eigenvalue calculations that the closed-loop system becomes unstable when the feedback law in (4) is replaced by $u(t) = -y(t - h)$, where the constant $h > 0$ represents a small time delay. In Section IV we will show that this instability could be predicted by considering the open-loop transfer function and could be eliminated by choosing $\alpha = 1$ (see Corollary 2 in Section IV). We note that the case $\alpha = 1$ gives the natural boundary condition for (6), and this can be justified by considering the rate of change of the energy of the system. In Section IV we will show that even in the case $\alpha = 0$, by choosing appropriate dynamic boundary controllers, the instability with respect to time delays can be eliminated (see Corollary 2).

We propose the following dynamic boundary controllers to solve the stability problems stated above

$$\dot{z} = Az + bu, \tag{8}$$

$$v = c^T z + du \tag{9}$$

where $z \in \mathbf{R}^n$, for some natural number n, is the controller state, u is the controller input, $A \in \mathbf{R}^{n \times n}$ is a constant matrix, $b, c \in \mathbf{R}^n$ are constant column vectors, d is a constant real number, and the superscript T denotes transpose.

We first make the following assumptions concerning the controller given by (8) and (9) throughout this work.

Assumption 1: All eigenvalues of $A \in \mathbf{R}^{n \times n}$ have negative real parts.

Assumption 2: (A, b) is controllable, and (c, A) is observable.

Assumption 3: $d \geq 0$, and there exists a constant γ, $d \geq \gamma \geq 0$, such that the following holds

$$\mathcal{R}e\{g(j\omega)\} > \gamma, \omega \in \mathbf{R} \tag{10}$$

where the transfer function $g(s)$ is defined as

$$g(s) = d + c^T (sI - A)^{-1} b. \tag{11}$$

Moreover, in case $d > 0$, we require $\gamma > 0$ as well. □

We note that this type of controllers has been proposed for the stabilization of flexible structures. For the application to wave equation, see [11] and [13], and to the Euler–Bernoulli beam, see [12].

III. Stability Results

Let Assumptions 1)–3), stated above, hold. Then, it follows from the Meyer–Kalman–Yakubovich Lemma that given any symmetric positive definite matrix $Q \in \mathbf{R}^{n \times n}$, there exists a symmetric positive definite matrix $P \in \mathbf{R}^{n \times n}$, a vector $q \in \mathbf{R}^n$, and a constant $\epsilon > 0$ satisfying

$$A^T P + PA = -qq^T - \epsilon Q, \tag{12}$$

$$Pb - c = \sqrt{2(d - \gamma)}q \tag{13}$$

moreover, in case $d = 0$ in (9), we can take $q = 0$ and $\epsilon = 1$; see [15, p. 132].

To analyze the systems considered in this paper, we first define the function space $\mathcal{H}$ as follows

$$\mathcal{H} := \{(p \ r \ z\ )^T | p \in \mathbf{H}_0^1, r \in \mathbf{L}^2, z \in \mathbf{R}^n\} \tag{14}$$

where the spaces $\mathbf{L}^2$ and $\mathbf{H}_0^k$ are defined as follows

$$\mathbf{L}^2 = \{f : [0,1] \to \mathbf{R} | \int_0^1 f^2 dx < \infty\}, \tag{15}$$

$$\mathbf{H}_0^k = \{f \in \mathbf{L}^2 | f, f', f'', \cdots, f^{(k)} \in \mathbf{L}^2, f(0) = 0\}. \tag{16}$$

Equations (1), (2), (8), and (9) together with feedback control law (4) can be written in the following abstract form

$$\dot{m} = \mathcal{A}_1 m, \ m(0) \in \mathcal{H} \tag{17}$$

where $m = (w \, w_t z)^T \in \mathcal{H}$, the operator $\mathcal{A}_1 : \mathcal{H} \to \mathcal{H}$ is a linear unbounded operator defined as

$$\mathcal{A}_1 \begin{pmatrix} p \\ r \\ z \end{pmatrix} = \begin{pmatrix} r \\ p_{xx} - 2ar - a^2 p \\ Az - br(1) \end{pmatrix}. \tag{18}$$

The domain $D(\mathcal{A}_1)$ of the operator $\mathcal{A}_1$ is defined as

$$D(\mathcal{A}_1) := \{(p \ r \ z\ )^T \in \mathcal{H} | p \in \mathbf{H}_0^2, r \in \mathbf{H}_0^1, z \in \mathbf{R}^n \quad p_x(1) - c^T z + dr(1) = 0\}. \tag{19}$$

Let Assumptions 1)-3) hold, let $Q \in \mathbf{R}^{n \times n}$ be an arbitrary symmetric positive definite matrix, and let $P \in \mathbf{R}^{n \times n}$, $q \in \mathbf{R}^n$ be the solutions of (12) and (13) where P is also a symmetric and positive definite matrix. We define the following "energy" norm in $\mathcal{H}$

$$E(t) := \|m(t)\|_E^2 = \frac{1}{2} \int_0^1 w_t^2 dx + \frac{1}{2} \int_0^1 w_x^2 dx + \frac{1}{2} z^T Pz. \tag{20}$$

We note that one can define an inner-product which induces the norm given above; hence without loss of generality we may assume that $\mathcal{H}$ is a Hilbert space.

Theorem 1: Consider the system given by (17), where the operator $\mathcal{A}_1$ is given by (18). Assume that Assumptions 1)–3) are satisfied.

i) The operator $\mathcal{A}_1$ generates a C_0-semigroup $T(t)$ of contractions in $\mathcal{H}$, (for the terminology of the semigroup theory, the reader is referred to, e.g., [14]).
ii) For $a = 0$, $d = 0$, the semigroup $T(t)$ is asymptotically stable, i.e., the solutions of (17) asymptotically converge to zero.
iii) For $a + d > 0$, the semigroup $T(t)$ is exponentially stable.

Proof:

i) We first define the following new "energy" of the system

$$E_1(t) = E(t) + \frac{a^2}{2}\int_0^1 w^2 dx \tag{21}$$

where $E(t)$ is given by (20). Note that due to boundary condition (2) at the fixed end, the integral term in (21) can be embedded in $E(t)$. By differentiating (21) with respect to time, we obtain

$$\begin{aligned}\dot{E}_1(t) &= yv - 2a\int_0^1 w_t^2 dx + \frac{1}{2}z^T(A^TP + PA)z + z^TPbu \\ &= -2a\int_0^1 w_t^2 dx - \gamma w_t^2(1,t) \\ &\quad - \frac{1}{2}[\sqrt{2(d-\gamma)}w_t(1,t) + z^Tq]^2 - \frac{\epsilon}{2}z^TQz \end{aligned} \tag{22}$$

where in deriving the first equation we used integration by parts, (1), (2), (8), and (9), to obtain the second equation we used (4), (12), and (13). It follows from (22) that the operator $\mathcal{A}_1$ is dissipative. It can be shown that the operator $\lambda I - \mathcal{A}_1 : \mathcal{H} \to \mathcal{H}$ is onto for $\lambda > 0$ (see [11] and [13] for similar calculations). Hence from Lumer–Phillips theorem we conclude that $\mathcal{A}_1$ generates a C_0-semigroup of contractions on $\mathcal{H}$; see [14].

ii) See [11] and [13].

iii) For $a = 0$ and $d > 0$, see [11]. Hence we consider the case $a > 0$, $d \geq 0$. It is known that the operator $\mathcal{A}_1$ has compact resolvent when $a = 0$; see [11]. Since the terms containing a can be considered as a bounded perturbation to this operator, it can easily be shown that the operator $\mathcal{A}_1$ has compact resolvent for $a > 0$ as well. This implies that the operator $\mathcal{A}_1$ has point spectrum. By using (22) it can be shown that $\mathcal{A}_1$ cannot have an eigenvalue on the imaginary axis. Since $\mathcal{A}_1$ has point spectrum, it follows that the imaginary axis belongs to the resolvent of $\mathcal{A}_1$.

To obtain an estimate of the resolvent on the imaginary axis, let $y = (f\ h\ r\ )^T \in \mathcal{H}$ be given. We have to find $m = (p\ r\ z\ )^T \in D(\mathcal{A}_1)$ such that

$$(j\omega I - \mathcal{A}_1)m = y. \tag{23}$$

By using (18) in (23), after some straightforward calculations we conclude that $\|(j\omega I - \mathcal{A}_1)^{-1}\| < \infty$ for ω sufficiently large (see [11] for similar estimates). Since the imaginary axis belongs to the resolvent set $\rho(\mathcal{A}_1)$ of the operator $\mathcal{A}_1$, and since for each $\lambda \in \rho(\mathcal{A}_1)$, the operator $(\lambda I - \mathcal{A}_1)^{-1}$ is compact, it follows that for any $\Omega < \infty$, the following estimate holds

$$\sup_{\omega \leq \Omega} \|(j\omega I - \mathcal{A}_1)^{-1}\| < \infty. \tag{24}$$

By combining these results we conclude that estimate (24) holds for all ω. Hence, it follows from a result of [9] that the C_0-semigroup $T(t)$ generated by the operator $\mathcal{A}_1$ is exponentially stable. □

Now we consider the system given by (6) and (7), with the controller (8) and (9), and the feedback law (4). This system can be written in the following abstract form

$$\dot{m} = \mathcal{A}_2 m, m(0) \in \mathcal{H} \tag{25}$$

where $m = (w\ w_t z)^T \in \mathcal{H}$, the operator $\mathcal{A}_2 : \mathcal{H} \to \mathcal{H}$ is a linear unbounded operator defined as

$$\mathcal{A}_2\begin{pmatrix} p \\ r \\ z \end{pmatrix} = \begin{pmatrix} r \\ p_{xx} + ar_{xx} \\ Az - br(1) \end{pmatrix} \tag{26}$$

where $(p r z)^T \in \mathcal{H}$. The domain $D(\mathcal{A}_2)$ of the operator $\mathcal{A}_2$ is defined as

$$\begin{aligned} D(\mathcal{A}_2) := \{(p\ r\ z\ )^T \in \mathcal{H} | p \in \mathbf{H}_0^2, r \in \mathbf{H}_0^2, z \in \mathbf{R}^n; \\ p_x(1) + \alpha a r_x(1) - c^T z + dr(1) = 0\}. \end{aligned} \tag{27}$$

Theorem 2: Consider the system given by (25). Assume that Assumptions 1)–3) are satisfied. Then we have the following:

i) The operator $\mathcal{A}_2$ generates a C_0-semigroup $T(t)$ of contractions for each one of the following cases:
   i.1) for $\alpha = 1$,
   i.2) for $\alpha = 0$ and $d = 0$,
   i.3) for $\alpha = 0$ and $d > 0$, provided that $c^T b$ is sufficiently large or a is sufficiently small.
ii) For $a = 0$, $d = 0$, the semigroup $T(t)$ is asymptotically stable, i.e., the solutions of (25) asymptotically converge to zero.
iii) For $a + d > 0$, the semigroup $T(t)$ is exponentially stable.

Proof:

i) For case i.1), consider the "energy" $E(t)$ given by (20). By differentiating (20) with respect to time and by using (4), (6)-(9), it can be shown that $\dot{E} \leq 0$, hence the operator $\mathcal{A}_2$ is dissipative for the case i.1).
For case i.2), we again consider the "energy" $E(t)$ given by (20). Note that in this case since $d = 0$, without loss of generality we can take $q = 0$ and $\epsilon = 1$ in (12) and (13); see [15, p. 132]. By differentiating (20) with respect to time and by using (4), (6)-(9) and some straightforward inequalities, it can be shown that $\dot{E} \leq 0$, hence the operator $\mathcal{A}_2$ is dissipative for case i.2).
Finally, for case i.3), we choose the following "energy" function $E_2(t)$

$$E_2(t) = E(t) + \frac{1}{2}adw_t^2(1,t) \tag{28}$$

where $E(t)$ is given by (20). By differentiating (28) with respect to time, by using (4), (6)-(9), and following the analysis for the case i.2), it can be shown that $\dot{E}_2$ can be made negative if a is sufficiently small or $c^T b$ is sufficiently large. Provided that, we conclude that operator $\mathcal{A}_2$ is dissipative for the case i.3).
It can easily be shown that in all cases the operator $\lambda I - \mathcal{A}_2 : \mathcal{H} \to \mathcal{H}$ is onto for $\lambda > 0$ (see [11] for similar results). It then follows from the Lumer–Phillips Theorem that the operator $\mathcal{A}_2$ generates a C_0 semigroup of contractions in $\mathcal{H}$.

ii) See [11].

iii) The case $a = 0$, $d > 0$ was proved in [11]. Hence, we consider the case $a > 0$ and $d \geq 0$. It is known that for the uncontrolled case (i.e., (6) and (7) with $v \equiv 0$), the resulting system generates an exponentially decaying analytic semigroup. Since the controller given by (8) and (9) is essentially finite dimensional, it can be shown that the operator $\mathcal{A}_2$ generates an analytic semigroup when $b \equiv 0$. The term multiplying b can be considered as a perturbation, and it can easily be shown that $\mathcal{A}_2$ generates an analytic semigroup when

$||b||$ is sufficiently small (see [14, p. 80-81]). Note that for any $k > 0$, we can rescale b and c as $\tilde{b} = kb$, $\tilde{c} = \frac{1}{k}c$ without changing the transfer function $g(s)$ given by (11). Hence, without loss of generality we can assume that $||b||$ could be selected as small as desired; hence the operator $\mathcal{A}_2$ generates an analytic semigroup. Since the semigroup $T(t)$ generated by $\mathcal{A}_2$ is a contraction semigroup, it follows that the imaginary axis belongs to its resolvent set and that estimate (24) holds for the operator $\mathcal{A}_2$, for all ω; see [14, pp. 61-62]. Therefore, from [9] it follows that the semigroup $T(t)$ generated by $\mathcal{A}_2$ is exponentially decaying. □

IV. Robust Stability with Respect to Small Time Delays

In this section we analyze the stability of the systems (1), (2) and (6), (7) together with the controller (8), (9) and the delayed feedback law $u(t) = -y(t-h)$. To analyze input–output stability of this system, we use the frequency domain approach. The terminology used here is borrowed from [10]. Let $H(s)$ denote the transfer function of a single-input/single-output plant between its input u and its output y. $H(s)$ is said to be well posed if it is bounded on some right-half plane and is said to be regular if it has a limit at $+\infty$ along the real axis. If we apply the unity feedback and set $u = r - y$, where r is the new input, then the closed-loop transfer function between r and y becomes $G^0(s) = H(s)(1+H(s))^{-1}$. When there is a small time delay by ϵ in the feedback loop, the new transfer function $G^\epsilon(s)$ from r to y becomes $G^\epsilon(s) = H(s)(1+e^{-\epsilon s}H(s))^{-1}$. We say that G^0 is robustly stable with respect to delays if there is an $\epsilon_0 > 0$ such that for any $\epsilon \in [0, \epsilon_0]$, G^ϵ is L_2-stable. If this property does not hold, then arbitrary small time delays destabilize G^0.

Let the transfer function $H(s)$ be meromorphic (i.e., analytic except at its poles) on the half plane $C_0 = \{s \in \mathbf{C} | Re\{s\} > 0\}$. Let $\mathcal{B}$ denote the (discrete) set of poles of H in C_0, and let γ^* be defined as

$$\gamma^* = \lim_{|s|\to\infty} \sup_{s \in C_0 - \mathcal{B}} |H(s)|. \tag{29}$$

Theorem 4: Let $H(s)$ be a regular transfer function and assume that $G^0 = H(1+H)^{-1}$ is L_2-stable. Let γ^* be defined as in (29).

i) If $\gamma^* < 1$, then G^0 is robustly stable with respect to delays.

ii) If $\gamma^* > 1$, then G^0 is not robustly stable with respect to delays.

Proof: See [10]. For a different version of this result, see [4]. □

Now consider the system given by (1), (2), (8), and (9). An easy calculation shows that the (open loop) transfer function $H(s)$ from u to y is

$$H(s) = \frac{sg(s)}{(s+a)} \frac{1 - e^{-2(s+a)}}{1 + e^{-2(s+a)}} \tag{30}$$

where $g(s)$ is given by (11) (see also [1] and [10] for the case $g(s) = k$, where $k > 0$ is a constant). Since the system is exponentially stable for the case $a + d > 0$ (see Theorem 1), it follows that G^0 is L_2- stable; hence Theorem 4 is applicable. Note that when $d = 0$, both $g(s)$ and $H(s)$ are strictly proper. As is shown below, this is important for the stability robustness with respect to small delays.

Corollary 1: Consider the system given by (1), (2), (8), and (9). Let the assumptions of Theorem 1 be satisfied. Assume that $a > 0$.

i) If $d < \frac{e^{2a}-1}{e^{2a}+1}$, then G^0 is robustly stable with respect to time delays,

ii) If $d > \frac{e^{2a}-1}{e^{2a}+1}$, then G^0 is not robustly stable with respect to time delays.

Proof: From the formulation it is obvious that Theorem 4 is applicable; hence we need to compute γ^* given by (29). Note that $|g(s)|$ is bounded on C_0 and $g(s) = d + o(1/s)$ for large s. By using this and the results of [10], it can be shown that

$$\gamma^* = d\frac{e^{2a}+1}{e^{2a}-1}. \tag{31}$$

To see this, following [10], first note that for $s \in C_0$, we have $|1 - e^{-2(s+a)}| \leq 1 + e^{-2a}$, and $|1 + e^{-2(s+a)}| \geq 1 - e^{-2a}$, and $|s/(s+a)| \leq 1$. This shows that $\gamma^* \leq d\frac{e^{2a}+1}{e^{2a}-1}$. To prove the reverse inequality, we choose $s_n = 1/n + j(2n+1)\pi/2$ for $n \in \mathbf{N}$. It can easily be shown that $\lim_{n\to\infty} H(s_n) = d\frac{e^{2a}+1}{e^{2a}-1}$ which proves that (31) is satisfied. The claims of Corollary 1 now follows from Theorem 4. □

Remark 1: This result has been known for the nondynamic controller case (i.e., when $g(s) = k$, where $k > 0$ is a constant); see [5], [1], [10], and [7]. Hence Corollary 1 can be considered as a generalization of the similar results presented in the references mentioned above. Note, however, that Corollary 1 is still valid when $d = 0$, in which then case i) is trivially satisfied, hence the corresponding G^0 is always robustly stable with respect to small time delays for all $a > 0$. Moreover by Theorem 1, for the case $d = 0$, the closed-loop system is exponentially stable for $a > 0$ and is asymptotically stable for $a = 0$. Hence, the controller given by (8) and (9) solves the problems stated in the introduction. Moreover, for the case $d = 0$, both the corresponding controller transfer function $g(s)$ and the open-loop transfer function $H(s)$ are strictly proper; see (11) and (30). These points are important for actual implementation of $g(s)$ and for the well posedness of the model; see [8]. □

Next, we consider the system given by (6)-(9). An easy calculation shows that the open-loop transfer function from u to y is

$$H(s) = \frac{sg(s)}{\beta(1+\alpha as)} \frac{1 - e^{-2\beta}}{1 + e^{-2\beta}} \tag{32}$$

where β is given by

$$\beta^2 = \frac{s^2}{(1+as)} \tag{33}$$

and $g(s)$ is given by (11). Since the system is exponentially stable for the case $a + d > 0$ (see Theorem 2), it follows that $G^0 = H(1+H)^{-1}$ is L_2- stable; hence Theorem 4 is applicable. We have the following corollary.

Corollary 2: Consider the system given by (6)-(9). Let the conditions in Theorem 3 are satisfied. Assume that $a > 0$.

i) If $\alpha = 1$, then $\gamma^* = 0$, hence G^0 is robustly stable with respect to small time delays.

ii) If $\alpha = 0$ and $d = 0$, then $\gamma^* = 0$, hence G^0 is robustly stable with respect to small time delays.

iii) If $\alpha = 0$ and $d > 0$, then $\gamma^* = +\infty$, hence G^0 is not robustly stable with respect to small time delays.

Proof: From the formulation it is obvious that Theorem 4 is applicable, hence we need to compute γ^* given by (29). For $s \in C_0$, it follows from (33) that $\beta \in C_0$ as well; hence we have $|1 - e^{-2\beta}| \leq 2$ for $s \in C_0$. Next we show that $\inf_{s\in C_0} |1 + e^{-2\beta}| > 0$. To show this, first we define the set $C_M = \{s \in C_0 | \, |s| > M\}$ for $M > 0$. From (33) it follows that for $|s|$ sufficiently large we have $\beta \sim \sqrt{s/a}$, hence one can easily show that $|1 + e^{-2\beta}| \geq 1 - e^{-\sqrt{2M/a}}$ for $s \in C_M$, provided that M is sufficiently large. An easy calculation also shows that all zeroes of $1 + e^{-2\beta} = 0$ and (33) are in the left half of the complex plane and are all bounded away from the imaginary axis. Hence it follows easily that $\inf_{s\in C_0 - C_M} |1 + e^{-2\beta}| > 0$, for otherwise there must be a zero in $C_0 - C_M$, which is a contradiction. From these arguments it follows that $\frac{1-e^{-2\beta}}{1+e^{-2\beta}}$ is bounded on C_0. Since

$g(s)$ is bounded on C_0 and $g(s) \sim d + o(1/s)$ for $|s|$ sufficiently large, it follows from (29) and (32) that for $\alpha = 1$, we have $\gamma^* = 0$. For the case $\alpha = 0$ and $d = 0$, note that $g(s) \sim o(1/s)$ for large s, hence we have $\gamma^* = 0$. For the case $\alpha = 0$ and $d > 0$, it follows that $\gamma^* \sim \sqrt{s}$ for large s, hence we have $\gamma^* = +\infty$. Now, Corollary 2 follows from Theorem 4. □

Remark 2: Example 2 was first introduced in [7]. The controller proposed in [7] was nondynamic, i.e., (8)-(9) are not present and the controller was given by (3). It can be shown that the conclusions of Theorem 2 and Corollary 2 are valid in this case as well; hence, Theorem 2 and Corollary 2 can be considered as a generalization of similar results presented in [7]. Moreover, as stated in case ii) of Corollary 2, the use of strictly proper controllers (i.e., $d = 0$) eliminates the instability due to small time delays. □

V. Conclusion

In this paper we considered two different damped wave equations, both parameterized by a damping constant $a \geq 0$. When $a = 0$, these equations reduce to the standard conservative wave equation. We assumed that the system is fixed at one end and is controlled at the other end. We studied two problems: stabilization of these models for $a \geq 0$ and robust stabilization against small time delays in the feedback law for $a > 0$. To solve these problems we posed a class of dynamic boundary controllers. Under some assumptions, one of which is the strict positive realness of the controller transfer functions, we obtained various stability results. In particular we showed that the proposed controllers stabilize the models considered for $a \geq 0$ and that robustly stabilizes the same models against small time delays in the feedback loop for $a > 0$. The examples presented here clearly indicates that while strict positive realness of the controller transfer functions is important for stability, the strict properness is important for robustness against small time delays (for the case $a > 0$).

Finally, the ideas presented here can be extended to other flexible structures, such as flexible beams under various modeling assumptions. This will be the subject of a forthcoming paper.

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Optimal Routing of Customers with General Independent Interarrival Times in Deterministic Parallel Queues

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***Abstract*—A queueing system consisting of two parallel heterogeneous servers is considered. Customers can arrive at discrete-valued instants and, upon their arrivals, they are immediately routed to one of the server buffers. The interarrival times are assumed to be integer, independent, identically distributed random variables, whereas the service times of the servers are assumed to be integer and deterministic. The optimization problem considered is the minimization of the customer mean flow time over an infinite horizon. The existence of a stationary optimal policy with a switchover structure is established.**

I. Introduction

In recent years, optimal control of queueing systems has received considerable attention (see, for instance, [1]–[3]). Within this framework, the present paper deals with the optimal on-line assignment (routing) of incoming customers to one of two parallel heterogeneous deterministic servers. Two different decisional models have been proposed in the literature. In the first model, incoming customers are queued in a single waiting line and an assignment decision is made only whenever a server becomes free [4]–[6]. In the second model, assignment decisions are made immediately upon the arrivals of customers. In this case, each server has its own queue. This model was considered in [7] for Poisson arrivals and exponential service times, with the objective of minimizing the discounted number of customers in the system. The optimality of the shortest-line discipline was established. For two identical (exponential) servers, in [8] a different optimization objective was chosen, concerning the total completion time for customers that arrived over a given time interval,

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