Author's Reply by Man Zhihong

I. INTRODUCTION

I thank Dr. Park and Dr. Lee for their valuable comments in on our paper.[1] Indeed, to guarantee that the output tracking error ε_i converges to zero on the terminal sliding mode in a finite time for all bounded initial values, (2.11) in Remark 2.1[1] should be modified as

$$p_1 = 2k+1 \quad k = 0, 1, 2, \ldots \quad k < m. \tag{1}$$

In addition, as remarked above, the controller proposed in our paper[1] cannot generate a bounded control signal at the point $\varepsilon_i = 0$ and $\dot{\varepsilon} \neq 0$ before the error dynamics reach the terminal sliding mode. To deal with this problem, the controller in (3.1)[1] can be modified as follows:

$$u = \begin{cases} -\frac{S}{a_1\|S\|}w & \text{for } \|S\| \neq 0 \text{ with } \varepsilon_i \neq 0 \\ 0 & \text{for } \|S\| = 0 \text{ or } \|S\| \neq 0 \text{ with } \varepsilon_i = 0. \end{cases} \tag{2}$$

It is easy to see that using the above modified control law to replace the law in (3.1),[1] the control signal is bounded in error space. Especially at the point $\varepsilon_i = 0$ and $\varepsilon_i \neq 0$, with $\|S\| \neq 0$, the control $u = 0$ can take the error ε_i away from $\varepsilon_i = 0$, and then another part of the variable structure control law will drive the sliding variable vector S to the terminal sliding mode $S = 0$. The desired error dynamics can then be obtained on the terminal sliding mode according to (2.17) in our paper.[1]

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Comments Regarding "On Stability of Interval Matrices"[1]

Amit Bhaya and Eugenius Kaszkurewicz

Abstract—**We note that several of the results on Schur-stability of interval matrices[1] can be viewed as special cases of results obtained in a more general framework, and some of the latter were previously reported in the literature.**

I. PRELIMINARIES

This section introduces some definitions that are not used in the above-mentioned paper[1] which we follow for notation.

Given a set $\mathcal{A}$ of k real $n \times n$ matrices $\{A_i\}_{i=1}^k$, their convex hull, $\mathcal{P}$, is the set defined in the standard way as the set of all convex combinations of the matrices A_i. The set $\mathcal{P}$, also denoted Conv $(\mathcal{A})$, is called a matrix polytope, and the matrices A_i are called its vertex matrices. It is easy to show that every $n \times n$ interval matrix can be expressed as a polytope (with at most 2^{n^2} vertices), so that theoretically, the stability of an interval matrix is equivalent to the stability of a rank-one matrix polytope, using the terminology of [1].

A class of linear time-varying systems usually referred to as an interval system (or difference inclusion, in the Russian literature) can be associated to an interval matrix as follows: $x(k+1) = A(k)x(k)$, $A(k) \in \mathcal{A}$, $\forall\, k$.

A class of Lyapunov functions frequently used in this context is the class of weighted vector norms: given positive diagonal weighting matrices D and P, $D = \operatorname{diag}(d_1, \cdots, d_n)$, $d_i > 0, \forall\, i$, $P = \operatorname{diag}(p_1, \cdots, p_n)$, $p_i > 0, \forall\, i$, we have $\|x\|_\infty^D := \|D^{-1}x\|_\infty := \max_i \{d_i^{-1}|x_i|\}$, $\|x\|_1^P := \|Dx\|_1 := \sum_i d_i|x_i|$, and $\|x\|_2^P := \|P^{1/2}x\|_2 = \sqrt{\sum_i p_i x_i^2}$ which induce the matrix norms $\|A\|_\infty^D = \|D^{-1}AD\|_\infty$, $\|A\|_1^D = \|DAD^{-1}\|_1$, and $\|A\|_2^P = \|P^{1/2}AP^{-(1/2)}\|_2$, where $P^{1/2} = \operatorname{diag}(\sqrt{p_1}, \cdots, \sqrt{p_n})$ and $P^{1/2} = (P^{1/2})^{-1}$. The functions $\|x\|_\infty^D$, $\|x\|_1^D$ and $(\|x\|_2^P)^2 = x^T Px$ are also known as the diagonal-type weighted infinity-norm, one-norm Lyapunov functions, and diagonal quadratic Lyapunov functions, respectively. For brevity, they are also collectively referred to as diagonal-type Lyapunov functions. Note that the Lyapunov equation, $A^T PA - P = -Q$ (with Q positive definite), admits a positive diagonal solution P if and only if $\|A\|_2^P < 1$, and in this case matrix A is said to be diagonally stable and it will be convenient to denote this as $A \in \mathcal{D}(P)$. Clearly, results stated in terms of diagonal-type Lyapunov functions discussed in this paragraph can be restated as results in terms of diagonally weighted infinity-, one- or two-norms.

The concept of stability and asymptotic stability of a compact set of matrices was introduced in [2]–[4], and a particular case of this concept was later studied in [5]. Following these references, we say that a set $\mathcal{A} = \{A_1, \cdots, A_n\}$ of real $n \times n$ matrices is called simultaneously asymptotically stable if there exists a compact, balanced neighborhood U of the origin in $\mathbb{R}^n$ such that for some $\alpha > 1$ and for all i, $\alpha A_i U \subset U$. It is shown in [4] that this is equivalent to the existence of a single Lyapunov function $V(\cdot)$ that establishes the asymptotic stability of the matrices $A_1, \cdots, A_n$; in this case $V(\cdot)$ is called a simultaneous Lyapunov function (SLF). Note that in the definition in [5], an SLF was restricted to be a quadratic Lyapunov function; following [4] this is not required below, since a set of matrices $\mathcal{A}$ may be simultaneously asymptotically stable and yet not admit a quadratic SLF. Also, in [3], an SLF was referred to as a common Lyapunov function—in this note we are merely using current terminology.

II. COMMENTS

The comments below are organized by reference to the respective sections of the above-mentioned paper.[1]

A. On Section III[1]

Theorem 1[1] and Corollary 1.3[1] are generalized below and given a simple proof via a quadratic diagonal SLF. It is also pointed out that ii) $\Leftrightarrow$ iii) of Theorem 1[1] as well as its Corollaries 1.1[1] and 1.2,[1] were published earlier in the literature in a more general form.

We first restate Theorem 1 of Section III[1] below.

Theorem 1: Let $\pm V$ be a Morishima matrix defined below. Then the following are equivalent:

i) V is Schur-stable.
ii) $|V|$ is Schur-stable.
iii) $\mathcal{A}(-|V|, |V|)$ is Schur-stable.

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[1]M. E. Sezer and D. D. Šiljak, *IEEE Trans. Automat. Contr.*, vol. 39, pp. 368–371.

The equivalence of ii) and iii) holds for any matrix V and not just for the class of Morishima matrices, and this result, as well as Corollaries 1.1[1] and 1.2,[1] were obtained earlier in [11, Theorem 1, Corollaries 1 and 2], where, in fact, both bounded-input bounded-output (BIBO) and asymptotic stability were proven for the linear time-varying case. Note that the proof of ii)[1] implies iii)[1] and relies on localization of eigenvalues using Gerschgorin's theorem so that it may appear it is only valid for the time-invariant case. Indeed, for the case of asymptotic stability, Corollary 1.2[1] had been obtained even earlier in [12, Theorem 3].

Thus the new result in Section III[1] is Corollary 1.3[1] which uses the fact that a Morishima matrix A is similar to the matrix $|A|$; so A is stable if and only if $I - |A|$ is an M-matrix. Therefore, this corollary gives a previously unknown result, vis-à-vis the results in [11] and [12], in the case when the test matrix $\widehat{A}$ is unstable. In this case, once the Morishima sign pattern of the interval matrix being checked is identified, instability can be ensured, and this is explained by Lemma 2.3 below. Thus, if it is known that the interval matrix to be tested is Morishima, then the test matrix $\widetilde{A}$ is the correct test matrix to use rather than $\widehat{A}$ as shown by Lemma 2.3 below.

In Section III[1] the idea of choosing the nominal matrix at the center of the uncertainty hyperrectangle is used to reduce conservatism of the results presented. In fact, this idea was used earlier in [12, Theorem 1, Corollaries 1–3]; in particular, [12, Corollary 2 of Theorem 1] can be used to prove the stability of the interval matrix given at the end of Section III.[1] Finally, it should be pointed out that a survey of many of these ideas can be found in [13].

To generalize Corollary 1.3[1] and Theorem 1,[1] we first make some preliminary observations.

The class of Morishima matrices is a subclass of the class of checkerboard matrices, introduced in [6], and the latter class is defined as follows: given a real matrix A, if there exist two diagonal matrices, K_1 and K_2 whose diagonal elements are all ± 1 (also called signature matrices) such that $K_1 A K_2 = |A|$, then A is said to be checkerboard (if $K_1 = K_2, A$ is called Morishima). A is said to be a stable checkerboard matrix if $|A|$ is stable. It was shown in [7] (later published as [8]) that a stable checkerboard matrix admits a diagonal solution to its Lyapunov equation. Therefore we have the following facts.

Fact 2.1: A Schur-stable Morishima matrix is diagonally stable.□

Fact 2.2: The positive diagonal matrix P which is a solution for the Lyapunov equation for A (Morishima) is also a solution for this equation for $|A|$, i.e., A Morishima and $A \in \mathcal{D}(P)$ implies $|A| \in \mathcal{D}(P)$.

Proof: Since A is a Schur-stable Morishima matrix, there exists $P > 0$, diagonal, such that

$$A^T PA - P < 0$$

and consequently, for the nonsingular signature matrix S that transforms A to $|A|$, we have

$$\begin{aligned} 0 &> S^{-1}A^T SPSAS^{-1} - S^{-1}PS^{-1} \\ &= |A|^T P|A| - P. \end{aligned}$$

■

Considering the above, we can state the following generalization that includes Theorem 1[1] and Corollary 1.3[1].

Lemma 2.3: Assume that for all k, $A(k)$ belongs to a class of Morishima matrices having the same sign pattern with S denoting the signature matrix that transforms any matrix B in this class to the nonnegative matrix $|B|$. Also assume that $\underline{A} \leq A(k) \leq \overline{A}$. The following matrices and classes of systems are then associated to the Morishima interval matrix, denoted $\mathcal{A}(\underline{A}, \overline{A})$:

$$\begin{aligned} \widehat{A} &= (\hat{a}_{ij}) = \max\{|\underline{a}_{ij}|, |\overline{a}_{ij}|\} \\ \widetilde{A} &= S^{-1}\widehat{A}S, \text{ thus } \widetilde{A} \in \mathcal{A}(\underline{A}, \overline{A}) \\ x(k+1) &= |A(k)|x(k), \quad A(k) \in \mathcal{A}(\underline{A}, \overline{A}) \qquad (1) \\ x(k+1) &= A(k)x(k), \quad A(k) \in \mathcal{A}(\underline{A}, \overline{A}). \qquad (2) \end{aligned}$$

The following statements are equivalent:

a) $\widehat{A}$ is Schur-stable.
b) $|A|$ is Schur-stable for all $A \in \mathcal{A}(\underline{A}, \overline{A})$.
c) $\widehat{A}$ is diagonally stable, i.e., $\widehat{A} \in \mathcal{D}(P)$.
d) The class of systems (1) possesses a globally exponentially stable zero solution, and $V(x) = |x|^T P|x|$ is a quadratic diagonal SLF that establishes this fact.
e) $\widetilde{A}$ is Schur-stable.
f) A is Schur-stable for all $A \in \mathcal{A}(\underline{A}, \overline{A})$.
g) $\widetilde{A}$ is diagonally stable, i.e., $\widetilde{A} \in \mathcal{D}(P)$.
h) The class of systems (2) possesses a globally exponentially stable zero solution, and $V(x) = |x|^T P|x|$ is a quadratic diagonal SLF that establishes this fact.

Proof: Consider the quadratic diagonal SLF $V(x) = |x|^T P|x|$ for the interval system (2). Note the following equality:

$$\begin{aligned} V(x) &= |x|^T P|x| \\ &= |x|^T SPS|x| \end{aligned}$$

where the diagonal matrix S is the signature matrix corresponding to the transformation $A \to |A|$. Now

$$\begin{aligned} V[x(k+1)] &= |A(k)x(k)|^T SPS|A(k)x(k)| \\ &\leq |x(k)|^T |A(k)|^T SPS|A(k)||x(k)| \\ &\leq |x(k)|^T \widehat{A}^T SPS\widehat{A}|x(k)| \\ &= |x(k)|^T S\widetilde{A}^T S^{-1}SPSS^{-1}\widetilde{A}S|x(k)| \\ &= |x(k)|^T S\widetilde{A}^T P\widetilde{A}S|x(k)|. \end{aligned}$$

Thus

$$\Delta V[x(k)] \leq |x(k)|^T S(\widetilde{A}^T P\widetilde{A} - P)S|x(k)|$$

and since $\widetilde{A} \in \mathcal{D}(P)$, it follows that $\Delta V[x(k)]$ is negative definite. This proves g) $\Rightarrow$ h) $\Rightarrow$ f). The implication f) $\Rightarrow$ e) is trivial. The implication e) $\Rightarrow$ g) follows from Fact 2.1.

A similar development is used to show c) $\Rightarrow$ d) $\Rightarrow$ b) $\Rightarrow$ a) $\Rightarrow$ c) (see [10]). The equivalence g) $\Leftrightarrow$ c) is consequence of Fact 2.2. ■

From the above lemma, we see that whenever $A(k)$ is a Morishima matrix belonging to an interval of Morishima matrices $\mathcal{A}(\underline{A}, \overline{A})$, for all k, then $|A(k)| = SA(k)S^{-1}$ is the corresponding nonnegative matrix (S being the associated diagonal signature similarity transformation). Notice that for the nonnegative matrix $|A(k)|$, $\widehat{A}$ is a test matrix. Correspondingly, the matrix $\widetilde{A} = S^{-1}\widehat{A}S$ is a test matrix for the Morishima sign-pattern interval matrix $\mathcal{A}(\underline{A}, \overline{A})$.

Remark: One of the reviewers called our attention to a connective stability result [9, Theorem 2]. The equivalence c) $\Leftrightarrow$ h) of our Lemma 2.3 corresponds to a discrete-time version of [9, Theorem 2] which, however, was not stated in the context of interval systems or SLF's. In view of our results above, it may be said that the connective stability of a system with scalar subsystems is equivalent to the stability of a special interval system that is constructed from the nominal matrix.

B. On Section V[1]

This subsection has the following objectives: i) to put the results of Section V[1] on Schur stability of convex combinations of matrices in the more general context of simultaneous stability of polytopes of matrices; ii) to show that the concepts of proportional dominance, positive quasidiagonal dominance (PQDD), and connective stability[1] are all subsumed by the concepts of SLF's and simultaneous stability; and iii) to give a new result on the relationship between the existence of different types of SLF's for polytopes of nonnegative matrices that clarifies several points about the examples in the above-mentioned paper.[1]

The Brayton and Tong result [4] (see also [14] and the special cases in [15] and [16]) can be stated in the present context as follows: given a polytope of matrices $\mathcal{A}$, the class of time-varying interval systems $x(k+1) = A(k)x(k)$, $A(k) \in \mathcal{A}$, $\forall\, k$ is asymptotically stable iff there exists a single norm in which all vertex matrices have norm strictly bounded below unity. This norm is an SLF for all matrices in $\mathcal{A}$. Note that in this case, clearly every matrix in the convex hull of the vertex matrices is asymptotically stable. In fact, it can be shown that all finite products (of a certain length or greater) of matrices in the convex hull have norm less than unity in any induced matrix norm [17], [18]. This result and applications to nonlinear time-varying systems were discussed in [7] and [18].

From the point of view of this result, the concept of row (respectively, column) proportional dominance[1] can be interpreted as a sufficient condition for the existence of a diagonal type weighted infinity-norm (respectively, one-norm) SLF for polytopes of nonnegative matrices. Indeed, from this perspective, Theorem 3[1] is a direct consequence of convexity as follows:

$$\|A_i\|_\infty^D < 1, \text{ for all } \; i = 0, 1, \cdots, N$$

implies that

$$\left\|\sum_i^N \alpha_i A_i\right\|_\infty^D \le \sum_i^N \alpha_i \|A_i\|_\infty^D < \sum_i^N \alpha_i = 1.$$

In this context, consider the following example.

Example 2.4:

$$A_0 = \begin{bmatrix} 0.1 & 0.6 \\ 0.1 & 0.9 \end{bmatrix}$$
$$A_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.3 \end{bmatrix}$$

is a pair of matrices such that considering the matrices $I - A_0$ and $I - A_1$, neither one proportionally dominates the other, either rowwise or columnwise. However, $V(x) = \max\{1.4x_1, x_2\}$ is a diagonal-type weighted infinity-norm SLF for the pair that establishes the (simultaneous) asymptotic stability of convex combinations of A_0 and A_1. This example thus shows that proportional dominance is only a sufficient and not a necessary condition for the existence of a diagonal type infinity- or one-norm SLF.

The result below establishes the relationship between the existence of one-, two-, and ∞-norm SLF's for a polytope of nonnegative matrices using a result of Araki [23, Theorem 2] on M-matrices.

Proposition 2.5: Given a polytope of nonnegative matrices, $\mathcal{A} = \operatorname{Conv}\{A_1, \cdots, A_n\}$, a) implies b), where statements a) and b) are as follows:

a) There exist positive diagonal matrices D_r and D_c such that

$$\|D_r^{-1} A_i D_r\|_\infty < 1, \quad i = 1, \cdots, n \tag{3}$$
$$\|D_c^{-1} A_i^T D_c\|_\infty = \|D_c A_i D_c^{-1}\|_1 < 1, \quad i = 1, \cdots, n. \tag{4}$$

b) There exists a positive diagonal matrix P such that

$$\|P^{1/2} A_i P^{-(1/2)}\|_2 < 1, \quad i = 1, \cdots, n. \tag{5}$$

Furthermore, $P = D_c D_r^{-1}$.

Remark: It will be shown by example 2.7 below that statement b) does not imply statement a).

Proof: Note that if (3) and (4) hold, this is equivalent to asserting that the vectors d_r and d_c, which are the diagonals of the matrices D_r and D_c, respectively, are such that

$$u_i = (I - A_i) d_r > 0, \qquad i = 1, \cdots, n$$
$$v_i = (I - A_i)^T d_c > 0, \quad i = 1, \cdots, n.$$

Let $P = D_c D_r^{-1}$ as claimed. Then

$$\begin{aligned}(P - A_i^T P A_i) d_r &= P d_r - A_i^T P A_i d_r \\ &= d_c - A_i^T P (d_r - u_i) \\ &= (d_c - A_i^T d_c) + A_i^T P u_i \\ &= v_i + A_i^T P u_i, \quad i = 1, \cdots, n.\end{aligned}$$

But $v_i > 0$, and since $A_i^T \ge 0$, $P > 0$, $A_i^T P u_i \ge 0$. Thus $(P - A_i^T P A_i) d_r > 0$, i.e., $P - A_i^T P A_i$ is an M-matrix for all i. Since it is symmetric, it must be positive definite for all i. ■

In words, this proposition says that for polytopes of nonnegative matrices, the existence of both weighted infinity- and one-norm diagonal type SLF's implies the existence of a quadratic diagonal SLF.

Proposition 2.5 assures us that the only possibility that is ruled out is the existence of scalings D_r and D_c together with the inexistence of a scaling P. Alternatively, if P satisfying (5) does not exist, we may affirm that at most one of D_r and D_c exists.

In view of this proposition, we now examine two examples.

Example 2.6: Consider the following positive matrices[1]:

$$A_0 = \begin{bmatrix} 0.4 & 0.4 \\ 1.0 & 0.2 \end{bmatrix}$$
$$A_1 = \begin{bmatrix} 0.4 & 0.4 \\ 0.1 & 0.9 \end{bmatrix}$$

used to illustrate that

> "although $I - A_0$ and $I - A_1$ are simultaneously PQDD there exist no diagonal-type Lyapunov function that would prove simultaneous stability of A_0 and A_1."[1] The authors go on to say that "this fact points out the significance of connective stability which is diagonal-type simultaneous stability that can be established via simultaneous PQDD conditions."[1]

As noted above, the simultaneous PQDD condition ($= I - A_1$ proportionally dominating $I - A_0$) implies that there exists a positive diagonal weighting matrix D_r that defines a weighted infinity-norm SLF such that $\|D_r^{-1} A_i D_r\|_\infty < 1$, $i = 0, 1$. It is easy to show that there does not exist a positive diagonal matrix D_c satisfying (4), and that there also does not exist a positive diagonal P satisfying (5). It is therefore correct to say that there exists no quadratic diagonal SLF for A_0 and A_1, although there does exist a infinity-norm diagonal-type SLF for this pair. Reemphasizing what was said earlier in [7] and [19], both weighted two-norms and weighted infinity-norms are diagonal-type SLF's, and this is intimately related to many pleasant features enjoyed by these functions in the context of robust stability.

On the other hand, now we consider the following example.

Example 2.7:

$$A_0 = \begin{bmatrix} 0.6 & 0.1 \\ 0.7 & 0.5 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.3 \end{bmatrix}$$

is a pair of matrices such that, considering $I - A_0$ and $I - A_1$, neither one proportionally dominates the other, either rowwise or columnwise. In fact, there do not exist diagonal scalings satisfying either (3) or (4). Note also that the majorant matrix $\widehat{A}$ is unstable so that proving stability of the convex combination by "embedding" in (overbounding by) an interval matrix also does not work for this example. However, $P = \operatorname{diag}(0.9, 0.7)$ satisfies (5), so there exists a diagonal quadratic SLF that proves stability of convex combinations of A_0 and A_1, i.e., the global exponential stability of the zero solution of the difference inclusion $x(k+1) = A(k)x(k)$, $A(k) \in \operatorname{Conv}\{A_0, A_1\}$, $\forall\, k$. In other words, for this example of a robust stability problem, the general concept of simultaneous stability holds, even though connective stability cannot be established via simultaneous PQDD conditions.

Finally, note that stability of polytopes of matrices can be tested by checking that in any fixed matrix norm, all finite products of vertex matrices of a fixed length have norm strictly less than one (for a precise statement, see [18]). If this result is applied to the above example and the infinity norm is used, it turns out that all eight products of length three of the vertex matrices have infinity norm less than unity, proving stability of the difference inclusion $x(k+1) = A(k)x(k)$, $A(k) \in \operatorname{Conv}\{A_0, A_1\}$, $\forall\, k$.

In the context of time-varying uncertainties in hyperrectangles, it has been noted in [20, p. 680] that connective stability is a type of simultaneous stability in which stability of a single member of a class implies the stability of the whole class. Thus one may regard results on simultaneous stability of general matrix polytopes (see, for example, [4], [5], and [18], etc.) as generalizations of the concept of connective stability to polytopes. In the general case, however, it is no longer enough to check stability of one vertex and simultaneous stability of all vertices is required to ensure asymptotic stability (in the time-varying case). In conclusion, it may be said that the concept of simultaneous stability contains that of connective stability in the context of interval matrices.

For further discussion on the unifying role of simultaneous and diagonal stability in large scale system analysis and the relations between connective stability and diagonal stability, the reader is referred to [7], [21], and [22].

As a final comment, note that the computation of the infinity-norm SLF's in Proposition 2.5 is equivalent to finding a feasible point for each of two systems of linear inequalities derived from the rows and columns of the vertex matrices. This is interesting since the excellent monograph of Boyd and coworkers [24] shows that the problem of finding a quadratic SLF for a general polytope of matrices is a linear matrix inequality (LMI) problem that: a) does not, in general, admit an analytic solution; b) may, however, be seen to be a convex program and hence efficiently solvable. For the special case of nonnegative matrices, Proposition 2.5 shows that a simplification to the search for a feasible solution to a linear program may be possible. It should also be noted that several results have been obtained recently on the computational complexity of interval matrix stability and nonsingularity problems [1], [18], and [25], and, roughly speaking, it may be asserted that in general these problems are all hard in a technical sense.

III. Concluding Remarks

In the interim between the submission of the above-mentioned paper[1] and its publication, some closely related results have appeared which we take the opportunity to mention. In [10], there is a derivation of Corollary 1.2[1] using a diagonal quadratic SLF, a discussion of application to the case of nonlinear time-varying systems (this is mentioned as a topic of future research in the conclusions[1]) as well as an identification of other classes (not mentioned in the paper[1]) for which necessary and sufficient conditions for interval matrix stability are obtained. Finally, [18] contains additional material on simultaneous stability of polytopes of matrices and related concepts.

Authors' Reply by M. E. Sezer and D. D. Šiljak

We are grateful to the authors of the Comments for pointing out the fact that two of our corollaries were published in a conference paper that we did not acknowledge due to our oversight. Our response to other issues of the comments is as follows:

1) It is true that equivalence of (ii) and (iii) of our Theorem 1 holds for any matrix and not just for the class of Morishima matrices. Indeed, V being a Morishima matrix, is needed only in the proof of $(i) \Rightarrow (ii)$. We preferred to state in the same theorem $(ii) \Leftrightarrow (iii)$ for general V, and $(i) \Leftrightarrow (ii)$ for V Morishima, as our main purpose was to prove $(i) \Leftrightarrow (iii)$.
2) It is also true that Corollaries 1.1 and 1.2 of our paper were obtained earlier. We have no claim that these results are new; in fact, we acknowledge that Corollary 1.1 is the main result of [9] of our paper. Corollary 1.2, on the other hand, was stated only to pave the way to Corollary 1.3 which can be regarded as the main result of our paper.
3) The idea of choosing the nominal matrix at the center of the uncertainty hyperrectangle is nothing but a form of connective stability. Then, first, it is not fair to say that this idea was used previously by someone else. Second, any method based on connective stability can be used to prove stability of the interval matrix given at the end of Section III of our paper,[1] [12, Corollary 2] being one such method. However, more important than these facts is to realize that the problem considered in our paper is not just to find necessary and/or sufficient conditions for interval stability, but to do this in the spirit of connective stability by testing only one matrix (an extreme vertex) if possible. If such a test matrix is not available, then one can always try other ways to prove or disprove interval stability. Clearly, unavailability of such a convenience does not mean conservatism.
4) We disagree with the authors on their Lemma 2.3 being a generalization of our Theorem 1. Assuming that $A(k)$ belongs to a class of Morishima matrices, having the same sign pattern is the same thing as assuming $\underline{A}$ and $\overline{A}$ are Morishima matrices having the same sign pattern. This assumption makes the insertion of time in our result just as trivial as it would be in the case $\underline{A} > 0$ or $\overline{A} < 0$. It is then no surprise that one can find a quadratic SLF.
5) The authors of the comments criticize our discussion of simultaneous stability for being restricted to quadratic diagonal type and then state Proposition 2.5 to provide sufficient conditions for existence of a quadratic diagonal Lyapunov function. Now, if we are to remain within the framework of quadratic diagonal simultaneous stability, one can argue that their Proposition 2.5 does not solve the simultaneous stability problem but

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transforms it to another problem of finding suitable D_r and D_e which is no easier than the original problem itself. On the other hand, proportional dominance, which they criticize for being a special case, is a direct test. By including our Example 2.6 in their comments, they strengthen our point! Furthermore, if we are to allow other types of diagonal Lyapunov functions as in their Example 2.7, then we can as well be more general and treat the problem in the context of connective stability. The reader and the authors of the Comments can easily verify that P in authors' Example 2.7 guarantees connective stability of $A_N \pm eA_c$, where $A_N = (1/2)(A_0 + A_1)$, $A_c = (1/2)(A_0 - A_1)$ and $e \in [0, 1]$.

6) For the record, the connective stability was introduced at the Ninth Allerton Conference on Circuit and System Theory in 1971 with the following properties: i) It is interval stability because the coefficients of system matrix are allowed to vary independently within the hyperrectangle in the coefficient space; ii) It is simultaneous stability, because a single Lyapunov function (not necessarily quadratic), which is constructed for a system at an extreme vertex of the rectangle, establishes stability for the family of systems defined by the rectangle, and iii) It applies to nonlinear time-varying systems with stochastic, time delay, and singular perturbations.

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Comments on "Perturbation Bounds for Root-Clustering of Linear Systems in a Specified Second Order Subregion"[1]

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***Abstract*—This paper comments on the results of a recent paper.[1] We note that Theorems 3.1 and 4.1 are incorrectly stated, i.e., they are not valid for the non-Ω-transformable regions. The results[1] cannot cover the ride quality region listed in Table 1[1] since it is a non-Ω-transformable region.**

I. Introduction

In a recent paper,[1] the authors provide sufficient bounds for root-clustering in a specified second-order subregion of the complex plane for uncertain matrices with structured or unstructured uncertainties. Their results are less conservative for an Ω-transformable region[1].

In this paper, we note that there are some minor erroneous statements in the paper[1] and these are as follows. In Theorems 3.1 and 4.1,[1] the authors claim that the perturbed system matrix $A+E$ has all the eigenvalues located in the subregion Ω_2 described by (2.3),[1] i.e.,

$$\Omega_2 = \{(x, y) \mid \gamma_{00} + \gamma_{10}x + \gamma_{20}x^2 + \gamma_{02}y^2\} \tag{1}$$

where γ_{fg} is a real coefficient. However, there is one restriction on these main results of the paper[1] since the subregion Ω_2 described by (2.3)[1], i.e., (1), includes Ω-transformable and non-Ω-transformable

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