From (13) and property 1), $\dot{e}_i(t)$ are bounded, $1 \leq i \leq \rho$. By virtue of Barbalat's lemma [18], [21], [22] it follows that $\lim_{t\to\infty} e_i(t) = 0$, $1 \leq i \leq \rho$, which implies, in particular, that $\lim_{t\to\infty} [y(t) - y_r(t)] = 0$. □

Remark II.1: The adaptation gain γ in (14) is a positive real constant in the proof of Theorem II.1. Note that if we set $\gamma = 0$, that is the adaptation is switched off, the arguments used to show properties 1) and 2) still apply while the arguments used to show property 3) do not apply. This shows that adaptation may be switched off at any time still guaranteeing bounded signals and disturbance attenuation at the expense of not achieving asymptotic tracking even when disturbances are zero. □

Remark II.2: The result stated in Theorem II.1 may also be obtained for the class of nonlinear systems introduced in [15, Lemma 2.2], following the proofs of Lemma II.1 and Theorem II.1 with obvious modifications. □

III. Conclusions

We formulate in Definition II.1 the state feedback adaptive tracking problem with transient specifications and disturbance attenuation and provide a constructive solution to such a problem in Theorem II.1 for a class of nonlinear systems. The result obtained improves and generalizes those presented in [15], [16], and [23]. The proposed robust adaptive control (14) has the built-in property of still guaranteeing bounded signals and disturbance attenuation when the adaptation is switched off. Moreover, it offers the advantage of achieving zero asymptotic tracking error when disturbances are not present and time-varying parameters become constant; this is not guaranteed by the robust (nonadaptive) control presented in [16].

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Stabilization and Disturbance Rejection for the Wave Equation

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***Abstract*— We consider a system described by the one-dimensional linear wave equation in a bounded domain with appropriate boundary conditions. To stabilize the system, we propose a dynamic boundary controller applied at the free end of the system. The transfer function of the proposed controller is a proper rational function of the complex variable s and may contain a single pole at the origin and a pair of complex conjugate poles on the imaginary axis, provided that the residues corresponding to these poles are nonnegative; the rest of the transfer function is required to be a strictly positive real function. We then show that depending on the location of the pole on the imaginary axis, the closed-loop system is asymptotically stable. We also consider the case where the output of the controller is corrupted by a disturbance and show that it may be possible to attenuate the effect of the disturbance at the output if we choose the controller transfer function appropriately. We also present some numerical simulation results which support this argument.**

***Index Terms*— Boundary control systems, distributed parameter systems, disturbance rejection, semigroup theory, stability.**

I. Introduction

Many mechanical systems, such as spacecraft with flexible attachments or robots with flexible links, and many practical systems such as power systems and mass transport systems, contain certain parts whose dynamic behavior can be rigorously described only by partial differential equations (PDE's). In such systems, to achieve high precision demands, the dynamic effect of the system parts whose

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behavior are described by PDE's on the overall system has to be taken into account in designing the controllers.

In recent years, boundary control of systems represented by PDE's has become an important research area. This idea is first applied to the systems represented by the wave equation (e.g., elastic strings, cables; see, e.g., [1] and [5]) and recently extended to beam equations [2] and to the rotating flexible structures (see [10] and [11]). In particular, it has been shown that for a string which is clamped at one end and is free at the other end, a single *nondynamic* boundary control applied at the free end is sufficient to exponentially stabilize the system; see [1]. For more references on the subject the reader is referred to [7].

In this paper, we consider a linear time-invariant system which is represented by the one-dimensional wave equation in a bounded domain. We assume that the system is fixed at one end and a boundary control input is applied at the other end. For this system, we propose a finite-dimensional *dynamic* boundary controller. This introduces extra degrees of freedom in designing controllers which could be exploited in solving a variety of control problems, such as disturbance rejection, pole assignment, etc., while maintaining stability. The transfer function of the controller is a proper rational function of the complex variable s and may contain a single pole at $s = 0$ and a pair of complex conjugate poles at $s = \pm j\omega_1, \omega_1 \neq 0$, provided that the residues corresponding to these poles are nonnegative; the rest of the transfer function is required to be a strictly positive real function. The class of strictly positive real transfer functions (e.g., excluding the poles on the imaginary axis) has been proposed before for the stabilization of flexible structures; see [12] for the beam equation, [13] for the wave equation, and [14] for disturbance rejection. We then show that if $\omega_1 \neq m\pi$ for some natural number $m \in \boldsymbol{N}$, then the closed-loop system is asymptotically stable. We also consider the case where the output of the controller is corrupted by a disturbance. We show that if the structure of the disturbance is known (i.e., the frequency spectrum), then it may be possible to choose the controller accordingly to attenuate the effect of the disturbance at the system output. To support this idea, we also present some numerical simulation results.

This paper is organized as follows. In the next section we introduce the system considered and propose a class of controller for stabilization. In Section III we give some stability results. In Section IV we consider the disturbance rejection problem, and in Section V we present some numerical simulation results. Finally, we give some concluding remarks.

II. Problem Statement

We consider a string as an example of a system whose behavior is modeled by the wave equation. Without loss of generality, we assume that the string length, mass density, and the string tension are given as $L = 1, \rho = 1$, and $T = 1$, respectively. We denote the displacement of the string by $y(x,t)$ at $x \in (0,1)$ and $t \geq 0$. Furthermore, we assume that the string is fixed at one end and stabilized by *dynamic* boundary control at the other end. Thus, the system under consideration is represented by

$$y_{tt}(x,t) = y_{xx}(x,t), \qquad x \in (0,1) \qquad t \geq 0 \tag{1}$$

$$y(0,t) = 0, \qquad t \geq 0 \tag{2}$$

$$y_x(1,t) = -f(t), \qquad t \geq 0 \tag{3}$$

$$y(x,0) = w_0(x), \quad y_t(x,0) = w_1(x) \tag{4}$$

where a subscript, as in y_t, denotes a partial differential with respect to the corresponding variable, $f(\cdot)$: $\boldsymbol{R}_+ \to \boldsymbol{R}$ is the boundary control force applied at the free end of the string, and initial conditions $w_0(\cdot)$ and $w_1(\cdot)$ belong to appropriate function spaces which will be introduced later [see (18)–(21)]. We note that the systems represented by (1)–(4) are not restricted to strings; for example, vibrations of long cables, the longitudinal motion, and the torsional vibrations of elastic beams can also be represented by these equations (see, e.g., [9]).

It is well known that if we apply the following boundary controller:

$$f(t) = dy_t(1,t), \qquad d > 0 \tag{5}$$

then the closed-loop system given by (1)–(5) is exponentially stable. That is, the energy $E(t)$ associated with the solutions of (1)–(5), which is given by

$$E(t) = \tfrac{1}{2}\int_0^1 (y_t^2 + y_x^2)\, dx \tag{6}$$

decays uniformly exponentially as follows:

$$E(t) \leq Me^{-\delta t}E(0), \qquad t \geq 0 \tag{7}$$

for some $M > 0$ and $\delta > 0$. Moreover, for $d \neq 1$, the decay rate δ is given by

$$\delta = -\frac{1}{2}\ln\left|\frac{1-d}{1+d}\right| \tag{8}$$

(see, e.g., [7]). For $d = 1$, the estimate (8) will not be valid, but in this case all solutions become zero for $t > 2$; see, e.g., [8]. This result can also be obtained by using the so-called wave propagation method; see e.g., [3]. The solution of the wave equation can be written as the sum of two waves traveling in opposite directions. These waves will be reflected at the boundaries, and at any particular time t and position x, the solution $y(x,t)$ is the sum of these waves. It can be shown that the reflection coefficient σ at the boundary $x = 1$ is $\sigma = (1-d)/(1+d)$ (i.e., the incoming wave amplitude is multiplied by σ to find the amplitude of the reflected wave). Since for $d = 1$ we have $\sigma = 0$, and since any traveling wave will reach the boundary $x = 1$ at most 2 s after the initial time $t = 0$, it follows that for $t > 2$ we have $y(x,t) = 0$; for details see [3], [7], and [8]. Hence, for the controller given by (5), the best choice of d for stabilization seems to be $d = 1$. However, we will show later that when the system is subjected to a disturbance, due to measurements and actuation, this choice may not be a good one.

The problem we consider in this paper is to choose the controller which generates $f(t)$ appropriately to make the closed-loop system stable in some sense. Later we will analyze the effect of this controller on the output of the system $(y_t(1,t))$ when the controller is corrupted by a disturbance.

We assume that $f(t)$ is given by the following equations:

$$\dot{z}_1 = Az_1 + by_t(1,t) \tag{9}$$

$$\dot{x}_1 = \omega_1 x_2 \tag{10}$$

$$\dot{x}_2 = -\omega_1 x_1 + y_t(1,t) \tag{11}$$

$$f(t) = c^T z_1 + dy_t(1,t) + k_1 y(1,t) + k_2 x_2 \tag{12}$$

where $z_1 \in \boldsymbol{R}^n$ for some natural number n is the actuator state, $A \in \boldsymbol{R}^{n\times n}$ is a constant matrix, $b, c \in \boldsymbol{R}^n$ are constant column vectors, $d \in \boldsymbol{R}$, and the superscript T denotes transpose.

We make the following assumptions concerning the actuator given by (9)–(12) throughout this work.

Assumption 1: All eigenvalues of $A \in \boldsymbol{R}^{n\times n}$ have negative real parts.

Assumption 2: (A, b) is controllable and (c, A) is observable.

Assumption 3: $d \geq 0, k_1 \geq 0, k_2 \geq 0$, and there exists a constant $\gamma, d \geq \gamma \geq 0$, such that the following holds:

$$d + \mathcal{R}e\{c^T(j\omega I - A)^{-1}b\} > \gamma, \qquad \omega \in \boldsymbol{R}. \tag{13}$$

Moreover, for $d > 0$, we assume $\gamma > 0$ as well.

Remark 1: If we take the Laplace transform of (9)–(12) and use zero initial conditions for the actuator (i.e., $z_1(0) = 0, x_1(0) = x_2(0) = 0$), then we obtain

$$\hat{f}(s) = g(s)\hat{y}_t(1,s) + \frac{k_1}{s}y(1,0) \tag{14}$$

where a hat denotes the Laplace transform of the corresponding variable and

$$\begin{aligned} g(s) &= g_1(s) + \frac{k_1}{s} + \frac{k_2 s}{s^2+\omega_1^2} \\ g_1(s) &= c^T(sI-A)^{-1}b + d. \end{aligned} \tag{15}$$

Then, Assumptions 1)-3) imply that $g_1(s)$ is a strictly positive real transfer function, and if $k_1 > 0$ and/or $k_2 > 0$, then $g(s)$ is only a positive real (but not strictly) transfer function; see, e.g., [17]. □

III. Stability Results

Let Assumptions 1)–3) stated above hold. Then, since the transfer function $g_1(s) = d + c^T(sI - A)^{-1}b$ is strictly positive real, it follows from the Meyer–Kalman–Yakubovich lemma that given any symmetric positive definite matrix $Q \in \boldsymbol{R}^{n\times n}$, there exists a symmetric positive definite matrix $P \in \boldsymbol{R}^{n\times n}$, a vector $q \in \boldsymbol{R}^n$, and a constant $\epsilon > 0$ satisfying

$$A^TP + PA = -qq^T - \epsilon Q \tag{16}$$

$$Pb - c = \sqrt{2(d-\gamma)}q \tag{17}$$

(see [17, p. 133]).

To analyze the system given by (1)–(4) and (9)–(12), we first define the function space $\mathcal{H}$ as follows:

$$\begin{aligned} \mathcal{H} := \{&(uvz_1x_1x_2)^T | u \in \boldsymbol{H}^1, v \in \boldsymbol{L}^2, z_1 \in \boldsymbol{R}^n, \\ &x_1, x_2 \in \boldsymbol{R}, u(0) = 0\} \end{aligned} \tag{18}$$

where the spaces $\boldsymbol{L}^2$ and $\boldsymbol{H}^k$ are defined as follows:

$$\boldsymbol{L}^2 = \left\{ f\colon [0,L] \to \boldsymbol{R} \middle| \int_0^1 f^2\, dx < \infty \right\} \tag{19}$$

$$\boldsymbol{H}^k = \{ f \in \boldsymbol{L}^2 | f, f', f'', \cdots, f^{(k)} \in \boldsymbol{L}^2 \}. \tag{20}$$

System (1)–(4), (9)–(12) can be written in the following abstract form:

$$\dot{z} = Lz, \qquad z(0) \in \mathcal{H} \tag{21}$$

where $z = (yy_tz_1x_1x_2)^T \in \mathcal{H}$, the operator $L\colon \mathcal{H} \to \mathcal{H}$ is a linear unbounded operator defined as

$$L\begin{pmatrix} u \\ v \\ z_1 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} \\ Az_1 + bv(1) \\ \omega_1 x_2 \\ -\omega_1 x_1 + v(1) \end{pmatrix}. \tag{22}$$

The domain $D(L)$ of the operator L is defined as

$$\begin{aligned} D(L) := \{&(uvz_1x_1x_2)^T \in \mathcal{H} | u \in \boldsymbol{H}^2, v \in \boldsymbol{H}^1, z_1 \in \boldsymbol{R}^n, \\ &x_1, x_2 \in \boldsymbol{R}, u(0) = 0, v(0) = 0; \\ &u_x(1) + c^Tz_1 + dv(1) + k_1u(1) + k_2x_2 = 0\}. \end{aligned} \tag{23}$$

Let Assumptions 1)–3) hold, let $Q \in \boldsymbol{R}^{n\times n}$ be an arbitrary symmetric positive definite matrix, and let $P \in \boldsymbol{R}^{n\times n}, q \in \boldsymbol{R}^n$ be the solutions of (16) and (17) where P is also a symmetric and positive definite matrix. In $\mathcal{H}$, we define the following "energy" inner-product:

$$\begin{aligned} \langle z, \tilde{z} \rangle_E = &\tfrac{1}{2}\int_0^1 y_t\tilde{y}_t\, dx + \tfrac{1}{2}\int_0^1 y_x\tilde{y}_x\, dx \\ &+ \tfrac{1}{2}k_1y(1)\tilde{y}(1) + \tfrac{1}{2}\tilde{z}_1^TPz_1 \\ &+ \tfrac{1}{2}k_2(x_1\tilde{x}_1 + x_2\tilde{x}_2) \end{aligned} \tag{24}$$

where $z = (yy_tz_1x_1x_2)^T, \tilde{z} = (\tilde{y}\tilde{y}_t\tilde{z}_1\tilde{x}_1\tilde{x}_2)^T$. It can be shown that $\mathcal{H}$, together with the energy inner-product given by (24), becomes a Hilbert space. The "energy" norm induced by (24) is

$$\begin{aligned} E(t) := \|z(t)\|_E^2 = &\tfrac{1}{2}\int_0^1 y_t^2\, dx + \tfrac{1}{2}\int_0^1 y_x^2\, dx \\ &+ \tfrac{1}{2}k_1y^2(1,t) + \tfrac{1}{2}z_1^TPz_1 \\ &+ \tfrac{1}{2}k_2(x_1^2 + x_2^2). \end{aligned} \tag{25}$$

Theorem 1: Consider the system given by (21).

i: The operator L generates a C_0-semigroup of contractions $T(t)$ in $\mathcal{H}$ (for the terminology of semigroup theory, the reader is referred to [15]).

ii: If $\omega_1 \neq m\pi$ for some natural number $m \in \boldsymbol{N}$, then the semigroup $T(t)$ generated by L is asymptotically stable, that is all solutions of (21) asymptotically converge to zero.

Proof of i: We use the Lumer–Phillips theorem (see [15, p. 14]) to prove the assertion *i*. To prove that L is dissipative, we differentiate (25) with respect to time. Then by using (1)–(3), (9)–(12), integrating by parts, and using (16), (17), we obtain

$$\begin{aligned} \dot{E} = &\int_0^1 y_ty_{tt}\, dx + \int_0^1 y_xy_{xt}\, dx + \frac{1}{2}z_1^T(A^TP + PA)z_1 \\ &+ z_1^TPby_t(1,t) + k_1y(1,t)y_t(1,t) \\ &+ k_2y_t(1,t)x_2 \\ = &-y_t(1,t)f(t) + \frac{1}{2}z_1^T(A^TP+PA)z_1 \\ &+ z_1^TPby_t(1,t) + +k_1y(1,t)y_t(1,t) \\ &+ k_2y_t(1,t)x_2 \\ = &-c^Tz_1y_t(1,t) - dy_t^2(1,t) + z_1^TPby_t(1,t) \\ &- \frac{1}{2}z_1^Tqq^Tz_1 - \frac{\epsilon}{2}z_1^TQz_1 \\ = &-\gamma y_t^2(1,t) - \frac{1}{2}[\sqrt{2(d-\gamma)}y_t(1,t) - z_1^Tq]^2 \\ &- \frac{\epsilon}{2}z_1^TQz_1. \end{aligned} \tag{26}$$

Since $\dot{E} \leq 0$, it follows that L is dissipative [see (24)–(26)].

Let $z = (lhrr_1r_2)^T \in \mathcal{H}$ be given. To show that $\lambda I - L\colon \mathcal{H} \to \mathcal{H}$ is onto for $\lambda > 0$, one must show that for some $w = (uvz_1x_1x_2)^T \in D(L)$, we must have $(\lambda I - L)w = z$; hence the following equations should hold:

$$\lambda u - v = l, \quad \lambda v - u_{xx} = h \tag{27}$$

$$\begin{aligned} &\lambda z_1 - Az_1 - bv(1) = r, \quad \lambda x_1 - \omega_1 x_2 = r_1 \\ &\lambda x_2 + \omega_1 x_1 - v(1) = r_2 \end{aligned} \tag{28}$$

$$\begin{aligned} u(0) = 0, u_x(1) + c^Tz_1 + dv(1) \\ + k_1u(1) + k_2x_2 = 0. \end{aligned} \tag{29}$$

It can easily be shown that the solution $u(\cdot)$ of (27) satisfying $u(0) = 0$ is given by

$$\begin{aligned} u(x) = c_1 \sinh \lambda x - \frac{1}{\lambda}\int_0^x (h(s) \\ + \lambda l(s)) \sinh \lambda(x-s)\, ds \end{aligned} \tag{30}$$

where c_1 is a constant and $\sinh(\cdot)$ is the hyperbolic sine function. By using (28) and (29), after some straightforward calculations, we obtain

$$\begin{aligned}\lambda(\cosh\lambda + g(\lambda)\sinh\lambda)c_1 &= \int_0^1 (h(s)+\lambda l(s))(\cosh\lambda(1-s)+g(s)\sinh \\ &\quad \cdot \lambda(1-s))\,ds + \left(g(\lambda)-\frac{k_1}{\lambda}\right)l(1) \\ &\quad - c^T(\lambda I - A)^{-1}r - \frac{k_2\lambda}{\lambda^2+\omega_1^2}\left(r_2 - \frac{\omega_1 r_1}{\lambda}\right)\end{aligned} \tag{31}$$

where $g(\cdot)$ is given by (15). Since $g(\cdot)$ is a positive real transfer function, it follows that $g(\lambda) > 0$ for all $\lambda > 0$; see, e.g., [17, pp. 129–130]. It then follows that $\cosh\lambda + g(\lambda)\sinh\lambda > 0$ for all $\lambda > 0$, hence the constant c_1 can be uniquely determined from (31). The remaining unknowns v, z_1, x_1, x_2 can be found from (27) and (28). Hence it follows that $\lambda I - L$: $\mathcal{H} \to \mathcal{H}$ is onto for all $\lambda > 0$. Then, it follows from the Lumer–Phillips theorem that L generates a C_0-semigroup of contractions $T(t)$ on $\mathcal{H}$.

ii: To prove assertion *ii*, we use LaSalle's invariance principle, extended to infinite-dimensional systems; see [16, p. 78]. According to this principle, all solutions of (21) asymptotically tend to the maximal invariant subset of the following set:

$$\mathcal{S} = \{z \in \mathcal{H} | \dot{E} = 0\} \tag{32}$$

provided that the solution trajectories for $t \geq 0$ are *precompact* in $\mathcal{H}$. Since the operator $L : \mathcal{H} \to \mathcal{H}$ generates a C_0-semigroup of *contractions* on $\mathcal{H}$ (hence the solution trajectories are *bounded* on $\mathcal{H}$ for $t \geq 0$), the precompactness of the solution trajectories are guaranteed if the operator $(\lambda I - L)^{-1} : \mathcal{H} \to \mathcal{H}$ is compact for some $\lambda > 0$; see [16, p. 241]. To prove the last property, we first show that L^{-1} exists and is a compact operator on $\mathcal{H}$. To see this, let $z = (lhrr_1r_2)^T \in \mathcal{H}$ be given. We want to solve the equation $Lw = z$ for w, where $w = (uvz_1x_1x_2)^T \in D(L)$. The solution of this equation can easily be found as

$$u(x) = -\int_0^x \int_0^\xi h(\sigma)\,d\sigma\,d\xi + cx, \qquad x \in (0,1) \tag{33}$$

$$v(x) = -l(x), \qquad x \in (0,1) \tag{34}$$

$$z_1 = A^{-1}(l(1)b - r) \tag{35}$$

$$x_1 = \frac{l(1) - r_2}{\omega_1} \tag{36}$$

$$x_2 = -\frac{r_1}{\omega_1} \tag{37}$$

where the constant c can be uniquely determined from (12). It follows that L^{-1} exists and maps $\mathcal{H}$ into $\boldsymbol{H}^2 \times \boldsymbol{H}^1 \times \boldsymbol{R}^n \times \boldsymbol{R} \times \boldsymbol{R}$; moreover, $(uvz_1x_1x_2)^T \in D(L)$. Since $z = (lhrr_1r_2)^T \in \mathcal{H}$, it follows that $l(0) = 0$ and that $l \in \boldsymbol{H}^1$; see (18). Hence, if $\|z\|$ is bounded in $\mathcal{H}$, it follows easily that $l(1)$ is bounded as well. Therefore, L^{-1} maps the bounded sets of $\mathcal{H}$ into the bounded sets of $\boldsymbol{H}^2 \times \boldsymbol{H}^1 \times \boldsymbol{R}^n \times \boldsymbol{R} \times \boldsymbol{R}$. Since the embedding of the latter into $\mathcal{H}$ is compact (see [18, p. 14]), it follows that L^{-1} is a compact operator. This also proves that the spectrum of L consists entirely of isolated eigenvalues and that for any λ in the resolvent set of L, the operator $(\lambda I - L)^{-1}$: $\mathcal{H} \to \mathcal{H}$ is a compact operator; see [6, p. 187]. Furthermore, our argument above shows that $\lambda = 0$ is not an eigenvalue of L. Since the operator L generates a C_0-semigroup of contractions on $\mathcal{H}$, by the argument given above it follows that the solution trajectories of (21) are precompact in $\mathcal{H}$ for $t \geq 0$; hence by LaSalle's invariance principle, the solutions asymptotically tend to the maximal invariant subset of $\mathcal{S}$ [see (32)]. Hence, to prove that all solutions of (21) asymptotically tend to the zero solution, it suffices to show that $\mathcal{S}$ contains only the zero solution, which is a typical procedure in the application of LaSalle's invariance principle.

To prove that $\mathcal{S}$ contains only the zero solution, we set $\dot{E} = 0$ in (26), which results in $z_1 = 0$. This implies that $\dot{z}_1 = 0$; hence by using (9) and (12) we obtain $y_t(1,t) = 0, f(t) = k_1 y(1,t) + k_2 x_2$. Hence, all solutions of (21) in $\mathcal{S}$ satisfy the following equations:

$$y_{tt} = y_{xx} \tag{38}$$

$$\dot{x}_1 = \omega_1 x_2, \qquad \dot{x}_2 = -\omega_1 x_1 \tag{39}$$

$$y(0,t) = 0, \qquad y_t(1,t) = 0 \tag{40}$$

$$y_x(1,t) = -k_1 y(1,t) - k_2 x_2. \tag{41}$$

The solution x_2 of (39) can be written as

$$x_2 = c_1 \cos\omega_1 t + c_2 \sin\omega_1 t \tag{42}$$

where c_1 and c_2 are arbitrary constants.

Since the boundary conditions in (40) are separable, the solution of (38) can be found by using separation of variables; see [9]. That is, the solution of (38) and (40) assumes the following form:

$$y(x,t) = A(t)B(x) \qquad t \geq 0, \quad x \in [0,1] \tag{43}$$

where the functions A: $\boldsymbol{R}_+ \to \boldsymbol{R}$ and B: $[0,1] \to \boldsymbol{R}$ are twice differentiable functions to be determined from the boundary conditions (40). We distinguish the following cases.

a: $\dot{A} \equiv 0$. In this case, the solution of (38) is $y(x,t) = c_3 + c_4 x$. From (40) it follows that $c_3 = 0$, and by using this in (41) we find that $c_1 = c_2 = c_4 = 0$ as well. Hence, the only possible solution is $y(x,t) \equiv 0$.

b: $\dot{A} \neq 0$. In this case, the solution of (38) is in the following form:

$$\begin{aligned}A(t) &= c_3\cos\beta t + c_4 \sin\beta t \\ B(x) &= c_5 \cos\beta x + c_6 \sin\beta x\end{aligned} \tag{44}$$

where $c_3, \cdots, c_6$ are arbitrary constants. From (40) we obtain $c_5 = 0$ and $c_6 \sin\beta = 0$. (For otherwise $y_t(1,t) = 0$ would yield only the trivial solution $y(x,t) \equiv 0$.) Hence, for a nontrivial solution, β should satisfy $\sin\beta = 0$, hence $\beta = m\pi$ for some natural number $m \in \boldsymbol{N}$. Now using the arguments in ***a***, it can easily be shown that $\beta \neq 0$. By using these in (41), it can easily be shown that to have a nontrivial solution, we must have $\beta = \omega_1$, i.e., $\omega_1 = m\pi$ for some natural number $m \in \boldsymbol{N}$. For otherwise, the only solution in $\mathcal{S}$ would yield $c_1 = c_2 = \cdots = c_6 = 0$, which yields $y(x,t) \equiv 0$. Therefore, if $\omega_1 \neq m\pi$ for some natural number $m \in \boldsymbol{N}$, we conclude that the only solution of (21) which lies in the set $\mathcal{S}$ given by (32) is the zero solution. Hence, by LaSalle's invariance principle, we conclude that the solutions of (21) asymptotically tend to the zero solution. □

Remark 2: It was proven in [13] that for $k_2 = 0$, if $d > 0$, then the closed-loop system (1)–(4), (9), and (12) is exponentially stable, and if $d = 0$, then the same system is asymptotically stable. Since the subsystem (10), (11) is essentially finite-dimensional, one might expect the same conclusions to hold for the case $k_2 > 0$ as well. However, this point needs further investigation. □

IV. Disturbance Rejection

In this section we show the effect of the proposed control law given by (9)–(12) on the solutions of the system given by (1)–(4) when the output of the controller is corrupted by a disturbance $n(t)$, that is (12) has the following form:

$$f(t) = c^T z_1 + dy_t(1,t) + k_1 y(1,t) + k_2 x_2 + n(t) \tag{45}$$

or equivalently (14) has the following form:

$$\hat{f}(s) = g(s)\hat{y}_t(1,s) + \hat{n}(s) \tag{46}$$

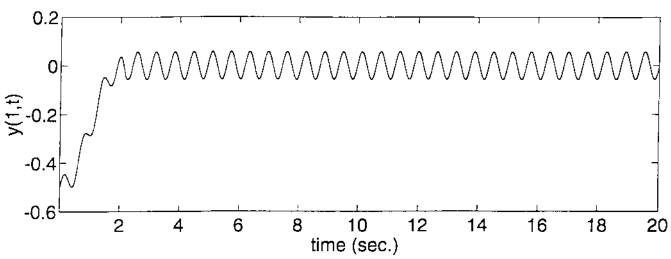

Fig. 1. Tip displacement for Case i.1.

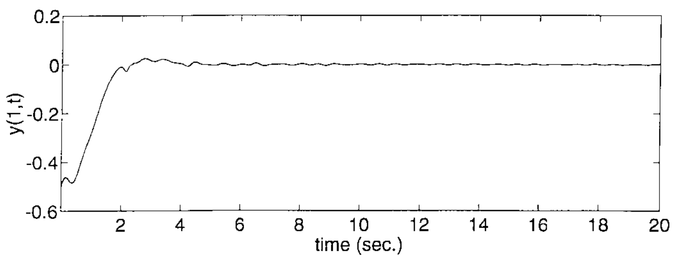

Fig. 2. Tip displacement for Case i.2.

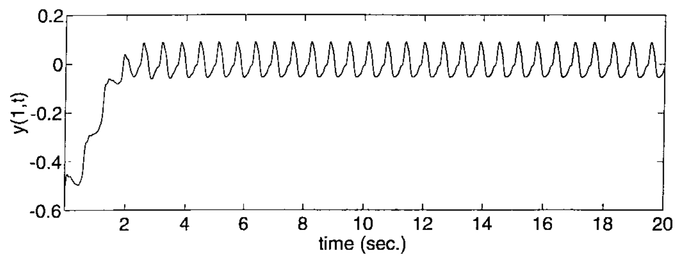

Fig. 3. Tip displacement for Case ii.1.

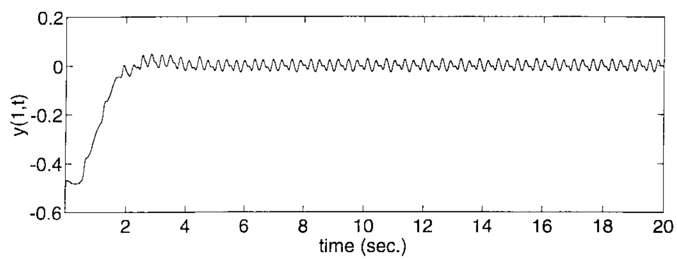

Fig. 4. Tip displacement for Case ii.2.

where $\hat{n}(s)$ is the Laplace transform of the disturbance $n(t)$. Note that in deriving (46), we neglected the effect of $y(1,0)$, cf., (14). Since $y(1,0)$ is a constant, we may include it as an additional term in the disturbance.

To find the transfer function from $n(t)$ to $y_t(1,t)$, we take the Laplace transform of (1)–(3), and set initial conditions to zero. Then, the solution of (1), (2) becomes

$$y(x,s) = c \sinh xs \tag{47}$$

where c is a constant and $\sinh$ is the hyperbolic sine function. By using (3) and (46), we obtain

$$c = -\frac{1}{s(\cosh s + g(s)\sinh s)}\hat{n}(s) \tag{48}$$

$$\hat{y}_t(1,s) = -\frac{\sinh s}{\cosh s + g(s)\sinh s}\hat{n}(s). \tag{49}$$

Now, consider the controller given by (5). It was stated in Section II that, without disturbance, this system is exponentially stable and that by choosing d appropriately, one can achieve arbitrary decay rates [see (7)]. Moreover $g(s) = d = 1$ is the best choice since in this case all solutions become zero for $t > 2$. However, from (49) one can easily see that this is not a good choice for disturbance rejection. To see this, first note that in this case the controller transfer function $g(s)$ is given by $g(s) = d = 1$ [see (5), (14), and (46)]. Hence, (49) becomes

$$\hat{y}_t(1,s) = \tfrac{1}{2}(e^{-2s} - 1)\hat{n}(s) \tag{50}$$

which yields, in time domain

$$y_t(1,t) = \tfrac{1}{2}(n(t-2) - n(t)). \tag{51}$$

In case $n(t)$ is sinusoidal, from (51) it follows that $y_t(1,t)$ is sinusoidal and its magnitude is not small (i.e., half of the magnitude of the disturbance). Hence the case $g(s) = d = 1$ is not a good choice for disturbance rejection. It can be shown that $d \neq 1, d \in \boldsymbol{R}$ yields similar results.

Another choice for disturbance rejection is the use of dynamic controllers [e.g., of the type (15) or (9)–(12)]. From (49) we can also derive a procedure to design $g(s)$ if we know the structure of $n(t)$. For example if $n(t)$ has a band-limited frequency spectrum (i.e., has frequency components in an interval of frequencies $[\Omega_1, \Omega_2]$), then we can choose $g(s)$ to minimize

$$c(\omega) = \left|\frac{\sinh j\omega}{\cosh j\omega + g(j\omega)\sinh j\omega}\right|, \qquad \omega \in [\Omega_1, \Omega_2].$$

Note that to ensure the stability of the closed-loop system, $g_1(s)$ should be a strictly positive real function as well [see (15)]. As a simple example, assume that $n(t) = a\cos\omega_0(t)$. Then we may choose $g(s)$ in the form (15) with $\omega_1 = \omega_0$. Provided that Assumptions 1)–3) are satisfied and that $\omega_0 \neq m\pi$ for some natural number $m \in \boldsymbol{N}$, the closed-loop system is asymptotically stable (see Theorem 1). Moreover, if $k_2 > 0$, then $g(s)$ given by (15) has a pair of complex conjugate poles at $s = \pm\omega_0$, hence $c(\omega)$ given above satisfies $c(\omega_0) = 0$. From (49) we may conclude that this eliminates the effect of the disturbance at the output $y_t(1,t)$.

V. Simulation Results

In this section, we show the effect of the proposed control law given by (46), where $g(s)$ is given by (15) on the solutions of the system (1)–(3), by means of some numerical simulation results.

For simulations, we first obtain a state-space representation [i.e., (9) and (12)] for the compensator given by (15). For this purpose we choose the well-known controllable canonical representation of $g(s)$. Then we use the finite difference technique with N point spatial discretization, approximating the spatial derivatives by using a central difference formula; see [4]. The resulting equations can be written in the form

$$\dot{z} = Fz + bd(t)$$

where the vector z contains the displacements and the velocities at N points and n compensator states. Hence $F \in \boldsymbol{R}^{m\times m}$ and $b \in \boldsymbol{R}^m, m = 2N + n$. This equation is then simulated by using a trapezoidal type algorithm. In the simulations we choose $N = 50$. As for the initial conditions we choose the initial displacement along the first mode of the uncontrolled system (i.e., (1)–(3) with $f(t) \equiv 0$), and we set

$$y(x,0) = -0.5\sin(0.5\pi x), \quad y_t(x,0) = 0, \qquad 0 \le x \le 1.$$

Initial conditions for the controller states are set to zero.

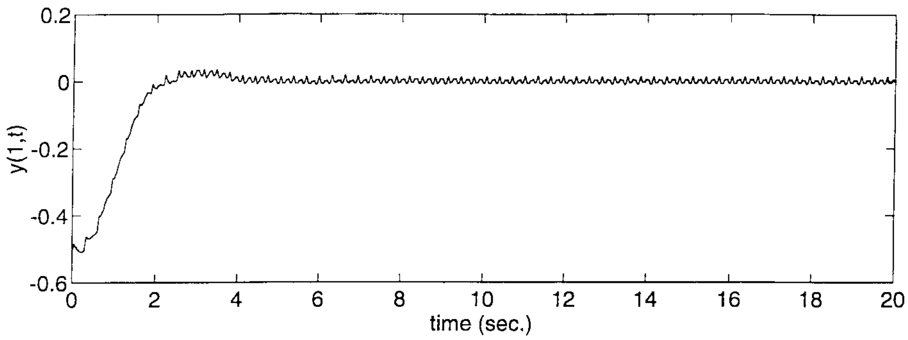

Fig. 5. Tip displacement for Case ii.3.

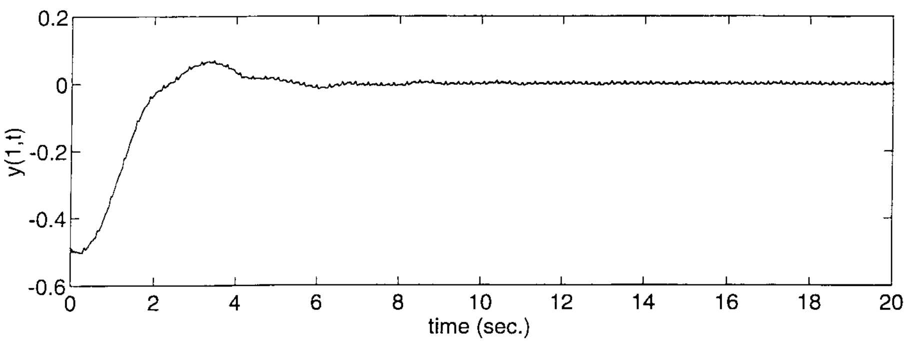

Fig. 6. Tip displacement for Case ii.4.

For the disturbance $n(t)$, we choose two different types of waveforms which are given below.

i: $n(t) = \cos 10t$.

For this disturbance, we choose the following controller:

$$g(s) = d + \frac{k_2 s}{s^2 + \omega_1^2}. \tag{52}$$

Note that in (15), for simplicity we choose $g_1(s) = 0$, and it can easily be shown that the conclusions of Theorem 1 are still valid in this case as well. For the controller we use the following sets of parameters.

Case i.1: $d = 1, k_2 = 0, \omega_1 = 10$.

Case i.2: $d = 1, k_2 = 10, \omega_1 = 10$.

The resulting endpoint positions $y(1,t)$ for Cases i.1 and i.2 are shown in Figs. 1 and 2, respectively. Obviously, with $k_2 = 0$, the controller given by (52) reduces to the controller given by (5), with $g(s) = d = 1$. As explained in Section II, the best choice for stability is $g(s) = d = 1$; see (8). However, as explained in Section IV, this is not a good choice for disturbance rejection, and Fig. 1 confirms this point. Also, Fig. 2 shows that the effect of disturbance can be attenuated by use of an appropriate dynamic controller.

ii: $n(t) = \Sigma_{k=1}^{5} (\cos 10kt/k)$.

The purpose of this choice of disturbance is to investigate the effect of the controller given by (52) on the system response when $n(t)$ contains harmonics of a fundamental frequency as well. For the controller given by (52), we choose the following sets of parameters.

Case ii.1: $d = 1, k_2 = 0, \omega_1 = 10$.

Case ii.2: $d = 1, k_2 = 10, \omega_1 = 10$.

The resulting endpoint positions $y(1,t)$ for Cases ii.1 and ii.2 are shown in Figs. 3 and 4, respectively. As can be seen from these figures, the disturbance rejection is better for $k_2 > 0$.

Note that in Case ii.2, the controller is "tuned" to eliminate the fundamental harmonic of the disturbance. We can put additional terms in $g(s)$ to eliminate the higher harmonics as well. For this, we choose the following controller:

$$g(s) = d + \frac{k_2 s}{s^2 + \omega_1^2} + \frac{k_3 s}{s^2 + \omega_2^2}. \tag{53}$$

For this controller, we choose the following sets of parameters.

Case ii.3: $d = 1, k_2 = 0, \omega_1 = 10, k_3 = 100, \omega_2 = 20$.

Case ii.4: $d = 1, k_2 = 10, \omega_1 = 10, k_3 = 50, \omega_2 = 20$.

The resulting endpoint positions $y(1,t)$ for Cases ii.3 and ii.4 are shown in the Figs. 5 and 6, respectively. As can be seen in these figures, the additional terms attenuate the effect of higher harmonics as well.

These simulations suggest that by using dynamic compensators one may improve the system response in case the system is subject to a disturbance. Moreover, once the compensator transfer function is parameterized [see (52)], optimum values for these parameters to shape the system response may be obtained (e.g., to decrease the overshoot, to decrease the rise time, etc.). However, this point needs further investigation.

VI. Conclusion

In this paper, we considered a linear time-invariant system which is represented by the one-dimensional wave equation in a bounded domain. We assumed that the system is fixed at one end and a boundary control input is applied at the other end. For this system, we proposed a finite-dimensional *dynamic* boundary controller. This introduces extra degrees of freedom in designing controllers which could be exploited in solving a variety of control problems, such as disturbance rejection, pole assignment, etc., while maintaining stability. The transfer function of the controller is a proper rational function of the complex variable s and may contain a single pole at $s = 0$ and another pair of complex conjugate poles at $s = \pm j\omega_1, \omega_1 \neq 0$, provided that the residues corresponding to these poles are nonnegative; the rest of the transfer function is required to be a strictly positive real function. We then proved that the closed-loop system is asymptotically stable provided that $\omega_1 \neq m\pi$ for some natural number $m \in \boldsymbol{N}$. We also studied the case where the output of the controller is corrupted by a disturbance. We showed that if the frequency spectrum of the disturbance is known, then by choosing the controller appropriately we can obtain better disturbance rejection. To support this idea, we presented some numerical simulation results.

We note that the ideas presented here can also be applied to other flexible structures (e.g., flexible beams). The work on this subject is still in progress and the results will be presented elsewhere.

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Mixed H_2/H_∞-Control of Discrete-Time Markovian Jump Linear Systems

Oswaldo L. V. Costa and Ricardo P. Marques

***Abstract*—In this paper we consider the mixed H_2/H_∞-control problem for the class of discrete-time linear systems with parameters subject to Markovian jump linear systems (MJLS's). It is assumed that both the state variable and the jump variable are available to the controller. The transition probability matrix may not be exactly known, but belongs to an appropriate convex set. For this controlled discrete-time Markovian jump linear system, the problem of interest can be stated in the following way. Find a robust (with respect to the uncertainty on the transition Markov probability matrix) mean-square stabilizing state and jump feedback controller that minimizes an upper bound for the H_2-norm, under the restriction that the H_∞-norm is less than a prespecified value δ. The problem of the determination of the smallest H_∞-norm is also addressed. We present an approximate version of these problems via linear matrix inequality optimization.**

***Index Terms*—Coupled Lyapunov equations, LMI optimization, Markovian jump systems, mixed H_2/H_∞-control.**

I. Introduction

A great deal of attention has been given nowadays to a class of stochastic linear systems subject to abrupt variations, namely, Markovian jump linear systems (MJLS's). This family of systems is modeled by a set of linear systems with the transitions between the models determined by a Markov chain taking values in a finite set. Due to a large number of applications in control engineering, several results on this field can be found in the current literature, regarding applications, stability conditions, and optimal control problems (see, for instance, [1]–[11], [13]–[18], and [21]–[28]).

The mixed H_2/H_∞ and H_∞ control problems for time-invariant discrete-time linear systems has been studied in the current literature, usually using a state-space approach, leading to nonstandard algebraic Riccati equations and Lyapunov-like equations (see, for instance, [12], [19], and [20]). The H_2- and H_∞-control problems for MJLS's have recently been analyzed in [5], [6], and [11]. For the H_2-control problem, a convex programming approach was applied in [5] and numerical algorithms developed. In this paper we study the mixed H_2/H_∞-control and H_∞-control problems of a discrete-time MJLS's. We will assume that the transition probability matrix for the Markov chain is not exactly known, but belongs to an appropriate convex set. In this case a robust mean-square (state and jump feedback) stabilizing controller is defined as a state-feedback controller, which also depends on the jump Markov variable, that stabilizes in the mean-square sense the MJLS for every appropriate Markov transition probability matrix. This kind of concept was first introduced by Rami and El Ghaoui in [27] for continuous-time MJLS's. Under these conditions, the mixed H_2/H_∞-control problem of an MJLS's can be formulated as follows: we are interested in finding a robust mean-square stabilizing controller that minimizes an upper bound for the H_2-norm, under the restriction that the H_∞-norm is less than a prespecified value δ. The problem of minimizing

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