Comments on "An Unstable Plant With No Poles"

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***Abstract*—In the above paper, it was shown by way of an example that there exist bounded-input–bounded-output (BIBO) unstable linear systems whose transfer functions are analytic in the finite plane. We note that this result could easily be shown by using some examples already present in the literature.**

***Index Terms*—Input/output stability, Laplace transform, poles.**

For single-input–single-output (SISO) linear systems given by a convolution with causal locally integrable impulse function, the following problem was considered in the above paper[1].

Problem 1: Find, if possible, an impulse function such that i) it is not absolutely integrable, and ii) its Laplace transform is an everywhere analytic function in the finite plane.

A solution is provided in[1] to this problem by way of an example. Note that, as is well known, the absolute integrability of impulse function is equivalent to bounded-input–bounded-output (BIBO) stability of the systems considered, see e.g., [1]. (In the sequel, the functions in time domain are given by small letters as $f, \ldots$ and are assumed to be zero for $t < 0$; the functions with capital letters as F denote their Laplace transform.)

It was shown in the above paper[1] that the function $h_1(t) = \sin(t^2/2)$ is a solution to problem 1. As is noted in[1], this example was considered in [5, p. 406]. Although the fact that $h_1(t)$ is not absolutely integrable, which is a known result, was not considered in [5], it was shown there that $H_1(s)$ can be extended to an everywhere analytic function in the finite plane, see [5, pp. 403–406]. The concern in [5] is to investigate whether "the study of poles and zeros of Laplace transform can shed light on interesting stability problems," where the stability is not in BIBO sense. Instead, the following problem was considered by [5] and by some others:

Problem 2: Find, if possible, a locally integrable (time) function such that i) it does not converge to zero as $t \to \infty$, and ii) its Laplace transform is an everywhere analytic function in the finite plane.

Although the Problem 1 is not addressed explicitly in the literature, Problem 2 is considered in many places. Note that a solution of Problem 2 is not necessarily a solution of Problem 1, since there exist unbounded yet absolutely integrable functions, see, e.g., [1, p. 386]. However, it is a trivial matter to show that most of the solutions to Problem 2 provided in the literature are also solutions to Problem 1. Next, we will provide some of these examples already present in the literature.

The function $h_1(t) = \sin(t^2/2)$ was considered in [5], as a solution to Problem 2. As shown in[1], this example also provides a solution to Problem 1. The function $h_2(t) = t\sin(t^2/2)$ was also given in [5, p. 406] as a solution to Problem 2, and since $|h_1(t)| \le |h_2(t)|$ for $t \ge 1$, $h_2(t)$ is also not absolutely integrable, and hence provides a solution to problem 1. Similarly, it can easily be shown that the functions $h_3(t) = t^m \sin(t^2/2)$ for any positive integer m are also solutions to both problems.

The function $h_4(t) = \sin t^\alpha$, $(\alpha > 1)$ was given in [2, p. 218] as a solution to Problem 2. (By using standard integral tables, e.g., [4, p. 399], it can easily be shown that $H_4(s)$ is analytic for $\Re\{s\} \ge 0$, where $\Re$ denotes the real part). It is a simple matter to show that this function is also not absolutely integrable (see[1] for similar calculations), hence provides another solution to Problem 1. This example is interesting at least in two respects. First, the example $h_1(t)$ given above may be considered as a special case of $h_4(t)$ with $\alpha = 2$. Secondly, for $\alpha = 1$, the corresponding Laplace transform has simple poles at $s = \pm j$, yet for $\alpha > 1$, these poles disappear. This aspect makes this example interesting from both mathematical and physical standpoint, see [2, p. 218]. Also note that the functions $h_5(t) = t^m h_4(t)$, where m is an arbitrary positive integer, also provides solutions to both problems, see [2, p. 218].

The function $h_6(t) = \pi e^t \sin(\pi e^t)$ was given in [3, p. 29] as a solution to Problem 2. It can easily be shown that this function is also not absolutely integrable, hence provides a solution to Problem 1 as well.

Finally, the following interesting function was shown to be a solution of both Problem 1 and 2 in [3, p. 18]:

$$h_7(t) = \begin{cases} 0 & 0 \le t < \ln\ln 3 \\ (-1)^n e^{0.5e^t} & \ln\ln n \le t < \ln\ln(n+1), \\ & \quad n = 3, 4, 5, \ldots. \end{cases} \tag{1}$$

It was shown in [3, p. 18] that this function has a Laplace transform which converges everywhere, yet nowhere absolutely. Hence, the domain of convergence for $h_7(t)$ is the whole plane, and since a Laplace transform is an analytic function in the interior of its domain of convergence, see [3, Th. 6.1], it follows that $H_7(s)$ is analytic everywhere. Evaluating the integrals in [3, p. 18] for $s = 0$, we see that $h_7(t)$ is not absolutely integrable, hence provides a solution to both problems.

Note that in presence of such examples, both in [5] and [2], it was concluded that to decide on stability problems related to a time function $h(t)$, not only the singularities of $H(s)$ in the finite plane is of importance, but also the behavior of $|H(s)|$ for $|s| \to \infty$ should be investigated, since, as noted in [2, p. 219], "on the latter depends whether or not the straight line path of integration in the complex inversion formula can be replaced by an angular path."

References

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