Now it is possible to choose such d that for any $y_0 \in W$ the inequality $(M/\gamma)[\mu d + C|y_0|_*] \le d$ is true. This means that operator $\mathcal{P}$ transforms the space $\mathcal{U}$ into itself. Similarly

$$\begin{aligned}\exp(\gamma t/\mu)&|\mathcal{P}(\mathcal{H})(t, \hat{x}, \mu) - \mathcal{P}(\overline{\mathcal{H}})(t, \hat{x}, \mu)| \\ &\le \exp(\gamma t/\mu) \int_t^\infty |\Phi(\theta, \phi(\theta, \mu) + \mathcal{H}, y(\theta, \mu), \mu) \\ &\qquad - \Phi(\theta, \phi(\theta, \mu) + \overline{\mathcal{H}}, y(\theta, \mu), \mu)|\, d\theta \\ &\le \exp(\gamma t/\mu) \int_t^\infty M|\mathcal{H} - \overline{\mathcal{H}}|\, d\theta \le \mu \frac{M}{\gamma} \rho(\mathcal{H}, \overline{\mathcal{H}})\end{aligned}$$

which means that operator $\mathcal{P}$ is a contraction operator on $\mathcal{U}$. Then, the operator $\mathcal{P}$ has the unique fixed point corresponding to the function $\Pi x = \mathcal{H}(t, \hat{x}, \mu)$. Moreover, from (37), one can conclude that the inequality $|\mathcal{H}(t, \hat{x}, \mu))| < \mu\, d \exp(-\gamma t/\mu)$ holds for all $(t, \hat{x}, \mu) \in \mathbf{R}^+ \times \mathbf{R}^n \times (0, \mu_0]$.

References

[1] D. V. Anosov, "On stability of equilibrium points of relay systems" (in Russian), *Automat. Remote Control*, vol. 10, pp. 135–149, Feb. 1959.

[2] G. Bartolini, A. Ferrara, and E. Usai, "Chattering avoidance by second-order sliding-mode control," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 241–246, Feb. 1998.

[3] A. G. Bondarev, S. A. Bondarev, N. Ye. Kostylyeva, and V. I. Utkin, "Sliding modes in systems with asymptotic state observers," *Automat. Remote Control*, vol. 46, pp. 679–684, May 1985.

[4] A. F. Filippov, *Differential Equations With Discontinuous Right Hand Side*. Dodrecht, The Netherlands: Kluwer, 1988.

[5] L. M. Fridman, "Singular extension of the definition of discontinuous systems and stability," *Diff. Equations*, vol. 26, pp. 1307–1312, Oct. 1990.

[6] L. Fridman, "An averaging approach to chattering," *IEEE Trans. Automat. Contr.*, vol. 46, pp. 1260–1265, Aug. 2001.

[7] L. Fridman and A. Levant, "Higher order sliding modes," in *Sliding Mode Control in Engineering*. ser. Control Engineering, no. 11, J. P. Barbot and W. Perruguetti, Eds. New York: Marcel Dekker, 2002, pp. 53–102.

[8] K. H. Johansson, A. Rantzer, and K. J. Astrom, "Fast switches in relay feedback systems," *Automatica*, vol. 35, pp. 539–552, 1999.

[9] P. V. Kokotovic, H. K. Khalil, and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design*. London, U.K.: Academic, 1986.

[10] A. Levant, "Robust exact differentiation via sliding mode technique," *Automatica*, vol. 34, pp. 379–384, 1998.

[11] H. Sira-Ramires, "Sliding regimes in general nonlinear systems: A relative degree approach," *Int. J. Control*, vol. 50, pp. 1487–1506, 1989.

[12] I. A. Shkolnikov and Y. B. Shtessel, "Tracking a class of nonminimum phase systems with nonlinear internal dynamics via sliding-mode control using method of system center," *Automatica*, vol. 38, pp. 837–842, 2002.

[13] V. I. Utkin, *Sliding Modes in Control and Optimization*. Berlin, Germany: Springer-Verlag, 1992.

[14] A. B. Vasil'eva, V. F. Butusov, and L. A. Kalachev, *The Boundary Layer Method for Singular Perturbation Problems*. Philadelphia, PA: SIAM, 1995.

Two-Channel Decentralized Integral-Action Controller Design

A. N. Gündeş and A. B. Özgüler

***Abstract*—We propose a systematic controller design method that provides integral-action in linear time-invariant two-channel decentralized control systems. Each channel of the plant is single-input–single-output, with any number of poles at the origin but no other poles in the instability region. An explicit parametrization of all decentralized stabilizing controllers incorporating the integral-action requirement is provided for this special case of plants. The main result is a design methodology that constructs simple low-order controllers in the cascaded form of proportional-integral and first-order blocks.**

***Index Terms*—Decentralized control, integral-action, stability.**

I. Introduction

We consider decentralized controller design with integral-action for linear time-invariant (LTI) plants, whose unstable poles can only occur at the origin. These plant models occur in many applications and are common in process control [7]. The decentralized controller structure is preferred for simplicity of implementation and the integral-action in the controllers achieves asymptotic tracking of step-input references applied at each input. We apply and explicitly define the parametrization of all decentralized controllers and incorporate integral-action into the controllers for this important class of plants, where the 2×2 plant transfer-function matrix may have simple or multiple poles at the origin in any or all of its entries.

The theory of decentralized control has produced relatively few systematic and explicit design methods despite the wide practical demand. The main difficulty is that the decentralized structure imposed on the free parameter of the set of all stabilizing controllers renders the optimization problem nonconvex [10]. Alternatively, when viewed as a problem of making the plant stabilizable and detectable from one of its channels, the decentralized stabilizing controllers are constructed relying on genericity arguments [2], [9], [12]. The decentralized controller parametrizations obtained previously (see, for example [5] and [8]) all characterize controllers at the conceptual level and do not provide explicit descriptions. The usual computational methods that would be used to convert such conceptual designs to explicit descriptions would typically produce unnecessarily high-order controllers since the standard (robust) control designs are not tailored to special type of plants as considered here.

The integral-action problem for the case of stable plants has been considered in the decentralized setting with single-input–single-output channels in [7], and [1], and design procedures were proposed for achieving reliable stability under the possible failure of controllers in [6]. For the case of unstable plants, controller designs were presented in [3] based on choosing the free design parameter to achieve a desired sensitivity function for a suitable diagonal or triangular model of the plant. However, explicit decentralized integral-action controller designs for plants with integrators are not available.

Manuscript received December 7, 2001; revised July 28, 2002. Recommended by Associate Editor P. Apkarian. This work was supported by the National Science Foundation under Grant ECS-9905729.

A. N. Gündeş is with Electrical and Computer Engineering Department, University of California, Davis, CA 95616 USA (e-mail: gundes@ece.ucdavis.edu).

A. B. Özgüler is with Electrical and Electronics Engineering Department, Bilkent University, Bilkent, TR-06533 Ankara, Turkey (e-mail: ozguler@ee.bilkent.edu.tr).

Digital Object Identifier 10.1109/TAC.2002.805671

Decentralized designs such as the reliable controller design described in [6] developed for stable plants obviously cannot be applied to plants with poles at the origin. Since only some of the entries of the plant transfer-function matrix may have poles at the origin or these poles may appear with different multiplicities, the integrators in the unstable plant cannot be extracted and incorporated into the controller, i.e., the plant P cannot simply be expressed as $P = (1/s)\hat{P}$ with $\hat{P}$ stable, and a controller of the form $(1/s)C$ cannot be designed for the resulting stable $\hat{P}$ following the methods in [6]. Note that $P = (1/s)\hat{P}$ would result in improper $\hat{P}$ except when P is strictly-proper, but more importantly, would generally mean $\hat{P}$ has transmission-zeros at the origin and cannot be (internally) stabilized by $(1/s)C$. Furthermore, reliable design as described in [6] assumes controllers may fail arbitrarily; the integrators of the channel with the failure would not be compensated by feedback and hence, reliable stabilization is not attempted in this case of unstable plants. Therefore, an entirely different methodology is developed here for this important class of plants with poles at the origin.

The main results here are the explicit parametrization of all decentralized controllers with integral-action (Theorem 1), and the completely systematic design procedure that defines all controller transfer-functions explicitly (Theorem 2). The significance and strength of the proposed design method can be explained as follows. 1) The set of all controllers (Theorem 1) is described based on two "semi-free" parameters. 2) A subclass of controllers is characterized with one parameter completely free (Theorem 2). 3) In each of its two channels, the "nominal controller" (Theorem 2) has no unstable poles other than at $s = 0$, which it contains by design to satisfy the integral-action requirement. The location of the stable poles is completely arbitrary. 4) The nominal controller in each of the two channels is in the form of one proportional-integral (PI) block cascaded with first-order blocks (lead or lag controllers). The number of these cascaded blocks depends on the number of integrators in the plant. 5) The nominal controller is a low-order controller, with order independent of the number of stable plant poles. 6) The parametrization of all decentralized controllers *without* integral-action derived from Theorem 1 leads to stable controllers so the proposed design achieves strong stabilization.

The design method is illustrated by an example, where the plant is the linearized model of a sugar mill process [3], [4]. Two of the entries of the 2×2 transfer-function matrix each have a simple pole at the origin. A PI controller is designed for the first channel and a PI cascaded with one lead block is designed for the second channel.

Notation: Let $\mathcal{U}$ be the extended closed right-half plane (for continuous-time systems) or the complement of the open unit-disk (for discrete-time systems). The sets of real numbers, proper rational functions with real coefficients, proper rational functions with no unstable poles are denoted by $\mathbb{R}$, $\mathrm{R_p}$, $\mathcal{S}$. The set of matrices with all entries in $\mathcal{S}$ is denoted by $\mathcal{M}(\mathcal{S})$; M is called stable iff $M \in \mathcal{M}(\mathcal{S})$; a square $M \in \mathcal{M}(\mathcal{S})$ is unimodular iff $M^{-1} \in \mathcal{M}(\mathcal{S})$; $m \in \mathcal{S}$ is a unit in $\mathcal{S}$ iff $m^{-1} \in \mathcal{S}$. A diagonal matrix whose entries are N_1 and N_2 is denoted by $\mathrm{diag}[N_1,\, N_2]$. For $M \in \mathcal{M}(\mathcal{S})$, the norm $\|\cdot\|$ is defined as $\|M\| = \sup_{s \in \partial\mathcal{U}} \overline{\sigma}(M(s))$, where $\overline{\sigma}$ denotes the maximum singular value and $\partial\mathcal{U}$ denotes the boundary of $\mathcal{U}$. For simplicity, the variable s is dropped and rational functions such as $P(s)$ are denoted by P.

Our discussion here is constrained to continuous-time systems although the results apply also to discrete-time systems with appropriate modifications.

II. Analysis

Consider the LTI, multiple-input–multiple-output, two-channel decentralized feedback system $\Sigma(P,\, C_D)$ shown in Fig. 1: $P \in \mathrm{R}_{\mathrm{p}}^{2\times 2}$ and $C_D \in \mathrm{R}_{\mathrm{p}}^{2\times 2}$ represent the transfer-functions of the plant and the decentralized controller, partitioned as

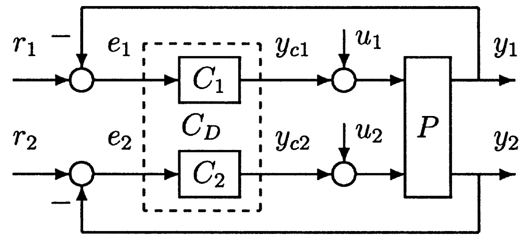

Fig. 1. The two-channel decentralized system $\Sigma(P,\, C_D)$.

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \qquad C_D = \mathrm{diag}[C_1,\, C_2]. \tag{1}$$

It is assumed that $\Sigma(P,\, C_D)$ is a well-posed system (i.e., all closed-loop transfer-functions are proper), and that P and C_D have no hidden modes corresponding to eigenvalues in $\mathcal{U}$. The plant $P \in \mathrm{R}_{\mathrm{p}}^{2\times 2}$ may have poles at $s = 0$ but it does not have any other $\mathcal{U}$-poles. Let $\alpha > 0$ be an arbitrary but fixed real number and define $Z \in \mathcal{S}$ as

$$Z = \frac{s}{s+\alpha}. \tag{2}$$

Since the only $\mathcal{U}$-poles are at $s = 0$, the plant P has a left-coprime factorization (LCF) $P = D^{-1}N$ of the form

$$P = \begin{bmatrix} Z^{m-1} & 0 \\ D_{21} & Z^{w-1} \end{bmatrix}^{-1} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{3}$$

where $m \geq 1$, $w \geq 1$ are integers, $N,\, D \in \mathcal{M}(\mathcal{S})$, D is in lower-triangular Hermite-form [11].

A decentralized controller $C_D = \mathrm{diag}[C_1,\, C_2]$ is said to be an *integral-action controller* iff C_D stabilizes P and $\hat{D}_c(0) = 0$ for any right-coprime factorization (RCF) $C_D = N_c\hat{D}_c^{-1}$ [11], [7]. Let $C_D = N_c\hat{D}_c^{-1} := \mathrm{diag}[N_1,\, N_2]\mathrm{diag}[\hat{D}_1^{-1},\, \hat{D}_2^{-1}]$, $\hat{D}_j(\infty) \neq 0$, be any RCF over $\mathcal{S}$ of $C_D = \mathrm{diag}[C_1,\, C_2]$. Therefore, $C_D = N_c\hat{D}_c^{-1}$ is an integral-action controller if and only if $\hat{D}_c = ZD_c$ for some $D_c := \mathrm{diag}[D_1,\, D_2] \in \mathcal{M}(\mathcal{S})$. This implies $C_D = N_c(ZD_c)^{-1}$ is a decentralized integral-action controller for P if and only if $N_cD_c^{-1}$ is a decentralized stabilizing controller for $Z^{-1}P$.

Lemma 1: An integral-action controller exists for $P = D^{-1}N$ if and only if $N(0)$ is nonsingular. △

By Lemma 1, a necessary condition due to the integral-action requirement is that $\mathrm{rank}N(0) = 2$. The decentralized integral-action controller $C_D = \mathrm{diag}[C_1,\, C_2]$, $C_j = N_j(ZD_j)^{-1}$, stabilizes the plant P if and only if T in (4) is unimodular

$$\begin{aligned} T &:= ZD\mathrm{diag}[D_1,\, D_2] + N\mathrm{diag}[N_1,\, N_2] \\ &= \begin{bmatrix} Z^mD_1 + N_{11}N_1 & N_{12}N_2 \\ ZD_{21}D_1 + N_{21}N_1 & Z^wD_2 + N_{22}N_2 \end{bmatrix}. \end{aligned} \tag{4}$$

The controller design problem here is to determine $D_j,\, N_j \in \mathcal{S}$ such that T in (4) is unimodular. The following lemma is used to construct simple explicit solutions for $D_j,\, N_j \in \mathcal{S}$ and these solutions are used in parametrizing all decentralized integral-action controllers for P.

Lemma 2: Let $G \in \mathcal{S}^{r\times\rho}$. For any integer $q \geq 1$, there exists $X \in \mathcal{S}^{\rho\times r}$ such that $Z^qI + GX$ is unimodular if and only if $\mathrm{rank}G(0) = r$.

III. Design

In this section, we propose design methods for two-channel decentralized integral-action controllers. The necessary condition $\mathrm{rank}N(0) = 2$, i.e., P has no transmission-zeros at $s = 0$, implies $(N_{11}N_{22} - N_{12}N_{21})(0) \neq 0$. In (3), the diagonal entry N_{11} may or may not be identically zero or zero at $s = 0$. If $N_{11} \neq 0$, then it is expressed as $N_{11} = Z^n G_1$ for some $G_1 \in \mathcal{S}$, $G_1(0) \neq 0$, where $n \geq 0$ is an integer corresponding to the number of zeros of N_{11} at $s = 0$; if $N_{11}(0) \neq 0$, then $G_1 = N_{11}$.

Theorem 1 gives a complete parametrization of all two-channel decentralized integral-action controllers for P, stated as two cases depending on the number of zeros of N_{11} at $s = 0$. If $N_{11} = 0$, then define $\beta := m$ and $G_1 = 0$. If $N_{11} \neq 0$, then let $N_{11} =: Z^n G_1$ for some $G_1 \in \mathcal{S}$, $G_1(0) \neq 0$. Define $\beta := \min\{n, m\}$, $q_1 := m - \beta$ and $q_2 := w + \beta$. Define $\tilde{N}_1, \tilde{D}_1 \in \mathcal{S}$ as follows.

i) If $\beta = m$, i.e., if $N_{11} = 0$ or if $m \leq n$, let

$$\tilde{N}_1 = \tilde{Q}_1 \quad \tilde{D}_1 = \left(1 - Z^{(n-m)} G_1 \tilde{Q}_1\right) \tag{5}$$

for some $\tilde{Q}_1 \in \mathcal{S}$ such that $\tilde{Q}_1(0) \neq 0$, and $\tilde{Q}_1(\infty) \neq G_1(\infty)^{-1}$.

ii) If $\beta = n < m$, let $X_1 \in \mathcal{S}$ be such that M_1 in (6) is a unit and let $\tilde{N}_1, \tilde{D}_1$ be as in (6)

$$M_1 := Z^{q_1} + G_1 X_1 \quad \tilde{N}_1 = X_1 M_1^{-1} \quad \tilde{D}_1 = M_1^{-1}. \tag{6}$$

With $\tilde{N}_1, \tilde{D}_1$ defined as (5) when $\beta = m$ or as (6) when $\beta = n$, define $G_2 \in \mathcal{S}$ as

$$G_2 := Z^\beta N_{22} - N_{12}(Z D_{21} \tilde{D}_1 + N_{21} \tilde{N}_1). \tag{7}$$

Let $X_2 \in \mathcal{S}$ be such that M_2 in (8) is a unit; let $Y \in \mathcal{S}$ be defined as (9) and let unimodular matrices $U_1, U_2 \in \mathcal{S}^{2\times 2}$ be defined as in (10)

$$M_2 := Z^{q_2} + G_2 X_2 \tag{8}$$

$$Y := N_{12}(Z D_{21} G_1 - Z^{(m-n)} N_{21}) \tag{9}$$

$$U_1 = \begin{bmatrix} \tilde{D}_1 & \tilde{N}_1 \\ -Z^{(n-\beta)}(G_1 + Y \tilde{D}_1 X_2 M_2^{-1}) & Z^{q_1} - Z^{(n-\beta)} Y \tilde{N}_1 X_2 M_2^{-1} \end{bmatrix}$$

$$U_2 = \begin{bmatrix} M_2^{-1} & X_2 M_2^{-1} \\ -G_2 & Z^{q_2} \end{bmatrix}. \tag{10}$$

Theorem 1 (All Decentralized Integral-Action Controllers): Let $P \in \mathrm{R}_{\mathrm{p}}^{2\times 2}$, $P = D^{-1}N$ be an LCF as (3), and $\operatorname{rank} N(0) = 2$. Let $U_1, U_2 \in \mathcal{S}^{2\times 2}$ be defined as in (10). Then, all decentralized integral-action controllers $C_D = \operatorname{diag}[C_1, C_2]$ are given by $C_j = N_j (Z D_j)^{-1}$ as in (11) below for $j = 1, 2$, where $R_{jj}, R_j \in \mathcal{S}$ are such that W in (11) is a unit

$$[D_j \quad N_j] = [R_{jj} \quad R_j] U_j \quad W := R_{11} R_{22} - Z^{q_2} Y R_1 R_2. \tag{11}$$

The controllers C_j are proper if and only if R_{jj}, R_j further satisfy $D_1(\infty) = (\tilde{D}_1 R_{11} - (G_1 + Y \tilde{D}_1 X_2 M_2^{-1}) R_1)(\infty) \neq 0$, $D_2(\infty) = (\tilde{D}_2 R_{22} - G_2 R_2)(\infty) \neq 0$. △

In Theorem 2, a careful choice of the parameters R_{jj}, R_j gives a particularly simple subclass of decentralized controllers based on the cascaded form of simple PI and first-order blocks. The "conditionally free" parameters of Theorem 1 are now replaced by a completely free parameter Q_2 and a conditionally free Q_1. The construction in the proof of Lemma 2 is crucial in this design. Under the same assumptions as in Theorem 1, the procedure is based on the following steps

Step 1)

i) If $\beta = m$, choose any $\tilde{Q}_1 \in \mathcal{S}$ such that $\tilde{Q}_1(0) \neq 0$, and $\tilde{Q}_1(\infty) \neq G_1(\infty)^{-1}$. Define $\tilde{N}_1 = \tilde{Q}_1$, $\tilde{D}_1 = (1 - Z^{(n-m)} G_1 \tilde{Q}_1)$ as in (5).

ii) If $\beta = n$, construct $X_1 \in \mathcal{S}$ satisfying (6) as in (14) for $j = 1$. Define $\tilde{N}_1 = X_1 M_1^{-1}$, $\tilde{D}_1 = M_1^{-1}$ as in (6).

With $\tilde{N}_1, \tilde{D}_1$ defined as (5) when $\beta = m$ or as (6) when $\beta = n$, with X_1 is constructed as in (14) for $j = 1$, define G_2 as (7). Construct $X_2 \in \mathcal{S}$ satisfying (8) as in (15) for $j = 2$.

Step 2): Choose any $f_j \in \mathbb{R}$; define $H_j := f_j s + G_j(0)^{-1}$. Choose $h_{j1} \in \mathbb{R}$ satisfying (12)

$$0 < h_{j1} < \left\| s^{-1}(G_j H_j - 1) \right\|^{-1}. \tag{12}$$

If $q_j > 1$, for $v = 2, \ldots, q_j$, choose $h_{jv} \in \mathbb{R}$ satisfying (13); let $X_j, M_j \in \mathcal{S}$ be as in (14) and (15)

$$0 < h_{jv} < \left\| s^{-1} \left(1 + G_j H_j \frac{h_{j1}}{s^{v-1}} \prod_{i=2}^{v-1} (s + h_{ji})\right)^{-1} \right\|^{-1} \tag{13}$$

$$X_j = \frac{1}{s+\alpha} h_{j1} H_j \prod_{i=2}^{q_j} \frac{(s + h_{ji})}{(s+\alpha)} \tag{14}$$

$$H_j := f_j s + G_j(0)^{-1} \quad M_j := Z^{q_j} + G_j X_j. \tag{15}$$

Theorem 2 (Decentralized Integral-Action Controller Design): Let the assumptions of Theorem 1 hold. A class of decentralized integral-action controllers $\{C_D = \operatorname{diag}[C_1, C_2]\}$ is obtained as follows: if $\beta = m$, design C_1 as

$$C_1 = \frac{(s+\alpha)}{s} \tilde{N}_1 \tilde{D}_1^{-1} = \frac{(s+\alpha)}{s} \tilde{Q}_1 \left(1 - Z^{(n-m)} G_1 \tilde{Q}_1\right)^{-1} \tag{16}$$

where $\tilde{Q}_1 \in \mathcal{S}$ is such that $\tilde{Q}_1(0) \neq 0$, and $\tilde{Q}_1(\infty) \neq G_1(\infty)^{-1}$. If $\beta = n$, design C_1 as in (17) for $j = 1$. In both cases, design C_2 as in (17) for $j = 2$:

$$\begin{aligned} C_j &= \frac{(s+\alpha)}{s} (X_j + Z^{q_j} Q_j)(1 - G_j Q_j)^{-1} \\ &= \frac{H_j h_{j1}}{s} \prod_{i=2}^{q_j} \frac{(s+h_{ji})}{(s+\alpha)} + \frac{(s+\alpha)}{s} M_j Q_j (1 - G_j Q_j)^{-1} \end{aligned} \tag{17}$$

where $Q_1, Q_2 \in \mathcal{S}$; $Q_1 \in \mathcal{S}$ is also such that

$$\tilde{W} := 1 + Y(X_2 + Z^{q_2} Q_2) M_2^{-1} M_1^{-1} Q_1 \tag{18}$$

is a unit. The controller C_j is proper if and only if $Q_j(\infty) \neq G_j(\infty)^{-1}$ for $j = 1, 2$. △

Comments 1:

1) *PI and first-order cascade structure of the controllers:* Let C_j in (17) obtained by setting $Q_j = 0$ be called the "nominal controller" C_{jo} shown in (19)

$$C_{jo} := \frac{(s+\alpha)}{s} X_j = \frac{H_j h_{j1}}{s} \prod_{j=2}^{q_j} \frac{(s+h_{ji})}{(s+\alpha)}. \tag{19}$$

This controller has important properties justifying the significance and strength of the proposed design. For $j = 1, 2$, C_{jo} is designed to have a pole at $s = 0$ to satisfy the integral-action requirement; C_{jo} has no other unstable poles and it has $(q_j - 1)$ poles at $s = -\alpha$, where α is completely free. If $n < m$, when $q_j = 1$, C_{jo} is simply a PI controller. In general, C_{jo} is in the form of one PI block $H_j h_{j1}/s = f_j h_{j1} + G_j(0)^{-1} h_{j1}/s$, cascaded with $(q_j - 1)$ first-order blocks $(s + h_{ji})/(s+\alpha)$, $i = 2, \ldots, q_j$, designed as needed when $q_j > 1$. The initial PI block can be designed as a pure integral controller $G_j(0)^{-1} h_{j1}/s$ by choosing $f_j = 0$. Each subsequent first-order block is minimum-phase, with a pole at $s = -\alpha$ and a zero at $-h_{ji}$; these may be interpreted as lead or lag controllers depending on α and h_{ji} [they would likely all be lead controllers since h_{ji} satisfying (13) are typically small and α can be chosen arbitrarily large at the beginning of the design procedure].

The order of C_{1o} is $q_1 = m - n$, which does not exceed the number of unstable poles of the plant in channel-one; the order of C_2 is $q_2 = m + \beta$, which does not exceed the total number of unstable poles of the plant in channel-one and channel-two (these unstable poles are all at $s = 0$ here). This low-order controller design where the controller order is independent of the number of stable poles of the plant has obvious advantages over full-order observer based controller designs.

2) *Properties of the proposed controller class:* The controllers C_j in (17), expressed as $C_j = C_{jo} + ((s+\alpha)/s)(Z^{q_j} + G_j X_j)Q_j(1 - G_j Q_j)^{-1}$, are biproper for any choice of the stable parameter Q_j because X_j is biproper by design. If $\beta = m < n$, C_1 in (16) is strictly-proper if and only if $Q_1 \in \mathcal{S}$ is strictly-proper. Due to the integral-action requirement, C_j have poles at $s = 0$ for any $Q \in \mathcal{S}$; C_j can be restricted to have no other unstable poles if and only if $Q_j \in \mathcal{S}$ is such that $(1 - G_j Q_j)$ is a unit; it is sufficient to take $\|Q_j\| < \|G_j\|^{-1}$. In the case that $\beta = m < n$, C_1 in (16) has no unstable poles other than at $s = 0$ if and only if $Q_1 \in \mathcal{S}$ is such that $(1 - Z^{(n-m)} G_1 Q_1)$ is a unit; it is sufficient to take $\|Q_1\| < \|G_1\|^{-1}$.

3) *Freedom in the design parameter:* The choice of the design parameter $Q_2 \in \mathcal{S}$ for C_2 in (17) is completely arbitrary [where C_2 is proper if and only if $Q_2(\infty) \neq G_2(\infty)^{-1}$]. This freedom may be used to satisfy other design objectives. The choice of the design parameter $Q_1 \in \mathcal{S}$ for C_1 in (17) is restricted so that $\tilde{W}$ is a unit [where C_1 is proper if and only if $Q_1(\infty) \neq G_1(\infty)^{-1}$]. While $Q_1 = 0$ obviously makes $\tilde{W}$ a unit, another sufficient condition is to choose $Q_1 \in \mathcal{S}$ such that $\|Q_1\| < \|Y(X_2 + Z^{q_2} Q_2) M_2^{-1} M_1^{-1}\|^{-1}$.

4) *Design without integral action in the controllers:* The integral-action in the controllers is due to the Z term in the denominators of C_j. It is obvious that the parametrization of all decentralized controllers *without* integral-action can be obtained from Theorem 1 simply by removing the Z^{-1} term from the controllers. We outline the parametrization and design for this case. The decentralized controller $C_D = \operatorname{diag}[C_1, C_2]$, $C_j = N_j D_j^{-1}$, stabilizes the plant P if and only if $\hat{T} := D\operatorname{diag}[D_1, D_2] + N\operatorname{diag}[N_1, N_2]$ is unimodular. Since dropping the integral-action requirement from the controllers reduces the number of integrators by one, in Theorems 1 and 2, substitute m by $(m-1)$, w by $(w-1)$, and re-define $G_2 := Z^{\beta} N_{22} - N_{12}(D_{21}\tilde{D}_1 + N_{21}\tilde{N}_1)$, $Y := N_{12}(D_{21}G_1 - Z^{(m-1-n)}N_{21})$. Then all decentralized controllers are obtained from (11). In Theorem 2, if $\beta = m - 1$, design $C_1 = \tilde{N}_1 \tilde{D}_1^{-1} = \tilde{Q}_1(1 - Z^{n-(m-1)} G_1 \tilde{Q}_1)^{-1}$, with $\tilde{Q}_1 \in \mathcal{S}$, $\tilde{Q}_1(0) \neq 0$, $\tilde{Q}_1(\infty) \neq G_1(\infty)^{-1}$. If $\beta = n$, design C_1 as in (20) for $j = 1$. In both cases, design C_2 as in (20)

$$C_j = (X_j + Z^{q_j} Q_j)(1 - G_j Q_j)^{-1} = \frac{H_j h_{j1}}{(s+\alpha)} \prod_{j=2}^{q_j} \frac{(s + h_{ji})}{(s+\alpha)} + M_j Q_j (1 - G_j Q_j)^{-1} \tag{20}$$

where, for $j = 1, 2$, $Q_j \in \mathcal{S}$, $Q_j(\infty) \neq G_j(\infty)^{-1}$, $Q_1 \in \mathcal{S}$ also satisfies $\tilde{W}$ in (18) is a unit. Since the term $(s+\alpha)/s$ is now removed from the controllers, the nominal decentralized controller $C_{jo} = X_j$ is stable, with q_j poles at $s = -\alpha$. This design is in the form of q_j cascaded stable first-order blocks. The initial block $H_j h_{j1}/(s+\alpha)$ has a zero at $s = -G_j(0)^{-1}/f_j$ [negative if we choose f_j with the same sign as $G_j(0)^{-1}$]. It is followed by $(q_j - 1)$ minimum-phase blocks $(s + h_{ji})/(s+\alpha)$, $i = 2, \ldots, q_j$, each with a pole at $s = -\alpha$ and a zero at $-h_{ji}$. These blocks may be interpreted as lead or lag controllers. The nominal controllers in this design are strongly stabilizing; they can even be made units by choosing f_j appropriately. △

Example 1 (Control of a Sugar Mill): We apply the design in Theorem 2 to the linearized model of a sugar mill process [3], [4]. The two-input–two-output plant and an LCF $P = D^{-1}N$ as (3) are

$$\begin{aligned} P &= \begin{bmatrix} \dfrac{-5}{25s+1} & \dfrac{s^2 - 0.005s - 0.005}{s(s+1)} \\ \dfrac{1}{25s+1} & \dfrac{-0.0023}{s} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{s}{s+\alpha} & 0 \\ -\frac{23}{50} & 1 \end{bmatrix}^{-1} \\ &\quad \cdot \begin{bmatrix} \dfrac{-5s}{(25s+1)(s+\alpha)} & \dfrac{s^2 - 0.005s - 0.005}{(s+\alpha)(s+1)} \\ \dfrac{165/50}{25s+1} & \dfrac{-23/50s}{(s+1)} \end{bmatrix} \end{aligned} \tag{21}$$

where $m = 2$, $w = 1$, $n = 1$, $G_1 = (-5/(25s+1))$, $G_1(0) = -5$. Since $\beta = n < m$, we design C_1 as in (17). Choosing $\alpha = 5$, $f_1 = -4.5$, $H_1 = -(4.5s + 0.2)$, condition (12) is satisfied for any $h_{11} \in \mathbb{R}$ such that $0 < h_{11} < 1/|5(f_1 + 5)|$; we choose $h_{11} = 0.38$. Since $q_1 = 1$, by (5), (14), $\tilde{N}_1 = X_1 M_1^{-1} = s^{-1} h_{11} H_1 (1 + s^{-1} G_1 h_{11} H_1)^{-1}$, $\tilde{D}_1 = M_1^{-1} = (s+\alpha)s^{-1}(1 + s^{-1} G_1 h_{11} H_1)^{-1}$, and $G_2 = ZN_{22} - N_{12}(ZD_{21}\tilde{D}_1 + N_{21}\tilde{N}_1)$, $G_2(0) = -0.0033/\alpha = -0.00066$. We choose $f_2 = -10$, $H_2 = -10(s + 1/0.0066)$; then $0 < h_{21} < 0.0637$ satisfies (12). With $h_{21} = 0.04$, $0 < h_{22} < 0.04$ satisfies (13); we choose $h_{22} = 0.039$. By (17)

$$\begin{aligned} C_1 &= \frac{-0.38(4.5s + 0.2)}{s} + \left(1 - 0.38 G_1 \frac{(4.5s + 0.2)}{s}\right) \\ &\quad \cdot Q_1 (1 - G_1 Q_1)^{-1} \\ C_2 &= \frac{-0.4(s + 1/0.0066)(s + 0.039)}{s(s+5)} + \left(\frac{s}{s+5} - 0.4 G_2 \right. \\ &\quad \left. \cdot \frac{(s + 1/0.0066)(s + 0.039)}{s(s+5)}\right) Q_2 (1 - G_2 Q_2)^{-1} \end{aligned}$$

where $Q_2 \in \mathcal{S}$ is completely free, and $Q_1 \in \mathcal{S}$ is such that (18) is a unit. For $Q_1 = 0$, the nominal controller C_{1o} is in the PI form; for $Q_2 = 0$, C_{2o} is the cascade of a PI and one first-order block, which is a lead controller since $\alpha > h_{22}$. The controllers C_1, C_2 are proper for all $Q_1 \in \mathcal{S}$, $Q_2 \in \mathcal{S}$ because G_j, $j = 1, 2$, are strictly-proper. The design parameters $\alpha, f_{11}, h_{11}, Q_1, f_{21}, h_{21}, h_{22}, Q_2$ (in that order) can be chosen within their respective constraints to change the closed-loop transfer-functions achieved using this design. △

IV. Conclusion

We presented a systematic method to explicitly design decentralized controllers with integral-action for two-channel plants that have integrators of any multiplicity in one or more entries of the 2×2 transfer-function matrix. The design achieves closed-loop stability and robust asymptotic tracking of step-input references. The nominal controller of the proposed class for each of the two channels has a pole at $s = 0$ but no other unstable poles. It is designed as a low-order controller in the form of one PI block cascaded with stable minimum-phase first-order blocks. Unlike most standard full-order observer-based controller designs, the controller order is independent of the number of stable plant poles. This low-order property and the simple explicit definition of the controllers without any computation makes this a very desirable straightforward design procedure.

In some cases the plant may have stable poles that could be considered undesirable. In Example 1, the plant pole at $s = -0.04$ appears as a pole in the closed-loop system as well since the instability region $\mathcal{U}$

is the extended closed right-half-plane. If the stability region is redefined to exclude such poles in order to achieve better performance, it may be possible to modify the design method and extend it to include plants with unstable poles in addition to those at the origin.

Other tractable extensions of the results presented here include the case of decentralized systems with more than two channels and multiple inputs and outputs in each channel.

APPENDIX

Proof of Lemma 1: If $N(0)$ is nonsingular, then $(DZ)^{-1}N$ is an LCF of $Z^{-1}P$. By standard results on decentralized fixed-modes [12], [9], it follows that $s = 0$ is not a decentralized fixed-mode of $(DZ)^{-1}N$. Hence, decentralized stabilizing controllers exist for $(DZ)^{-1}N$. The necessity follows from (4); if the decentralized integral-action controller C_D stabilizes the plant P, then T unimodular implies $\mathrm{rank}T(0) = \mathrm{rank}(NN_c)(0) = 2 = \mathrm{rank}N(0)$. △

Proof of Lemma 2: If $Z^qI + GX = M$ is unimodular, then $\mathrm{rank}M(0) = \mathrm{rank}G(0)X(0) = r \le \min\{\mathrm{rank}G(0), \mathrm{rank}X(0)\} \le r$. Conversely, if $\mathrm{rank}G(0) = r$, then X can be constructed as follows. Let $G(0)^R \in \mathbb{R}^{\rho\times r}$ denote (any) right-inverse of $G(0) \in \mathbb{R}^{r\times\rho}$. Choose any $F \in \mathbb{R}^{\rho\times r}$; define $H := Fs + G(0)^R$. Choose $h_1 \in \mathbb{R}$ and define $\tilde{M}_1$ as in (22)

$$\begin{aligned} &0 < h_1 < \|s^{-1}(GH - I)\|^{-1} \\ &\tilde{M}_1 := \frac{s}{s+h_1} I + G\frac{Hh_1}{s+h_1} =: ZI + G\tilde{X} \end{aligned} \tag{22}$$

then, for any h_1 satisfying (22), $\tilde{M}_1 = I + (s+h_1)^{-1}sh_1[s^{-1}(GH - I)] \in \mathcal{M}(\mathcal{S})$ is unimodular. If $q = 1$, then $X = (s+\alpha)^{-1}(s+h_1)\tilde{X} = (s+\alpha)^{-1}Hh_1$ and $M = (s+\alpha)^{-1}(s+h_1)\tilde{M}_1$. If $q > 1$, then construct a unimodular $\tilde{M}_2$ similarly, substituting $G\tilde{X}\tilde{M}_1^{-1}$ for G in (22), where $(G\tilde{X}\tilde{M}_1^{-1})(0) = I$. Choose $h_2 \in \mathbb{R}$ satisfying (23)

$$\begin{aligned} 0 < h_2 &< \|s^{-1}(G\tilde{X}\tilde{M}^{-1} - I)\|^{-1} \\ &= \|s^{-1}(I + s^{-1}GHh_1)^{-1}\|^{-1} \end{aligned} \tag{23}$$

then $\tilde{M}_2 := (s+h_2)^{-1}sI + G\tilde{X}\tilde{M}_1^{-1}h_2(s+h_2)^{-1}$ is unimodular for any h_2 satisfying (23). Therefore, the product $\tilde{M}_2\tilde{M}_1 = (s+h_2)^{-1}(s+h_1)^{-1}s^2I + G\tilde{X}$ is also unimodular. If $q = 2$, then $X = (s+\alpha)^{-2}(s+h_2)(s+h_1)\tilde{X} = (s+\alpha)^{-2}Hh_1(s+h_2)$ and $M = (s+\alpha)^{-2}(s+h_2)(s+h_1)\tilde{M}_2\tilde{M}_1 = Z^2I + G\tilde{X}$. Continue similarly if $q > 2$, i.e., for $v = 3, \ldots, q$, construct a unimodular $\tilde{M}_v$ similarly, substituting $G\tilde{X}\prod_{i=1}^{v-1}\tilde{M}_i^{-1}$ for G in (22), where $(G\tilde{X}\prod_{i=1}^{v-1}\tilde{M}_i^{-1})(0) = I$. For $v = 3, \ldots, q$, choose $h_v \in \mathbb{R}$ satisfying (24) and define $\tilde{M}_v$ similarly as in (22)

$$\begin{aligned} 0 < h_v &< \left\| s^{-1}\left(G\tilde{X}\prod_{i=1}^{v-1}\tilde{M}_{ji}^{-1} - I \right) \right\|^{-1} \\ &= \left\| s^{-1}\left(I + GH\frac{h_1}{s^{v-1}}\prod_{i=2}^{v-1}(s+h_i) \right)^{-1} \right\|^{-1} \end{aligned} \tag{24}$$

then $\tilde{M}_v := (s+h_v)^{-1}sI + G\tilde{X}\prod_{i=1}^{v-1}\tilde{M}_i^{-1}h_v(s+h_v)^{-1}$ is unimodular for any h_v satisfying (24). Therefore, the product $\prod_{i=0}^{v-1}\tilde{M}_{(v-i)} = \prod_{i=1}^{v}(s+h_i)^{-1}I + G\tilde{X}$ is also unimodular. Finally, for $v = q$, $X \in \mathcal{S}^{\rho\times r}$ and the unimodular $M \in \mathcal{S}^{r\times r}$ are

$$\begin{aligned} X &= \frac{\prod_{i=1}^{q}(s+h_i)}{(s+\alpha)^q}\tilde{X} = \frac{h_1H\prod_{i=2}^{q}(s+h_i)}{(s+\alpha)^q} \\ M &= \frac{\prod_{i=1}^{q}(s+h_i)}{(s+\alpha)^q}\prod_{i=0}^{q-1}\tilde{M}_{(q-i)} = Z^qI + GX. \end{aligned} \tag{25}$$

Therefore, there exists X such that $Z^qI + GX$ is unimodular for any integer q. △

Proof of Theorem 1: The equivalent parametrizations of all decentralized controllers given in [8], [5] can be applied to the plant in (3) by including the controller's poles at $s = 0$ in the augmented plant denominator and finding all decentralized controllers for $(ZD)^{-1}N$. Using the procedure in [8], all decentralized controllers $C_D = \mathrm{diag}[C_1, C_2]$ for $(ZD)^{-1}N$ are given by $C_j = N_jD_j^{-1}$, where D_j, N_j are as in (11) for $j = 1, 2$, with R_{jj}, R_j satisfying (11). The unimodularity of $U_j \in \mathcal{M}(\mathcal{S})$ in (10) are due to $\det U_1 = 1$ by (5) or (6) and $\det U_2 = 1$ by (8). Using U_j, the matrix B is in the Smith form $S = U_1BU_2^T = \mathrm{diag}[1\, Z^{q_2}Y]$ [11]

$$B = \begin{bmatrix} \det D & Z^{m-1}N_{22} - ZD_{21}N_{12} \\ Z^{w-1}N_{11} & \det N \end{bmatrix}.$$

For $j = 1, 2$, $X_j \in \mathcal{S}$ satisfying (6), (8) exist since $G_j(0) \neq 0$: When $\beta = m$, we choose $Q_1(0) \neq 0$; then $\det N(0) \neq 0$ implies $G_2(0) = -N_{12}N_{21}Q_1(0) \neq 0$. When $\beta = n < m$, $G_1(0) \neq 0$, and $G_2(0) = \det N(0)G_1(0)^{-1} \neq 0$ by assumption. By Lemma 2, there exists $X_j \in \mathcal{S}$ such that M_j in (15) is a unit. The controllers C_j are proper if and only if $D_j(\infty) \neq 0$. When $\beta = m$, this is equivalent to $\tilde{Q}_1(\infty) \neq G_j(\infty)^{-1}$. △

Proof of Theorem 2: The proposed controllers are obtained from Theorem 1 by choosing R_{jj}, R_j, $j = 1, 2$ as follows. If $\beta = m$, choose $R_{11} = 1$, $R_1 = 0$, $R_{22} = 1$, $R_2 = Q_2M_2^{-1}$; then $W = 1$ is a unit. If $\beta = n$, choose $R_{11} = 1 - Y\tilde{D}_1X_2M_2^{-1}Q_1$, $R_1 = Q_1\tilde{D}_1$, $R_{22} = 1$, $R_2 = Q_2M_2^{-1}$; then $W = 1 + Y(X_2 + Z^{q_2}Q_2)M_2^{-1}M_1^{-1}Q_1 = \hat{W}$ is a unit due to the choice of $Q_1 \in \mathcal{S}$. With this R_{jj}, R_j, we have $N_j = (X_j - Z^{q_2}Q_j)M_j^{-1}$, $D_j = (1 - G_jQ_j)M_j^{-1}$. It was shown in the proof of Theorem 1 that $G_j(0) \neq 0$, $j = 1, 2$. It follows that M_j is a unit for X_j in (14) constructed according to *Step 2)* by applying the proof of Lemma 2 to G_j, $j = 1, 2$ (where G_j is scalar, i.e., $\rho = r = 1$). △

REFERENCES

[1] P. J. Campo and M. Morari, "Achievable closed-loop properties of systems under decentralized control: Conditions involving the steady-state gain," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 932–943, May 1994.

[2] J. P. Corfmat and A. S. Morse, "Decentralized control of linear multivariable systems," *Automatica*, vol. 8, pp. 479–485, 1976.

[3] G. C. Goodwin, M. M. Seron, and M. E. Salgado, "H_2 design of decentralized controllers," in *Proc. Amer. Control Conf.*, vol. 6, 1999, pp. 3826–3830.

[4] G. C. Goodwin, S. F. Graebe, and M. E. Salgado, *Control System Design*. Upper Saddle River, NJ: Prentice-Hall, 2001.

[5] A. N. Gündeş and C. A. Desoer, *Algebraic Theory of Linear Feedback Systems With Full and Decentralized Compensators*. New York: Springer-Verlag, 1990, vol. 142.

[6] A. N. Gündeş and M. G. Kabuli, "Reliable decentralized integral-action controller design," *IEEE Trans. Automat. Contr.*, vol. 46, pp. 296–301, Feb. 2001.

[7] M. Morari and E. Zafiriou, *Robust Process Control*. Upper Saddle River, NJ: Prentice-Hall, 1989.

[8] A. B. Özgüler and M. Hıraoğlu, "Implications of a characterization result on strong and reliable decentralized control," in *Modeling, Robustness and Sensitivity Reduction in Control Systems*. ser. NATO ASI, R. F. Curtain, Ed. New York: Springer-Verlag, 1987, vol. F34, pp. 425–450.

[9] A. B. Özgüler, *Linear Multichannel Control: A System Matrix Approach*. Upper Saddle River, NJ: Prentice-Hall, 1994.

[10] D. D. Sourlas and V. Manousiouthakis, "Best achievable decentralized performance," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1858–1871, Nov. 1995.

[11] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*. Cambridge, MA: MIT Press, 1985.

[12] S. H. Wang and E. J. Davison, "On the stabilization of decentralized control systems," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 473–478, 1973.