Global Stabilization Via Local Stabilizing Actions

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***Abstract*—Stabilization of a linear, time-invariant system via stabilization of its main diagonal subsytems is the underlying problem in all diagonal dominance techniques for decentralized control. In these techniques as well as all Nyquist-based techniques, sufficient conditions are obtained under the assumption that the collection of the unstable poles of all diagonal subsystems is the same as the unstable poles of the overall system. We show that this assumption is by itself enough to construct a solution to the problem at least in cases where the diagonal subsystems have disjoint poles.**

***Index Terms*—Decentralized control, diagonal dominance, interconnected systems, stabilization.**

I. Introduction

Consider an N-channel $p \times m$ multivariable system in transfer matrix representation

$$Z(s) = \begin{bmatrix} Z_{11}(s) & \cdots & Z_{1N}(s) \\ \vdots & & \vdots \\ Z_{N1}(s) & \cdots & Z_{NN}(s) \end{bmatrix} \tag{1}$$

where $Z_{ij}(s)$ is $p_i \times m_j$ for $i, j \in \mathbf{N} := \{1, \ldots, N\}$ with $p = \sum_i p_i, m = \sum_j m_j$. A main problem in decentralized control is to determine whether the $p \times m$ system can be stabilized by local controllers $Z_{ci}(s), i \in \mathbf{N}$, each of size $m_i \times p_i$ and each stabilizing its own main diagonal subsystem represented by $Z_{ii}(s)$. Alternatively, the problem is finding a decentralized controller $Z_c = \mathrm{diag}\{Z_{c1}, Z_{c2}, \ldots, Z_{cN}\}$ simultaneously stabilizing Z and its diagonal part $Z_d := \mathrm{diag}\{Z_{11}, Z_{22}, \ldots, Z_{NN}\}$, [5]. This is a fundamental problem of decentralized action since it seeks an answer to the question "When do local solutions result in a similar global solution?"

The problem can be traced to the work of Rosenbrock [6], where well-known single-input–single-output (SISO) frequency domain design techniques are extended to multiloop systems satisfying certain "diagonal dominance" or "weak interaction" properties. A review of the existing results on the problem can be found in [2, Ch. 4]. In [15] and [3], related problems are studied. In [9], the problem for $N = 2$ has been posed as one of reliable stabilization of a feedforward interconnected system. In [10], a multichannel generalization of the problem has been posed as a decentralized concurrent stabilization problem and it has been established that the problem is equivalent to decentralized strong stabilization of a transformed plant resulting from an application of an initial decentralized controller to Z. The so called "decentralized blocking zeros" are in turn central to the solution of the decentralized strong stabilization problem, [10].

Let $\mathbf{P}$ denote the set of (proper) transfer functions and let $G(s) \in \mathbf{P}^{p \times m}$ be a transfer matrix. A number s_0 in the extended closed right half complex plane is called an *unstable blocking zero* of $G(s)$ if $G(s_0) = 0$. Similarly, s_0 is called an *unstable decentralized blocking zero of* Z if for some permutation $\{i_1, \ldots, i_N\}$ of $\mathbf{N}$ the following holds: $Z_{i_k i_l}(s_0) = 0, k = 1, \ldots, N, l = 1, \ldots, k$. A main result of [10] and [8] on decentralized strong stabilization problem is that Z admits a stable decentralized stabilizing controller if and only if there are an even number of poles of Z between evey pair of real unstable decentralized blocking zeros of Z. The constructive part of the proof of this result requires the assumptions

$$\begin{aligned} &Z \text{ is strongly connected, [1], and} \qquad \forall i, j \in \mathbf{N}, i \neq j \\ &\text{rank } Z_{ij} \geq 2 \text{ or rank } Z_{ji} \geq 2. \end{aligned} \tag{2}$$

The problem of simultaneously stabilizing Z and Z_d is equivalent to decentralized strong stabilization of some initially compensated system $\bar{Z}(Z_{c0})$, where $Z_{c0} = \mathrm{diag}\{Z_{c01}, \ldots, Z_{c0N}\}$ is a stabilizing controller for Z_d. While the decentralized blocking zeros of $\bar{Z}(Z_{c0})$ can be explicitly described independent of the initial controller Z_{c0}, its poles depend on the choice of Z_{c0}. This result of [10] is thus not as transparent as one would desire. Nevertheless, there are three important special cases, where the condition for solvability can be stated purely on the original plant [8]. The difference plant $Z_{\mathrm{off}} := Z - Z_d$ plays a major role in all these cases. This is natural to expect since our problem is one of simultaneous stabilization using a special structure controller and since, by [12, Lemma 5.4.20], some parity interlacing property of the difference plant is the main solvability condition for simultaneous stabilization of two plants.

i) In case the difference plant Z_{off} is stable, under any diagonally stabilizing initial controller Z_{c0}, the plant $\bar{Z}(Z_{c0})$ can be shown to be decentrally strong stabilizable. What makes the problem nontrivial is thus the unstable poles of the difference plant.
ii) Suppose that the diagonal subplants Z_{ii} are all stable. The problem has a solution if and only if Z_{off} is strong decentralized stabilizable, i.e., there are an even number of its poles between each pair of its real unstable decentralized blocking zeros.
iii) Suppose a minimal realization of Z is stabilizable and detectable from every channel $i = 1, \ldots, N$. If N is odd, then Z and Z_d can always be simultaneously stabilized by a decentralized controller. If N is even, then Z and Z_d can be simultaneously stabilized if and only if there are an even number of real poles of Z, counted with multiplicities, between each pair of real unstable decentralized blocking zeros of Z_{off}.

Note, with regard to iii), that, by definition, stabilizability and detectability from any one of the channels, say channel-1, would actually be sufficient to stabilize the overall system by a local stabilizing controller applied there. The diagonal subsystems of the resulting closed loop system would be also all stable. This would not however constitute a solution to our problem which assumes that the local controllers at channels $2, \ldots, N$ are "blind" to what goes on in channel-1. The local actions at channels $2, \ldots, N$ to stabilize the respective subsystems would hence, in general, destroy the stabilizing action taken by the local controller at channel-1.

We investigate, in the next section, another case for which the decentralized simultaneous stabilization of Z and Z_d is made possible by an assumption on Z_{off}. The main result, Theorem 2, can be obtained by investigating the decentralized strong stabilizability of $\bar{Z}(Z_{c0})$. However, we will give a direct proof, thereby eliminating the connectivity assumptions (2). We also focus on the case $N = 2$ and state and prove the results for the two-channel case only for notational clarity. All results of the next section, Lemma 1, and Theorems 1 and 2, are valid in the N-channel case, but details have to be worked out.

II. Diagonal Dominance Methods

All Nyquist array based (block) diagonal dominance methods to decentralized control, [14], and many of the "interaction measure" [2] techniques are based on the following assumption.

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$(\mathbf{A})Z$ and $Z_d = \mathrm{diag}\{Z_{11}, \ldots, Z_{NN}\}$ have the same number of unstable poles with multiplicities.

The assumption clearly concerns the difference plant Z_{off} and one expects that simultaneous stabilization problem will be easier to solve under such an assumption. However, the relevance of $\mathbf{A}$ to decentralized stabilization needs clarification. For instance, [11], it is neither implied by nor implies the lack of unstable decentralized fixed modes: Consider $Z = [Z_{ij}], i, j = 1, 2$ with $Z_{11} = (1)/(s-1), Z_{12} = 1, Z_{21} = (s)/((s-1)^2), Z_{22} = (1)/(s-1)$. Assumption $\mathbf{A}$ is satisfied for Z since both Z and $\mathrm{diag}\{Z_{11}, Z_{22}\}$ have a double pole at $s = 1$. However, Z has an unstable decentralized fixed mode at $s = 1$. Also, a plant can be devoid of unstable decentralized fixed modes while $\mathbf{A}$ fails. Let $Z_{11} = Z_{12} = Z_{22} = (1)/(s+1), Z_{21} = (1)/(s)$. The 2×2 plant $Z = [Z_{ij}]$ has no unstable decentralized fixed modes and $\mathbf{A}$ fails.

We clarify, in Theorem 1, the connection between $\mathbf{A}$ and unstable decentralized fixed modes in an important special case. We first need the following result in which $\mathbf{S}$ denotes the set of stable transfer functions.

Lemma 1: If $\mathbf{A}$ holds and if the unstable poles of the diagonal subsystems Z_{11} and Z_{22} are disjoint, then the overall transfer matrix Z has the bicoprime representation $Z =$

$$\begin{bmatrix} P_{11}C_1 \\ P_{22}C_2 \end{bmatrix} (D_1 D_{22} C_{22} C_1)^{-1} [D_1 R_{11} \ D_2 R_{22}] \tag{3}$$

for some nonsingular matrices D_i, C_i, D_{ii}, C_{ii} and matrices $P_{ii}, R_{ii}, i = 1, 2$ over $\mathbf{S}$ satisfying $D_1 D_{22} = D_2 D_{11}, C_{22} C_1 = C_{11} C_2$ and such that the following fractions are coprime:

$$P_{11} C_{22}^{-1}, D_{11}^{-1} R_{22}, P_{22} C_{11}^{-1}, D_{22}^{-1} R_{11}. \tag{4}$$

Proof: Consider an arbitrary bicoprime fractional representation $Z = [P_1' \ \ P_2']' Q^{-1} [R_1 \ \ R_2]$, where 'prime' denotes "transpose," over $\mathbf{S}$ in which, say, Q is $r \times r$. Let $C_i = \mathrm{gcrf}\,\{P_i, Q\}$ and write $P_i = P_{ii} C_i, Q = \bar{Q}_i C_i$ for $i = 1, 2$. Also let $D_i = \mathrm{gclf}\,\{\bar{Q}_i, R_i\}$ and write $\bar{Q}_i = D_i Q_i, R_i = D_i R_{ii}$ for $i = 1, 2$. Here, "gcr(l)f" denotes "greatest common right (left) factor" so that the matrices $(\bar{Q}_i, P_{ii})$ are right coprime and (Q_i, R_{ii}) are left coprime over $\mathbf{S}$, see [4]. Since we started out with a bicoprime representation, (D_1, D_2) is left coprime, (C_2, C_1) are right coprime, and we write $D_1^{-1} D_2 = D_{22} D_{11}^{-1}, C_2 C_1^{-1} = C_{11}^{-1} C_{22}$ for right coprime (D_{22}, D_{11}) and for left coprime (C_{11}, C_{22}) over $\mathbf{S}$. It follows that

$$\det C_i \simeq \det C_{ii}, \det D_i \simeq \det D_{ii}, \qquad i = 1, 2 \tag{5}$$

where, for $a, b \in \mathbf{S}, a \simeq b$ means that a and b are associates, i.e., they are equal upto multiplication by a unit of $\mathbf{S}$. Noting that $Q = D_1 Q_1 C_1 = D_2 Q_2 C_2$, we can then write

$$C_{22} Q_1^{-1} D_{22} = C_{11} Q_2^{-1} D_{11}. \tag{6}$$

Now, the diagonal subsytem transfer matrices are $Z_{ii} = P_{ii} Q_i^{-1} R_{ii}$ and are in bicoprime fractional representation for $i = 1, 2$. By hypothesis, they have disjoint poles for $i = 1$ and $i = 2$ so that $\det Q_1$ and $\det Q_2$ are coprime in $\mathbf{S}$, which implies that both sides in (6) must be matrices over $\mathbf{S}$. In other words, we can write

$$Q_i = \hat{D}_j \hat{C}_j, D_{jj} = \hat{D}_j U_j, C_{jj} = V_j \hat{C}_j \tag{7}$$

for suitable matrices over $\mathbf{S}$ and for $i, j = 1, 2, i \neq j$ such that $V_1 U_1 = V_2 U_2$. We now show that U_j and V_j are actually unimodular matirices as a consequence of hypothesis $\mathbf{A}$. In fact, by $\mathbf{A}$, we have $\det Q \simeq \det Q_1 \det Q_2$, which gives $\det Q_i \simeq \det C_j \det D_j =$ for $i, j = 1, 2$ and $i \neq j$. But then, using (5) and (7), $\det \hat{C}_j \det \hat{D}_j \simeq \det \hat{C}_j \det \hat{D}_j \det U_j \det V_j$, which gives $\det U_j \det V_j \simeq 1$, i.e., U_j, V_j are indeed unimodular matrices for $j = 1, 2$. Hence, by (7), $Q_i = D_{jj} (V_j U_j)^{-1} C_{jj}, i, j = 1, 2, i \neq j$. We can now redefine $D_{ii} \leftrightarrow D_{ii} (V_i U_i)^{-1}$ for $i = 1, 2$ so that still $D_1^{-1} D_2 = D_{22} D_{11}^{-1}$ by $V_1 U_1 = V_2 U_2$ and $Q_i = D_{jj} C_{jj}$ for $i \neq j$. □

Supposing (3) exists, let us express Z_{11} and Z_{22} in left and right coprime fractional representations. Let

$$\begin{aligned} D_{11}^{-1} R_{22} &= \tilde{R}_2 \tilde{D}_1^{-1}, D_{22}^{-1} R_{11} = \tilde{R}_1 \tilde{D}_2^{-1} \\ P_{11} C_{22}^{-1} &= \tilde{C}_2^{-1} \tilde{P}_1, P_{22} C_{11}^{-1} = \tilde{C}_1^{-1} \tilde{P}_2 \end{aligned} \tag{8}$$

for left coprime matrices $(\tilde{C}_i, \tilde{P}_j)$ and right coprime matrices $(\tilde{R}_j, \tilde{D}_i), i \neq j$. Also, let

$$\begin{aligned} \tilde{P}_1 D_{22}^{-1} &= \hat{D}_2^{-1} \hat{P}_1, \tilde{P}_2 D_{11}^{-1} = \hat{D}_1^{-1} \hat{P}_2 \\ C_{22}^{-1} \tilde{R}_1 &= \hat{R}_1 \hat{C}_2^{-1}, C_{11}^{-1} \tilde{R}_2 = \hat{R}_2 \hat{C}_1^{-1} \end{aligned} \tag{9}$$

where, for $i \neq j, (\hat{D}_i, \hat{P}_j)$ are left coprime and $(\hat{R}_i, \hat{C}_j)$ are right coprime. It follows, by various coprimeness conditions, that for $i, j = 1, 2, i \neq j$

$$Z_{ii} = (\hat{D}_j \tilde{C}_j)^{-1} \hat{P}_i R_{ii} = P_{ii} \hat{R}_i (\tilde{D}_j \hat{C}_j)^{-1} \tag{10}$$

are right and left coprime fractions over $\mathbf{S}$, respectively.

Theorem 1: If $\mathbf{A}$ holds and if unstable poles of the diagonal subsystems are disjoint, then Z has no unstable decentralized fixed modes.

Proof: By Lemma 1, a bicoprime fraction (3), where $Q = D_1 D_{22} C_{22} C_{21}$ is say $r \times r$, exists. By [1] and by [4, Ch. 7] Z is free of unstable decentralized fixed modes if and only if the matrices

$$\begin{bmatrix} D_1 D_{22} C_{11} C_2 & D_2 R_{22} \\ P_{11} C_1 & 0 \end{bmatrix}, \begin{bmatrix} D_2 D_{11} C_{22} C_1 & D_1 R_{11} \\ P_{22} C_2 & 0 \end{bmatrix} \tag{11}$$

are complete over $\mathbf{S}$, i.e., their first r invariant factors are units of $\mathbf{S}$. Using coprimeness of (4) and (8), it follows that the first matrix is complete if and only if

$$\begin{bmatrix} D_{22} C_{11} & D_{22} \tilde{R}_2 \\ \tilde{P}_1 C_{11} & 0 \end{bmatrix} \tag{12}$$

is complete. This is because the first matrix in (11) and (12) can be shown to have the same invariant factors over $\mathbf{S}$. By hypothesis, $(\det D_{22}, \det C_{11})$ are coprime since their unstable zeros belong to poles of different diagonal subsystems. By [4, Cor. (2.8)], the matrices (D_{22}, C_{11}) are skew prime over $\mathbf{S}$, i.e., there are matrices $\hat{D}_{22}, \hat{C}_{11}$ such that $D_{22} C_{11} = \hat{C}_{11} \hat{D}_{22}$ with $(C_{11}, \hat{D}_{22})$ right and $(D_{22}, \hat{C}_{11})$ left coprime over $\mathbf{S}$. Now, (9) implies that (12) is complete if and only if $\begin{bmatrix} I & \hat{D}_{22} \hat{R}_2 \\ \hat{P}_1 \hat{C}_{11} & 0 \end{bmatrix}$ is complete. But, this matrix is obviously complete over $\mathbf{S}$ as its first r invariant factors are unity. By similar arguments, the second matrix in (11) is also complete and the result follows.

Remark 1: When $\mathbf{A}$ holds and the diagonal subsystems have common unstable poles, the unstable decentralized fixed modes seem to arise from the common diagonal subsytem poles and (central) unstable zeros. It is easy to see this, using the results in [5], for the special case when Z is 2×2. Let $Z = [(n_{ij})/(m_{ij})] = M^{-1} N$, where $(m_{ij}, n_{ij}), i, j = 1, 2$ are coprime elements in $\mathbf{S}$ and (M, N) are 2×2 left coprime matrices over $\mathbf{S}$. Then, the common unstable zeros of $\{m_{11}, m_{22}, \det N\}$ are precisely the unstable decentralized fixed modes of Z. •

Since, plants free of unstable decentralized fixed modes can be stabilized by a decentralized controller, [13], Theorem 1 gives that plants satisfying assumption $\mathbf{A}$ and having disjoint unstable poles in the diagonal admit decentralized stabilizing controllers. We show now that, for such plants, we can do much better.

Theorem 2: Suppose $\mathbf{A}$ holds and the diagonal subsystems Z_{11} and Z_{22} have their unstable poles disjoint. Then, there exists a decentralized controller simultaneously stabilizing Z_d and Z.

Proof: By Lemma 1, we have the fraction (3). We now establish that *there exist controllers* $Z_{ci} := L_i K_i^{-1}$ *and matrices* $\tilde{L}_i$, for $i = 1, 2$, *satisfying the following conditions simultaneously:*

$$\tilde{C}_j K_i \hat{D}_j + \tilde{P}_i \tilde{R}_i \tilde{L}_i = I \qquad i, j = 1, 2, i \neq j \tag{13}$$

$$U := I - \tilde{P}_1 \tilde{R}_2 \tilde{L}_2 \tilde{P}_2 \tilde{R}_1 \tilde{L}_1 \text{ is unimodular, and} \tag{14}$$

$$L_i := \tilde{D}_j \tilde{L}_i \hat{D}_j^{-1}, i, j = 1, 2, i \neq j. \tag{15}$$

are matrices over $\mathbf{S}$. We can then rewrite (13) as

$$\hat{D}_j \tilde{C}_j K_i + \hat{P}_i R_{ii} L_i = I, i \neq j. \tag{16}$$

by (15) so that Z_{ci} is a stabilizing controller for Z_{ii} for $i = 1, 2$. Moreover, the closed-loop denominator matrix attained by Z_{ci}, i.e.,

$$Q_{11} = \begin{bmatrix} D_1 D_{22} C_{22} C_1 & D_1 R_{11} L_1 & D_2 R_{22} L_2 \\ -P_{11} C_1 & K_1 & 0 \\ -P_{22} C_2 & 0 & K_2 \end{bmatrix}$$

satisfies, $\mathrm{diag}\{I, \tilde{C}_2, \tilde{C}_1\} Q_{11} \mathrm{diag}\{I, \hat{D}_2, \hat{D}_1\} =$

$$\begin{bmatrix} D_1 D_{22} & 0 & 0 \\ -\tilde{P}_1 & I & 0 \\ -\tilde{P}_2 & 0 & I \end{bmatrix} \begin{bmatrix} I & \tilde{R}_1 \tilde{L}_1 & \tilde{R}_2 \tilde{L}_2 \\ 0 & I & \tilde{P}_1 \tilde{R}_2 \tilde{L}_2 \\ 0 & \tilde{P}_2 \tilde{R}_1 \tilde{L}_1 & I \end{bmatrix} \Lambda$$

with $\Lambda := \mathrm{diag}\{C_{22} C_1, I, I\}$, where (8), (9), and (13) are employed. Since $\det(D_1 D_{22}) = \det \hat{D}_1 \det \hat{D}_2$ and $\det(C_{22} C_1) = \det \tilde{C}_1 \det \tilde{C}_2$, it follows that $\det Q_{11}$ is equal to the determinant of the middle matrix, or

$$\det Q_{11} = \det (I - \tilde{P}_1 \tilde{R}_2 \tilde{L}_2 \tilde{P}_2 \tilde{R}_1 \tilde{L}_1) \tag{17}$$

which is a unit by (14). Therefore, $\mathrm{diag}\,\{Z_{c1}, Z_{c2}\}$ stabilizes the diagonal part Z_d as well as the overall system Z.

We now prove the italicized statement above to complete the proof. Let controllers in right coprime fraction $L_{0i} K_{0i}^{-1}$ satisfy

$$\hat{D}_j \tilde{C}_j K_{0i} + \hat{P}_i R_{ii} L_{0i} = I, \qquad i, j = 1, 2, \quad i \neq j. \tag{18}$$

Such controllers exist since, by (10), $(\hat{D}_j \tilde{C}_j, \hat{P}_i R_{ii})$ are left coprime for $i, j = 1, 2, i \neq j$. Let $d_i := \det \hat{D}_i \simeq \det D_i$ and $c_i := \det \tilde{C}_i \simeq \det C_i$ for $i = 1, 2$ and note that $(c_2 d_2, \hat{D}_1 \tilde{C}_1)$ is left coprime. We can hence choose L_{02} such that $c_2 d_2$ divides it, i.e., $L_{02} = c_2 d_2 H_{02}$ for some matrix H_{02} over $\mathbf{S}$. By (18), $\tilde{L}_{i0} := \tilde{D}_i^{-1} L_{i0} \hat{D}_i$ is a matrix over $\mathbf{S}$ and satisfies

$$\tilde{C}_j K_{0i} \hat{D}_j + \tilde{P}_i \tilde{R}_i \tilde{L}_{0i} = I, \qquad i, j = 1, 2, \quad i \neq j. \tag{19}$$

Consider, for arbitrary X_i over $\mathbf{S}$ and for $i, j = 1, 2, i \neq j$,

$$K_i = K_{0i} - P_i \hat{R}_i X_i \quad \tilde{L}_i = \tilde{L}_{0i} + \hat{C}_j X_i \hat{D}_j \tag{20}$$

which clearly still satisfy (19), i.e., $\tilde{C}_j K_i \hat{D}_j + \tilde{P}_i \tilde{R}_i \tilde{L}_i = I, i, j = 1, 2, i \neq j$. Also note that, if $X_2 = c_2 d_2 Y_2$ for some Y_2, then $\tilde{L}_2 = c_2 d_2 \tilde{H}_2$ for some H_2 over $\mathbf{S}$. It can be directly verified that the condition (15) is also satisfied for any X_1 and Y_2. We now choose Y_2 and X_1 to ensure that $\tilde{L}_i K_i^{-1}$ satisfy the condition (14). Let us first note that Y_2 can be fixed so as to make

$$M := I - \tilde{P}_1 \tilde{R}_2 \tilde{L}_2 \tilde{P}_2 \tilde{R}_1 \tilde{L}_{01} \tag{21}$$

nonsingular. In fact, let y be such that $Y_2 := y \hat{C}_2^{-1} \tilde{L}_{02} \hat{D}_2^{-1}$ is over $\mathbf{S}$ and at some real number $\alpha, y(\alpha) = 1$. (Choose, for instance, α to be any real number that is neither a zero of d_2 nor c_2 and define $y(s) = (c_2(s) d_2(s))/(c_2(\alpha) d_2(\alpha))$.) Now, we can write $M = I - (1 - y) \tilde{P}_1 \tilde{R}_2 \tilde{L}_{20} \tilde{P}_2 \tilde{R}_1 \tilde{L}_{01}$ which satisfies $M(\alpha) = I$ so that M is nonsingular. With Y_2, and hence X_2, so fixed we next choose X_1 such that $U = M - \tilde{P}_1 \tilde{R}_2 \tilde{L}_2 \tilde{P}_2 \tilde{R}_1 \hat{C}_2 X_1 \hat{D}_2 = I - \tilde{P}_1 \tilde{R}_2 \tilde{H}_2 \tilde{P}_2 \tilde{R}_1 c_2 \tilde{L}_{01} d_2 - \tilde{P}_1 \tilde{R}_2 \tilde{H}_2 \tilde{P}_2 \tilde{R}_1 c_2 \hat{C}_2 d_2 X_1 \hat{D}_2$ is unimodular, where the last expression is by $\tilde{L}_2 = c_2 d_2 \tilde{H}_2$. Let $T := \hat{D}_2 \tilde{P}_1 \tilde{R}_2 \tilde{H}_2 \tilde{P}_2 \tilde{R}_1 c_2, G := T \hat{C}_2 d_2$. We consider $\hat{U} := \hat{D}_2 U \hat{D}_2^{-1}$, which is $\hat{U} = I - T \tilde{L}_{01} \,\mathrm{adj}\, \hat{D}_2 - G X_1$. Clearly, U is unimodular if and only if $\hat{U}$ is. Referring to [12, Cor. 5.3.6 and Th. 4.4.2], we need only show that at the unstable zeros of the smallest invariant factor of G, i.e., at the unstable zeros of $sif(G), \det(I - T \tilde{L}_{01} \,\mathrm{adj}\, \hat{D}_2)$ has constant sign. For any square nonsingular matrix B and any rectangular A, it is easy to see that sif(AB) divides $\det B \,\mathrm{sif}(A)$. Thus, $\mathrm{sif}(G) \mathrm{divides} c_2 d_2 \,\mathrm{sif}(T)$. Now, if any complex number s_0 in the right half plane is either a zero of c_2 or a zero of $\mathrm{sif}(T)$, then $\det[I - T(s_0) \tilde{L}_{01}(s_0) \mathrm{adj} \hat{D}_2(s_0)] = \det I = 1$. Also, if s_0 is an unstable zero of d_2, then

$$\det[I - T(s_0) \tilde{L}_{01}(s_0) \,\mathrm{adj}\, \hat{D}_2(s_0)] = \det\{I - [\mathrm{adj}\, \hat{D}_2(s_0)] T(s_0) \tilde{L}_{01}(s_0)\} = 1$$

since d_2 divides $(\mathrm{adj}\, \hat{D}_2) T$. Therefore, at all unstable zeros of $c_2 d_2 \,\mathrm{sif}(T)$, and of $\mathrm{sif}(G), \det(I - T \tilde{L}_{01} \,\mathrm{adj}\, \hat{D}_2)$ has constant sign so that $\hat{U}$ and U can be made unimodular by a suitable X_1. □

Remark 2: If $\mathbf{A}$ holds but Z_{11} and Z_{22} have common unstable poles, then one can show, in the 2×2 case of Remark 1, that Z_d and Z can be simultaneously stabilized if and only if there are an even number of unstable zeros of $n_{11} n_{22} \det N$ between every pair of real unstable zeros of $gcf\{m_{11}, m_{22}\}$. This condition corresponds to $(gcf\{m_{11}, m_{22}\}, n_{11} n_{22} \det N)$ being coprime and having the parity interlacing property. In the multivariable case, one may thus expect that Z_d and Z can be simultaneously stabilized if and only if there are an even number of elements of the set $\{s : Z_{11}(s) = 0 \text{ or } Z_{22}(s) = 0 \text{ or } Z(s) = 0\}$ between every pair of common real, unstable diagonal subsytem poles. As a consequence, if $\det N$ is identically zero, then the lack of unstable common poles of Z_{11} and Z_{22} becomes a necessary condition for solution of the problem. •

The construction in Theorem 2 of local stabilizing controllers achieving global stability can be summarized as follows.

i) Obtain the fractional representation (3) for Z following the procedure of Lemma 1.
ii) Determine initial stabilizing local controllers $L_{0i} K_{0i}^{-1}$ satisfying (18). Make sure that $c_2 d_2$ divides L_{02}.
iii) Determine Y_2 such that M of (21) is nonsingular. Let $X_2 := c_2 d_2 Y_2$.
iv) Determine X_1 such that $\hat{U}$ is unimodular using [12, Cor. 5.3.6].
v) Compute $L_i K_i^{-1}$ according to (20) and (15), where X_1 and X_2 are substituted from ii) and iii).

The following simple example illustrates this procedure.

Example: Consider

$$Z = \begin{bmatrix} \frac{1}{2(s-1)} & -\frac{1}{2s} \\ \frac{1}{2(s-1)} & \frac{1}{2s} \end{bmatrix} = \begin{bmatrix} \frac{s}{s+1} & -\frac{s}{s+1} \\ -\frac{1}{s+1} & \frac{2s-1}{s+1} \end{bmatrix}^{-1} \begin{bmatrix} 0 & -\frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

in which

$$P_1 = [1 \;\; 0] \quad P_2 = [0 \;\; 1] \quad R_1 = \left[0 \;\; \frac{1}{s+1}\right]'$$

$$R_2 = \left[-\frac{1}{s+1} \, \frac{1}{s+1}\right]'$$

and Q is the denominator matrix shown previously.

i) We obtain (3) by $C_1 = C_2 = C_{11} = C_{22} = I_2$ and by

$$D_1 = \begin{bmatrix} \frac{s}{s+1} & 0 \\ 0 & 1 \end{bmatrix} \quad D_{22} = \begin{bmatrix} 1 & -1 \\ -\frac{1}{s+1} & \frac{2s-1}{s+1} \end{bmatrix}$$
$$R_{11} = \begin{bmatrix} 0 \\ \frac{1}{s+1} \end{bmatrix}, D_2 = \begin{bmatrix} 1 & 0 \\ -1 & \frac{s-1}{s+1} \end{bmatrix}$$
$$D_{11} = \begin{bmatrix} \frac{s}{s+1} & -\frac{s}{s+1} \\ 1 & 1 \end{bmatrix} \quad R_{22} = \begin{bmatrix} -\frac{1}{s+1} \\ 0 \end{bmatrix}.$$

It can be checked that $D_1 D_{22} = D_2 D_{11}, Q = D_1 D_{22} C_{22} C_1$ and (3) is obtained. Let us alo identify left and right coprime fractions for Z_{11} and Z_{22}. We have $D_{22}^{-1} R_{11} = \tilde{R}_1 \tilde{D}_2^{-1}$, and $D_{11}^{-1} R_{22} = \tilde{R}_2 \tilde{D}_1^{-1}$, where

$$\tilde{R}_1 = \begin{bmatrix} \frac{1}{2(s+1)} \\ \frac{1}{2(s+1)} \end{bmatrix} \quad \tilde{D}_2 = \frac{s-1}{s+1}$$
$$\tilde{R}_2 = \begin{bmatrix} -\frac{1}{2(s+1)} \\ \frac{1}{2(s+1)} \end{bmatrix} \quad \tilde{D}_1 = \frac{s}{s+1}.$$

Also, $P_{ii} = \tilde{P}_i = P_i, i = 1,2$ and $C_{ii} = \tilde{C}_i = C_i = I, i = 1,2$. Moreover, $\tilde{P}_1 D_{22}^{-1} = \hat{D}_2^{-1} \hat{P}_1$ and $\tilde{P}_2 D_{11}^{-1} = \hat{D}_1^{-1} \hat{P}_2$, where $\hat{D}_1 = s/(s+1)$

$$\hat{D}_2 = \frac{s-1}{s+1} \quad \hat{P}_1 = \left[\frac{2s-1}{2(s+1)} \frac{1}{2}\right] \quad \hat{P}_2 = \left[-\frac{1}{2} \frac{s}{2(s+1)}\right].$$

ii) Initial stabilizing controllers for

$$Z_{11} = (\hat{D}_2 \tilde{C}_2)^{-1} \hat{P}_1 \quad R_{11} = \left(\frac{s-1}{s+1}\right)^{-1} \frac{1}{2(s+1)}$$
$$Z_{22} = (\hat{D}_1 \tilde{C}_1)^{-1} \quad \hat{P}_2 R_{22} = \left(\frac{s}{s+1}\right)^{-1} \frac{1}{2(s+1)}$$

are easily computed (in this simple case by inspection) as $L_{0i} K_{0i}^{-1}, i = 1,2$ with $K_{01} = 1, \tilde{L}_{01} = L_{01} = 4, K_{02} = (s+3)/(s+1), \tilde{L}_{02} = L_{02} = -2(s-1)/(s+1)$, where $c_2 d_2 = (s-1)/(s+1)$ is a factor of L_{02}.

iii) Here, on letting $X_2 = (s-1)/(s+1)Y_2$, we have $M = 1+(1)/((s+1)^2)[-2(s-1)/(s+1)+(s(s-1))/((s+1)^2)Y_2]$ nonsingular for the choice $Y_2 = 0$. Thus, $U = 1 - (s-1)/(2(s+1)^3)(4 + (s-1)/(s+1)X_1)$.

iv) Note that when evaluated at the two unstable blocking zeros $s = 1$ and $s = \infty$ of the coefficient of $X_1, 1-(2(s-1))/((s+1)^3)$ is equal to 1. Restricting X_1 to be of first order, it can be computed, using e.g., root-locus, that U is unimodular for $X_1 = 0.286(s+5)/(s+0.1)$ with its zeros at $\{-3.0699, -0.1719 \pm 0.7416i, -0.3432 \pm 0.6666i\}$.

v) Fixing X_1 as in iv) and $X_2 = 0$, we obtain that

$$Z_c = \begin{bmatrix} \frac{4(s+1)(s+0.1)+0.286(s-1)(s+5)}{(s+1)(s+0.1)-0.143(s+5)} & 0 \\ 0 & -2\frac{s-1}{s+3} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{4.286s^2+5.544s-1.030}{s^2+0.957s-0.615} & 0 \\ 0 & -2\frac{s-1}{s+3} \end{bmatrix}$$

simultaneously stabilizes Z_d and Z.

Note that the construction of Theorem 2 is not necessarily efficient, i.e., it may involve more computations or yield higher order conrollers than necessary in some cases. In fact, for our example in which diagonal subsystems are scalar, the second controller need not contain the unstable zero at $s = 1$ and it is easy to check that $\mathrm{diag}\{4, 2\}$ is another solution to the problem. •

III. Conclusion

We have shown that the assumption $\mathbf{A}$ is a crucial one for stabilizing a system by stabilizing its main diagonal subsystems and somewhat trivializes the existence of a solution: $\mathbf{A}$ by itself ensures a solution to exist and no extra conditions such as diagonal dominance need be imposed. This has been established in Theorem 2 for the case in which the unstable poles of the diagonal subsytems are disjoint. In the general case, when diagonal subsystems have some common unstable poles, a similar result is expected. If Z is 2×2, for instance, one can show that it is possible to simultaneously stabilize Z_d and Z if and only if there are an even number of zeros of $Z_{11} Z_{22} Z$ between every pair of real, unstable, and common diagonal subsystem poles. Extension of this result to multivariable case is currently under investigation. In closing, we should mention that when stabilization is not the only concern and other design specifications are present, the diagonal dominance property is very useful as illustrated in [7].

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