

Partially Informed Agents can Form a Swarm in a Nash Equilibrium

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Abstract—Foraging swarms in one-dimensional motion with incomplete position information are studied in the context of a noncooperative differential game. In this game, the swarming individuals act with partial information as it is assumed that each agent knows the positions of only the adjacent ones. It is shown that a Nash equilibrium solution that exhibits many features of a foraging swarm such as movement coordination, self-organization, stability, and formation control exists.

Index Terms—Dynamic multi-agent systems, swarming behavior, social foraging, dynamic game theory, Nash equilibrium, finite horizon, artificial potentials.

I. INTRODUCTION

The motivations for collective movements such as schooling of fish, flocking of birds, and herding of sheep are having protection from predators, saving energy, and locating food sources with ease [1]. Such swarms have attracted attentions of scientists and engineers in many disciplines. The following features of a swarm are most remarkable [2]: i) No member in a swarm views the whole picture, but their decentralized actions result in a collective behavior. ii) Simple actions of the members described in [3] result in a complex behavior of the swarm. iii) There are no leaders commanding the others so that many swarms are self-propelled. iv) There is limited communication based on local information among members. Such features of swarms are expressed by the notions of coordinated group behavior, self organization, stability, collision avoidance and distributed control [4]. Engineers have based their designs of multi-robot or multi-vehicle systems mainly on these concepts [5], [6], [7], [8].

In recent years, swarm analysis techniques have focused on three principal methodologies; namely, model based approaches, Lyapunov analysis and simulations. Compared to model based approaches, simulation based approaches suffer from convergence, accuracy, and computational complexity issues. On the other hand, while Lyapunov based methods (e.g. [9], [10], [11]) remain confined to the stability (boundedness) analysis, a model based approach allows a more comprehensive theoretical analysis that may reveal important structural properties.

Noncooperative game theory, in particular the notion of Nash equilibrium, is ideally suited for studying collective behaviors that are caused by *decentralized* individual motives and actions. It thus seems that quests into the nature and the origin of collective behavior in swarms is a natural application area for game theoretical models; but, such studies are surprisingly rare. Currently, the application has mainly been limited to two-person games since the objective was mainly to understand the “motive formation” of animals, [12], [13]. In studying multi-robot, multi-vehicle systems *cooperative* game theory has been the main tool applied since the emphasis [14] is on the “design” of a swarm system, rather than an analysis which strives to “explain” collective behavior. Vehicle platooning or air traffic control in automated environments require conflict resolution so that game theory is used in [15], [16], [17] and [18] for the purpose of coordination.

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First studies, which demonstrate that a swarming behavior may result as a Nash equilibrium are [19] and [20]. A main assumption in both [19] and [20] is that each agent has a complete information of its pairwise distances to other agents. The main contribution of this article is to relax this assumption by considering that each agent has a partial information access and knows its pairwise distances to neighboring agents only. The assumption that a member interacts with (exchanges information with or has sensory perception of) all of the remaining members of a swarm may be a realistic assumption when the swarm size is not too large or when designing a swarm system from scratch. It may not, however, be realistic in large biological swarms or if the cost of communication is substantial. The swarm is thus assumed to have the structure of a line topology communication network as opposed to a complete topology network.

The paper is organized as follows. In Section II, the main non-cooperative dynamic game is introduced for the case where target location is exactly known by the agents. In the remaining part of Section II, main results and their implications are given. Section III is on conclusions and the proofs of the main theorems are given in the Appendix.

II. PROBLEM DEFINITION AND MAIN RESULTS

One dimensional motion of swarms with incomplete position information is modeled as a noncooperative infinite dimensional dynamic game in this section. Every agent in the swarm is assumed to know its distance to only the adjacent agents. Each swarm member minimizes the total work done in a time interval $[0, T]$ by controlling its velocity. The total work done by the i -th member of the swarm for $i = 1, \dots, N$ can be formulated as

$$L^i(u^i, x^{i-1}, x^i, x^{i+1}) := [x^i(T)]^2 f + \int_0^T \left\{ \frac{u^i(t)^2}{2} + \sum_{j=i-1, j \neq i}^{i+1} \left(\frac{a[x^i(t) - x^j(t)]^2}{2} - r|x^i(t) - x^j(t)| \right) \right\} dt, \quad (1)$$

with the convention that $x_0(t) = x_{N+1}(t) = 0$. Each agent is assumed to adjust its control so as to minimize this expression. Here, N is the number of agents, $x^i(t)$ is the position of the i^{th} member. The control input of agent- i is assumed to be its velocity

$$u^i(t) = \dot{x}^i(t). \quad (2)$$

The first component of the total work is the environment potential which monitors the toxicity or the amount of food source at position x . Here, it is selected as a quadratic profile as in [21]. The second component is the kinetic energy term which measures the total effort of the i^{th} member. The minimization of this effort term implies that the swarm members use their energy efficiently which is an essential feature of actual biological swarms [22]. The third term in the total work done is the attraction potential energy and the last term is the repulsion potential energy. The attraction and repulsion potentials are again chosen following [23], [21]. The parameters f, a , and r are the weights of the environment, attraction, and repulsion potential terms, respectively. These weights are thus assumed to be of the same value for all swarm members, which is a reasonable assumption for biological swarms consisting of the same species.

The optimization performed by the swarm members is

$$\min_{u^i} \{L^i\} \text{ subject to } \dot{x}^i = u^i, \forall i = 1, \dots, N. \quad (3)$$

This is thus a noncooperative dynamic game and we will investigate the existence and uniqueness of Nash equilibrium of this game. For a concise exposition, we give the main result only for the *specified terminal condition* case of $x^i(T) = 0$ for all $i = 1, \dots, N$ in (1). (See [24] for the general *free terminal condition* case.) The closed

form solution will be obtained in Appendix through the approaches outlined in [25] and [26].

We will see in Appendix that solution of the above problems requires solving nonlinear differential equations that do not obey any Lipschitz condition. Therefore, neither the existence nor the uniqueness of a Nash equilibrium is clear at the outset.

We now describe the main features of the solution to the game played by agents. Consider the position vector of the N agents $\mathbf{x}(t) := [x^1(t) \dots x^N(t)]'$, and the vector of pairwise distances and sum $\mathbf{y}(t) := [x^1(t) - x^2(t)] \dots [x^{N-1}(t) - x^N(t)] \sum_{j=1}^N x^j(t)'$ where “prime” denotes transpose. Let $M \in \mathbf{R}^{(N-1) \times N}$ be such that $M_{i,i} = 1$, $M_{i,i+1} = -1$, $M_{i,j} = 0$ for $i = 1, \dots, N-1$, $j = 1, \dots, N$, $i \neq j \neq i+1$. Thus, the i -th row of M has all zeros except a 1 and a -1 at its i -th and $(i+1)$ -st positions, respectively. Consider the singular value decomposition

$$M = U \Sigma V', \quad (4)$$

for unitary matrices $V \in \mathbf{R}^{N \times N}$, $U \in \mathbf{R}^{(N-1) \times (N-1)}$. The matrix M has one zero singular value and $N-1$ distinct singular values all in the open interval $(0, 2)$. The N singular values $\sigma_1 > \sigma_2 > \dots > \sigma_{N-1} > \sigma_N$ are non-degenerate so that the columns of U and of V are unique up to sign. Let

$$\sigma_k := 2 \cos\left(\frac{k\pi}{2N}\right), \quad \alpha_k := \sigma_k \sqrt{a}, \quad k = 1, \dots, N-1, \quad (5)$$

and $\sigma_N = \alpha_N := 0$. The time constants α_k^{-1} will determine how $\mathbf{x}(t)$ and $\mathbf{y}(t)$ evolve in time.

Define

$$\begin{aligned} b_k(t) &:= \frac{\sinh[\alpha_k(T-t)]}{\sinh(\alpha_k T)}, \\ c_k(t) &:= \frac{1}{\alpha_k^2} [1 - b_k(t) - \frac{\sinh(\alpha_k t)}{\sinh(\alpha_k T)}], \end{aligned} \quad (6)$$

and consider $B(t) := \text{diag}[b_1(t), \dots, b_{N-1}(t), \frac{T-t}{T}]$, $C(t) := r \text{diag}[c_1(t), \dots, c_{N-1}(t), \frac{(T-t)t}{2}]$,

$$Q := \text{diag}[U, 1], \quad \mathbf{r} = [1 \ 0 \ \dots \ 0 \ 1 \ 0 \ 0]' \in \mathbf{R}^N. \quad (7)$$

THEOREM 1. *Given any $r \in (0, \infty)$, there exists $a_0 \in (0, \infty)$ such that for each value $a \in (0, a_0)$ of the attraction parameter, a unique Nash equilibrium with specified terminal condition of the partial information game (1)-(3) exists. This Nash solution has the following properties:*

P1. *The initial ordering among the N agents in the queue is preserved during $[0, T]$.*

P2. *The vector of pairwise distances and sum at time t is given by*

$$\mathbf{y}(t) = QB(t)Q'\mathbf{y}(0) + QC(t)Q'\mathbf{r}. \quad (8)$$

P3. *For every T and as $T \rightarrow \infty$, the swarm size $d_{\max}(t) := \max_{i,j} |x^i(t) - x^j(t)|$ remains bounded in $[0, T]$.*

It follows that self-organized (no leader) agents, each individually optimizing its effort, end up in a coordinated movement towards the foraging location. Here, we emphasize as a fundamental feature of Nash equilibrium that if each agent minimizes its total work (1), which only requires the position information of agents adjacent to it and the knowledge that the location of food (or the least toxic region) is the origin, then the foraging swarm behavior characterized by P1-P3 is expected. The swarm that results from this decentralized action is such that the initial ordering among agents is preserved, it is stable (its size is bounded) by P3, and the distance between the consecutive agents can be computed by P2 at any given time. Also by P2, the last entry of $\mathbf{y}(t)$ gives the swarm-center $\bar{x}(t) := \frac{x^1(t) + \dots + x^N(t)}{N}$ as $\bar{x}(t) = \frac{T-t}{T} \bar{x}(0)$, which monotonically approaches the target location as $t \rightarrow T$ and ends up at the origin at T .

The proof of Theorem 1 in the Appendix (see Remark A.1) will show that if $a > 0$ is sufficiently large, then the existence conclusion of Theorem 1 also holds true. It is, however, still an open question whether a Nash solution exists without the assumption of a small or large attraction parameter a .

III. CONCLUSIONS

The results in this article complement the (more comprehensive) result of [20] that was based on the hypothesis of complete information. The main contribution in both has been to show that a collective behavior of foraging swarm can result from self-organized actions of individual agents. This is a large step in explaining the phenomena of biological swarms.

The prices paid in going from the complete to partial information assumptions are described in [24] and they can be summarized as: a slower convergence to the foraging location, more dependence on the initial conditions, and having to additionally assume an ordering relation such as the attraction parameter a is small (or large), or equivalently, that the repulsion parameter r is large (or small). The simulations carried out in [24] and our intuition indicate that the unique Nash solution of Theorem 1 is actually valid for all $a, r > 0$.

IV. APPENDIX

This section contains a proof of Theorem 1. The proof is rather technical and long because an essential task is to establish the “positivity” of certain time-varying matrices in the foraging interval $[0, T]$. We refer the reader to [24] for the result in the free terminal condition case and for details.

The optimal control problem that the i^{th} agent needs to solve, i.e., minimize (1) subject to (3), is first considered, [27]. Applying the necessary conditions of optimal control on Hamiltonian as in [20], we have the state equation given by

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} 0 & -I \\ -A & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{p}(t) \end{bmatrix} + r \begin{bmatrix} 0 \\ \mathbf{s}(t) \end{bmatrix}, \quad (9)$$

where $\mathbf{x} := [x^1 \dots x^N]'$, $\mathbf{p} := [p^1 \dots p^N]'$, $\mathbf{s} := [\sum_{j=0, j \neq 1}^2 \text{sgn}(x^1 - x^j) \dots \sum_{j=N-1, j \neq N}^{N+1} \text{sgn}(x^N - x^j)]'$.

The “signum vector” \mathbf{s} is piecewise-constant in the interval $[0, T]$ with each constant value obtained by a permutation of entries in $[1 \ 0 \ \dots \ 0 \ -1]'$. This is because its i^{th} entry $s^i = \sum_{j=i-1, j \neq i}^{i+1} \text{sgn}(x^i - x^j)$ is equal to 1 if agent i is leading the queue, -1 if i is the last in queue, and 0 otherwise. Also in (9), A is the symmetric tridiagonal matrix

$$A = a M' M = a \begin{bmatrix} 1 & -1 & 0 & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & 0 & -1 & 1 \end{bmatrix},$$

where M is as defined prior to (4) and a is the attraction parameter in (1). Note that the matrix V in (4) is such that $A = a M' M = V a \Sigma' \Sigma V' = V \text{diag}[D^2, 0] V'$, $D := \text{diag}[\alpha_1, \dots, \alpha_{N-1}]$. We will now obtain a solution to (9) under the assumption that $\mathbf{s}(t) = \mathbf{s}(0)$ for all $t \in [0, T]$. We first list certain properties of the matrix A .

Let $B_n(t)$ denote the n -th Bernoulli polynomial (see e.g., [28], Ch. 12).

Lemma A.1. *It holds that*

$$b_k(t) = \sum_{n=0}^{\infty} \beta_n(t) \sigma_k^{2n}, \quad c_k(t) = \sum_{m=0}^{\infty} \gamma_m(t) \sigma_k^{2m}, \quad (10)$$

where

$$\beta_n(t) = B_{2n+1}(1 - \frac{t}{2T}) \frac{2^{2n+1}}{(2n+1)!} T^{2n} a^n, \quad (11)$$

$$\gamma_m(t) = -[B_{2m+3}(1 - \frac{t}{2T}) + B_{2m+3}(\frac{1}{2} + \frac{t}{2T})] \frac{2^{2m+3}}{(2m+3)!} T^{2m+2} a^{m+1}.$$

Moreover,

$$\text{sign}\{\beta_n(t)\} = \text{sign}\{\gamma_m(t)\} = (-1)^n, \quad \forall t \in [0, T]. \quad (12)$$

Proof. By the defining equation for Bernoulli polynomials $\frac{ye^{xy}}{e^y - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{y^n}{n!}$, we can write $b_k(t)$ of (6) as

$$\frac{2\alpha_k T e^{(2\tau-1)\alpha_k T}}{e^{\alpha_k T} - e^{-\alpha_k T}} = \sum_{n=0}^{\infty} B_n(\tau) \frac{(2\alpha_k T)^n}{n!}, \quad (13)$$

$$\frac{2\alpha_k T e^{-(2\tau-1)\alpha_k T}}{e^{-\alpha_k T} - e^{\alpha_k T}} = \sum_{n=0}^{\infty} B_n(\tau) (-1)^n \frac{(2\alpha_k T)^n}{n!}.$$

Subtracting and dividing by $2\alpha_k T$, and letting $\tau := 1 - \frac{t}{2T}$, we get

$$\frac{e^{(2\tau-1)\alpha_k T} - e^{-(2\tau-1)\alpha_k T}}{e^{\alpha_k T} - e^{-\alpha_k T}} = \sum_{n=0}^{\infty} B_n(\tau) [1 - (-1)^n] \frac{(2\alpha_k T)^{n-1}}{n!},$$

which leads to the expression for $b_k(t)$ in (10) and (11). A similar procedure applied to $c_k(t)$ also leads to the expansion in (10), (11) for $c_k(t)$. The odd-numbered Bernoulli polynomials have constant sign in the interval $(0.5, 1)$ with $B_1(\tau)$ having positive sign (Chapter 23 of [29]). Now, $B'_{2n+1}(\tau) = (2n+1)B_{2n}(\tau)$ for $n \geq 1$ so that $\text{sign}\{B'_{2n+1}(1)\} = \text{sign}\{B_{2n}(1)\}$. Since $B_{2n}(1)$ are the second Bernoulli numbers, it follows that $\text{sign}\{B'_{2n+1}(1)\} = (-1)^{n-1}$ for $n \geq 1$. Thus, $B_{2n+1}(\tau)$ is decreasing to 0 as $\tau \rightarrow 1$ if and only if n is even, which gives $\text{sign}\{B_{2n+1}(\tau)\} = (-1)^n, \forall \tau \in (0.5, 1)$. This implies (12) by the expressions in (11). \square

We now turn to (9) with $\mathbf{s}(t) = \mathbf{s}(0)$ for all $t \in [0, T]$ and note that

$$\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{p}(t) \end{bmatrix} = \phi(t) \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{p}(0) \end{bmatrix} + \psi(t, 0) \mathbf{s}(0), \quad (14)$$

where

$$\phi(t) = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix} := \mathcal{L}^{-1} \left\{ \begin{bmatrix} sI & I \\ A & sI \end{bmatrix}^{-1} \right\},$$

$$\psi(t, t_0) := \int_{t_0}^t r \begin{bmatrix} \phi_{12}(t-\tau) \\ \phi_{22}(t-\tau) \end{bmatrix} d\tau,$$

for initial time $t_0 \geq 0$ with r being the repulsion parameter. The inverse Laplace transform of

$$\begin{bmatrix} sI & I \\ A & sI \end{bmatrix}^{-1} = \begin{bmatrix} s(s^2I - A)^{-1} & -(s^2I - A)^{-1} \\ -A(s^2I - A)^{-1} & s(s^2I - A)^{-1} \end{bmatrix},$$

gives that

$$\begin{aligned} \phi_{11}(t) &= \phi_{22}(t) = V \text{diag} [\cosh(\alpha_1 t), \dots, \cosh(\alpha_{N-1} t), 1] V', \\ \phi_{12}(t) &= -V \text{diag} \left[\frac{\sinh(\alpha_1 t)}{\alpha_1}, \dots, \frac{\sinh(\alpha_{N-1} t)}{\alpha_{N-1}}, t \right] V', \\ \phi_{21}(t) &= -V \text{diag} [\alpha_1 \sinh(\alpha_1 t), \dots, \alpha_{N-1} \sinh(\alpha_{N-1} t), 0] V'. \end{aligned} \quad (15)$$

Moreover, integration results in

$$\begin{aligned} \psi_1(t, 0) &= r V \text{diag} \left[\frac{1 - \cosh(\alpha_1 t)}{\alpha_1^2}, \dots, \frac{1 - \cosh(\alpha_{N-1} t)}{\alpha_{N-1}^2}, -\frac{t^2}{2} \right] V', \\ \psi_2(t, 0) &= r V \text{diag} \left[\frac{\sinh(\alpha_1 t)}{\alpha_1}, \dots, \frac{\sinh(\alpha_{N-1} t)}{\alpha_{N-1}}, t \right] V'. \end{aligned} \quad (16)$$

Using the boundary condition $\mathbf{x}(T) = 0$ in (14) for $t = T$ gives $\phi_{11}(T)\mathbf{x}(0) + \phi_{12}(T)\mathbf{p}(0) + [\psi_1(T, 0) - \psi_2(T, 0)]\mathbf{s}(0) = 0$ which can be solved for $\mathbf{p}(0)$ since $\phi_{12}(T)$ is nonsingular. It follows that there is a candidate solution of (9) for every $\mathbf{x}(0)$. This solution is

$$\begin{aligned} \mathbf{x}(t) &= \{\phi_{11}(t) - \phi_{12}(t)[\phi_{12}(T)]^{-1}\phi_{11}(T)\}\mathbf{x}(0) \\ &\quad + \{\psi_1(t, 0) - \phi_{12}(t)[\phi_{12}(T)]^{-1}\psi_1(T, 0)\}\mathbf{s}(0). \end{aligned} \quad (17)$$

Proof of THEOREM 1. Let us assume $x_1(t) > x_2(t) > \dots > x_N(t)$ without loss of generality so that $\mathbf{s}(0)' = \begin{bmatrix} 1 & 0 & \dots & 0 & -1 \end{bmatrix}$. Substituting (15) and (16) into (17) yields

$$\mathbf{x}(t) = VB(t)V'\mathbf{x}(0) + VC(t)V'\mathbf{s}(0). \quad (18)$$

Note that the nonsingular matrix $[M' \mathbf{w}]$, where $\mathbf{w} \in \mathbf{R}^N$ is a vector of all 1's, satisfies

$$\begin{bmatrix} M \\ \mathbf{w}' \end{bmatrix} V a \Sigma' \Sigma V' = Q \begin{bmatrix} D^2 & \mathbf{0} \\ \mathbf{0}' & 0 \end{bmatrix} Q' \begin{bmatrix} M \\ \mathbf{w}' \end{bmatrix},$$

and both $B(t)$ and $C(t)$ are matrix functions of $\Sigma' \Sigma$. Hence, the transformation $\mathbf{y}(t)' = \mathbf{x}(t)'[M' \mathbf{w}]$ applied to (18) gives

$$\begin{aligned} \mathbf{y}(t) &= \begin{bmatrix} M \\ \mathbf{w}' \end{bmatrix} VB(t)V' \begin{bmatrix} M \\ \mathbf{w}' \end{bmatrix}^{-1} \mathbf{y}(0) \\ &\quad + \begin{bmatrix} M \\ \mathbf{w}' \end{bmatrix} VC(t)V' \begin{bmatrix} M \\ \mathbf{w}' \end{bmatrix}^{-1} \mathbf{r}. \end{aligned} \quad (19)$$

This yields (8).

We now show that with such $\mathbf{y}(t)$, the ordering of the agents indeed remains the same, i.e., $\text{sign } y^i(t) = \text{sign } y^i(0)$ for all $i = 1, \dots, N-1$ and $t \in [0, T]$. We establish this for some (small enough) values of the attraction parameter $a > 0$. Let us consider the sub-vector $\mathbf{y}_d := [x_1 - x_2 \dots x_{N-1} - x_N]'$ of \mathbf{y} . Then, with $\mathbf{r}_d := [1 \ 0 \dots 0 \ 1]'$, (8) gives $\mathbf{y}_d(t) = K(t)\mathbf{y}_d(0) + L(t)\mathbf{r}_d$, $K(t) := U \text{diag}[b_1(t), \dots, b_{N-1}(t)]U'$, $L(t) := U \text{diag}[c_1(t), \dots, c_{N-1}(t)]U'$, which are both positive definite matrices for every $t \in [0, T]$ by the fact that $b_i(t)$ and $c_i(t)$ are positive functions of $t \in [0, T]$ for $i = 1, \dots, N-1$. The matrix U of (4) that occurs here can be written explicitly by [30], p. 514, as $U = [U_{ij}]$, $U_{ij} = \sqrt{\frac{2}{N}} \sin(\frac{(N-j)i\pi}{N})$, so that, for $i, j = 1, \dots, N-1$,

$$K_{ij}(t) = \frac{2}{N} \sum_{k=1}^{N-1} b_{N-k}(t) \sin(\frac{ki\pi}{N}) \sin(\frac{kj\pi}{N}),$$

$$L_{ij}(t) = \frac{2}{N} \sum_{k=1}^{N-1} c_{N-k}(t) \sin(\frac{ki\pi}{N}) \sin(\frac{kj\pi}{N}).$$

We now show that, there exist values for the attraction parameter $a > 0$ such that for all $i, j = 1, \dots, N-1$ and $t \in [0, T]$, $K_{ij}(t) > 0$ and $L_{ij}(t) > 0$. Consider

$$\begin{aligned} K_{ij}(t) &= \frac{2}{N} \sum_{k=1}^{N-1} b_k(t) \sin(\frac{(N-k)i\pi}{N}) \sin(\frac{(N-k)j\pi}{N}) \\ &= \frac{2}{N} (-1)^{i+j} \sum_{k=1}^{N-1} b_k(t) \sin(\frac{ki\pi}{N}) \sin(\frac{kj\pi}{N}) \\ &= \frac{(-1)^{i+j}}{N} (-1)^{i+j} \sum_{k=1}^{N-1} b_k(t) \{\cos[\frac{(i-j)k\pi}{N}] \\ &\quad - \cos[\frac{(i+j)k\pi}{N}]\}. \end{aligned} \quad (20)$$

By these expressions it follows that $K_{ij}(t) = K_{N-j, N-i}(t)$, $L_{ij}(t) = L_{N-j, N-i}(t)$ for all i, j , i.e., K and L are centrosymmetric (or bisymmetric) matrices, [31]. This allows us to only show the positivity of the entries with

$$j < i \leq N-j, \quad j = 1, \dots, \lfloor \frac{N-1}{2} \rfloor. \quad (21)$$

Substituting (10) into $K_{ij}(t)$ and employing the trigonometric identity

$$\cos^{2m}(\theta) = \frac{1}{2^{2m}} \binom{2m}{m} + \frac{2}{2^{2m}} \sum_{l=0}^{m-1} \binom{2m}{l} \cos[2(m-l)\theta],$$

we have

$$\begin{aligned} K_{ij}(t) &= \frac{(-1)^{i+j}}{N} \sum_{n=0}^{\infty} \beta_n \sum_{k=1}^{N-1} \sigma_k^{2n} \left\{ \cos\left[\frac{(i-j)k\pi}{N}\right] - \cos\left[\frac{(i+j)k\pi}{N}\right] \right\} \\ &= \frac{(-1)^{i+j}}{N} \sum_{n=0}^{\infty} \beta_n \sum_{k=1}^{N-1} 2^{2n} \cos^{2n}\left(\frac{k\pi}{2N}\right) \left\{ \cos\left[\frac{(i-j)k\pi}{N}\right] - \cos\left[\frac{(i+j)k\pi}{N}\right] \right\} \\ &= \frac{(-1)^{i+j}}{N} \sum_{n=0}^{\infty} \beta_n \left(\frac{2n}{n} \right) \sum_{k=1}^{N-1} \left\{ \cos\left[\frac{(i-j)k\pi}{N}\right] - \cos\left[\frac{(i+j)k\pi}{N}\right] \right\} \\ &\quad + \frac{2(-1)^{i+j}}{N} \sum_{n=0}^{\infty} \beta_n \sum_{l=0}^{n-1} \left(\frac{2n}{l} \right) \sum_{k=1}^{N-1} \cos\left[\frac{(n-l)k\pi}{N}\right] \\ &\quad \left\{ \cos\left[\frac{(i-j)k\pi}{N}\right] - \cos\left[\frac{(i+j)k\pi}{N}\right] \right\}. \end{aligned} \quad (22)$$

We now compute the finite sums over k and t . Let $\mathcal{E}(N)$ read as “an even multiple of N .” The first sum is

$$\begin{aligned} &\sum_{k=1}^{N-1} \left\{ \cos\left[\frac{(i-j)k\pi}{N}\right] - \cos\left[\frac{(i+j)k\pi}{N}\right] \right\} \\ &= \left\{ \begin{matrix} -1, & i-j \neq \mathcal{E}(N) \\ N-1, & i-j = \mathcal{E}(N) \end{matrix} \right\} - \left\{ \begin{matrix} -1, & i+j \neq \mathcal{E}(N) \\ N-1, & i+j = \mathcal{E}(N) \end{matrix} \right\} = 0, \end{aligned}$$

where the last equality is by (21). Let $t_1 := t - i + j$, $t_2 := t + i - j$, $t_3 := t + i + j$, $t_4 := t - i - j$. The second sum is

$$\begin{aligned} &\sum_{l=0}^{n-1} \left(\frac{2n}{l} \right) \sum_{k=1}^{N-1} \cos\left[\frac{(n-l)k\pi}{N}\right] \left\{ \cos\left[\frac{(i-j)k\pi}{N}\right] - \cos\left[\frac{(i+j)k\pi}{N}\right] \right\} \\ &= \sum_{t=1}^n \left(\frac{2n}{n-t} \right) \sum_{k=1}^{N-1} \cos\left(\frac{t k \pi}{N}\right) \left\{ \cos\left[\frac{(i-j)k\pi}{N}\right] - \cos\left[\frac{(i+j)k\pi}{N}\right] \right\} \\ &= \frac{1}{2} \sum_{t=1}^n \left(\frac{2n}{n-t} \right) \sum_{k=1}^{N-1} \left\{ \cos\left(\frac{t_1 k \pi}{N}\right) - \cos\left(\frac{t_2 k \pi}{N}\right) + \cos\left(\frac{t_3 k \pi}{N}\right) - \cos\left(\frac{t_4 k \pi}{N}\right) \right\} \\ &= \frac{1}{2} \sum_{t=1}^n \left(\frac{2n}{n-t} \right) \left\{ \begin{matrix} -1, & t_1 \neq \mathcal{E}(N) \\ N-1, & t_1 = \mathcal{E}(N) \end{matrix} \right\} - \left\{ \begin{matrix} -1, & t_2 \neq \mathcal{E}(N) \\ N-1, & t_2 = \mathcal{E}(N) \end{matrix} \right\} \\ &\quad + \left\{ \begin{matrix} -1, & t_3 \neq \mathcal{E}(N) \\ N-1, & t_3 = \mathcal{E}(N) \end{matrix} \right\} - \left\{ \begin{matrix} -1, & t_4 \neq \mathcal{E}(N) \\ N-1, & t_4 = \mathcal{E}(N) \end{matrix} \right\}. \end{aligned} \quad (23)$$

By (21), it is easy to see that if $t_l = \mathcal{E}(N)$ for some $l = 1, 2, 3, 4$, then $t_k \neq \mathcal{E}(N)$ for all three $k \neq l$. Therefore,

$$\begin{aligned} K_{ij}(t) &= (-1)^{i+j} \sum_{n=0}^{\infty} \beta_n \\ &\quad \left[\sum_{\substack{p=1 \\ p = \mathcal{E}(N) + i - j}}^n \left(\frac{2n}{n-p} \right) + \sum_{\substack{p=1 \\ p = \mathcal{E}(N) - i - j}}^n \left(\frac{2n}{n-p} \right) \right. \\ &\quad \left. - \sum_{\substack{p=1 \\ p = \mathcal{E}(N) - i + j}}^n \left(\frac{2n}{n-p} \right) - \sum_{\substack{p=1 \\ p = \mathcal{E}(N) + i + j}}^n \left(\frac{2n}{n-p} \right) \right]. \end{aligned} \quad (24)$$

At this stage, rather than $K_{ij}(t)$, it will be more convenient to consider the expression for $K_{i,N-j}(t)$ for

$$N - j < i \leq j, \quad j = \lfloor \frac{N+1}{2} \rfloor, \dots, N-1. \quad (25)$$

With this change of index, we are still considering the same subset of entries of K but their expressions will be simpler. Substituting $N - j$ for j in the above expression, we have

$$\begin{aligned} S &:= K_{i,N-j}(t) (-1)^N (-1)^{i+j} \\ &= \sum_{n=0}^{\infty} \beta_n \left[\sum_{\substack{p=1 \\ p = O(N) - i - j}}^n \left(\frac{2n}{n-p} \right) + \sum_{\substack{p=1 \\ p = O(N) - i + j}}^n \left(\frac{2n}{n-p} \right) \right. \\ &\quad \left. - \sum_{\substack{p=1 \\ p = O(N) - i + j}}^n \left(\frac{2n}{n-p} \right) - \sum_{\substack{p=1 \\ p = O(N) + i + j}}^n \left(\frac{2n}{n-p} \right) \right], \end{aligned} \quad (26)$$

where $O(N)$ reads “odd multiple of N .” Writing out a few terms of each summation in the expression of S , it is not difficult to see that

$$\begin{aligned} S &= \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \sum_{k=0}^{2N-1} \left\{ \beta_{mN-i-j+k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN-2(i+j)+2k}{2tN+k} \right. \\ &\quad + \beta_{mN-i+j+k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN-2(i-j)+2k}{2tN+k} \\ &\quad - \beta_{mN+i-j+k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN+2(i-j)+2k}{2tN+k} \\ &\quad \left. - \beta_{mN+i+j+k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN+2(i+j)+2k}{2tN+k} \right\}. \end{aligned} \quad (27)$$

We now separate the even and odd k in the summations with respect to k , to obtain

$$\begin{aligned} S &= \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \sum_{k=0}^{N-1} \left\{ \beta_{mN-i-j+2k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN-2(i+j)+4k}{2tN+2k} \right. \\ &\quad + \beta_{mN-i+j+2k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN-2(i-j)+4k}{2tN+2k} \\ &\quad - \beta_{mN+i-j+2k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN+2(i-j)+4k}{2tN+2k} \\ &\quad - \beta_{mN+i+j+2k} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN+2(i+j)+4k}{2tN+2k} \\ &\quad + \beta_{mN-i-j+2k+1} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN-2(i+j)+4k+2}{2tN+2k+1} \\ &\quad + \beta_{mN-i+j+2k+1} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN-2(i-j)+4k+2}{2tN+2k+1} \\ &\quad - \beta_{mN+i-j+2k+1} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN+2(i-j)+4k+2}{2tN+2k+1} \\ &\quad \left. - \beta_{mN+i+j+2k+1} \sum_{t=0}^{\frac{m-1}{2}} \binom{2mN+2(i+j)+4k+2}{2tN+2k+1} \right\} \end{aligned} \quad (28)$$

For fixed m and k , the smallest indexed β occurs in the first term in the brackets. By the expression in (11), the sign of S is determined by the sign of $\beta_{mN-i-j+2k}$ for small enough attraction parameter $a > 0$ because $\beta_{mN-i-j+2k}$ is divisible by the smallest power of a among all β that occur in the above expression. It follows by (12) that, $\text{sign}(\beta_{mN-i-j+2k}) = (-1)^{mN-i-j}$ for all $t \in [0, T]$. Since m is odd, we have $\text{sign}(S) = (-1)^N (-1)^{i+j}$. This establishes that, there exists $a > 0$ such that for all i, j as in (25), $K_{i,N-j}(t) > 0$, $t \in [0, T]$. The proof of positivity of the matrix L is obtained in exactly the same manner since $\gamma_n(t)$ of Lemma A.1 replacing $\beta_n(t)$ in the last expression above yields $L_{i,N-j}(t)$. This proves that there is a Nash equilibrium in which the initial ordering among the agents is preserved in the whole interval $[0, T]$.

Here, it will be shown that swarm size is bounded. Since $x_1(t) > x_2(t) > \dots > x_N(t)$, the swarm size is equal to $x^1(t) - x^N(t)$ which

is given by

$$\begin{aligned} x^1(t) - x^N(t) = & \sum_{m=1}^N \sum_{p=1}^N \sum_{n=1}^{N-1} q_{nm} q_{pm} y_p(0) b_m(t) \\ & + \sum_{m=1}^N \sum_{n=1}^{N-1} q_{nm} q_{1m} c_m(t) \\ & + \sum_{m=1}^N \sum_{n=1}^{N-1} q_{nm} q_{(N-1)m} c_m(t), \end{aligned} \quad (29)$$

that results from (8), where q_{ij} is the ij^{th} entry of the matrix Q of (7) and $\mathbf{y}(0)$, $\mathbf{b}(t)$, and $\mathbf{c}(t)$ are as defined in (6). Note that, by triangular inequality

$$\begin{aligned} x^1(t) - x^N(t) \leq & \sum_{m=1}^N \sum_{p=1}^N \sum_{n=1}^{N-1} |q_{nm}| |q_{pm}| |y_p(0)| \max_t |b_m(t)| \\ & + \sum_{m=1}^N \sum_{n=1}^{N-1} |q_{nm}| |q_{1m}| \max_t |c_m(t)| \\ & + \sum_{m=1}^N \sum_{n=1}^{N-1} |q_{nm}| |q_{(N-1)m}| \max_t |c_m(t)|. \end{aligned} \quad (30)$$

Considering the first and second derivatives of $b_m(t)$ and $c_m(t)$, it is easy to show that $\max_t |b_m(t)| = 1$ and $\max_t |c_m(t)| = \frac{1}{\alpha_m^2} \left[1 - \frac{1}{\cosh\left(\frac{\alpha_m T}{2}\right)} \right]$, where α_k is given in (5). Since all the terms in the right hand side have finite positive values, $x^1(t) - x^N(t)$ is also finite. This completes the boundedness proof. \square

We finally show that Nash solution is unique with respect to strategies that are continuous against initial positions. Suppose that there are changes in the ordering of the agents at the $n-1$ time instants $\{t_1, \dots, t_{n-1}\} \in (0, T)$, with $n \geq 2$. The integer n is finite since the terminal condition should be satisfied exactly, not asymptotically. Let $t_0 := 0$ and $t_n := T$. For $k = 1, \dots, n-1$, the response at $t \in (t_{k-1}, t_k)$ can be expressed in terms of the response at t_{k-1} as

$$\mathbf{z}(t) = \phi^k(t - t_{k-1}) \mathbf{z}(t_{k-1}) + \psi^k(t, t_{k-1}) \mathbf{s}_{k-1}, \quad t \in (t_{k-1}, t_k), \quad (31)$$

where $\phi^k(t - t_{k-1})$ is the state transition matrix for $t \in (t_{k-1}, t_k)$ and is related to the state transition matrix $\phi^{k-1}(t - t_{k-2})$, $t \in (t_{k-2}, t_{k-1})$, by

$$\phi^k(t - t_{k-1}) = \begin{bmatrix} P \phi_{11}^{k-1}(t - t_{k-2}) P' & P \phi_{12}^{k-1}(t - t_{k-2}) P' \\ P \phi_{21}^{k-1}(t - t_{k-2}) P' & P \phi_{22}^{k-1}(t - t_{k-2}) P' \end{bmatrix},$$

where P is a permutation matrix and the sizes of four partitions are all $N \times N$. The matrix $\psi^k(t, t_k)$ is

$$\psi^k(t, t_k) := \int_{t_k}^t r \begin{bmatrix} P \phi_{12}^{k-1}(t - t_{k-2}) P' \\ P \phi_{22}^{k-1}(t - t_{k-2}) P' \end{bmatrix} d\tau.$$

It follows that at $t_n = T$, we have

$$\begin{aligned} \mathbf{z}(T) = & \left[\prod_{l=1}^n \phi^l(t_l - t_{l-1}) \right] \mathbf{z}(0) \\ & + \sum_{l=1}^n \left[\prod_{m=l}^n \phi^m(t_m - t_{m-1}) \right] \psi^l(t_l, t_{l-1}) \mathbf{s}_{l-1}. \end{aligned} \quad (32)$$

Multiplying both sides on the left by $[I \ 0]$, where I has size N , we

have

$$\begin{aligned} [I \ 0] \mathbf{z}(T) = & [I \ 0] \left[\prod_{l=1}^n \phi^l(t_l - t_{l-1}) \right] \mathbf{z}(0) \\ & + [I \ 0] \sum_{l=1}^n \left[\prod_{m=l+1}^n \phi^m(t_m - t_{m-1}) \right] \psi^l(t_l, t_{l-1}) \mathbf{s}_{l-1}. \end{aligned} \quad (33)$$

We employ the boundary condition $\mathbf{x}(T) = \mathbf{0}$ and obtain $0 = \Omega_{11} \mathbf{x}(0) + \Omega_{12} \mathbf{p}(0) + \sum_{l=1}^n \Gamma_1^l \mathbf{s}_{l-1}$, where Ω_{ij} is the ij^{th} block of $\Omega := \prod_{l=1}^n \phi^l(t_l - t_{l-1})$ and Γ_i^l is the i^{th} block of

$\left[\prod_{m=l+1}^n \phi^m(t_m - t_{m-1}) \right] \psi^l(t_l, t_{l-1})$. We now show that Ω_{12} is nonsingular for small enough a so that $\mathbf{p}(0)$ is uniquely determined. In fact, as $a \rightarrow 0$, the state transition matrix in each interval $l = 1, \dots, n$ asymptotically approaches

$$\phi^l(t) \rightarrow \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} I & -tI \\ 0 & I \end{bmatrix} \begin{bmatrix} P' & 0 \\ 0 & P' \end{bmatrix} = \begin{bmatrix} I & -tI \\ 0 & I \end{bmatrix},$$

for the permutation matrix P that represents the ordering change passing from the interval $l-1$ to l . It follows that as $a \rightarrow 0$

$$\begin{aligned} \Omega \rightarrow & \begin{bmatrix} I & -(t_n - t_{n-1})I \\ 0 & I \end{bmatrix} \begin{bmatrix} I & -(t_{n-1} - t_{n-2})I \\ 0 & I \end{bmatrix} \\ \dots & \begin{bmatrix} I & -(t_1 - t_0)I \\ 0 & I \end{bmatrix}, \end{aligned} \quad (34)$$

so that $\Omega_{12} \rightarrow -\sum_{l=1}^n (t_l - t_{l-1})I = -T I$, which implies that Ω_{12} is nonsingular for sufficiently small $a > 0$. Therefore,

$$\mathbf{p}(0) = -\Omega_{12}^{-1} \left(\Omega_{11} \mathbf{x}(0) + \sum_{l=1}^n \Gamma_1^l \mathbf{s}_{l-1} \right). \quad (35)$$

Let us now consider the response in the vicinity of t_1 , the first change of ordering instant, at which (31) gives $\mathbf{x}(t) = [\phi_{11}^1(t) \mathbf{x}(0) + \phi_{12}^1(t) \mathbf{p}(0)] + \psi^1(t, 0) \mathbf{s}_0$. Suppose $x_i(t_1) = x_j(t_1)$, i.e., the i^{th} and the j^{th} agents change positions at t_1 . Substituting $\mathbf{p}(0)$ obtained in (35) and multiplying both sides of this equation by the row vector \mathbf{w}_{ij}^T , all entries of which are 0 except 1 in its i^{th} entry and -1 in its j^{th} entry, we obtain

$$\begin{aligned} x_i(t) - x_j(t) = & \mathbf{w}_{ij}^T (\phi_{11}^1(t) + \phi_{12}^1(t) \Omega_{12}^{-1} \Omega_{11}) \mathbf{x}(0) \\ & + \mathbf{w}_{ij}^T \phi_{12}^1(t) \Omega_{12}^{-1} \sum_{l=1}^n \Gamma_1^l \mathbf{s}_{l-1} + \mathbf{w}_{ij}^T \psi^1(t, 0) \mathbf{s}_0. \end{aligned} \quad (36)$$

For ϵ sufficiently small and $t \in (t_1, t_1 + \epsilon)$, the left hand side can be made as small as desired without any permutation in Ω and Γ_1^l since no change of ordering occurs in this time interval. By continuity of strategies with respect to $\mathbf{x}(0)$, $x_i(t) - x_j(t)$ and the first term on the right hand side vary continuously and can assume an infinity of values, whereas the last term can only take a finite number of values. It follows that (36) can not hold. This contradiction implies that the solution with no ordering change is unique for all $0 < a < a_0$ for some $a_0 > 0$. \square

Remark A.1. The infinite summation expression for S , crucially used in establishing the existence of Nash equilibrium, also indicates that $\text{sign}(S)$ is determined by the sign of $-\beta_{mN+i+j+2k+1}$ for large enough attraction parameter $a > 0$. This is because $\beta_{mN+i+j+2k+1}$ is divisible by the largest power of a among all β that occur in that expression. It follows by (12) that, $\text{sign}(-\beta_{mN+i+j+2k+1}) = (-1)^{mN+i+j}$ for all $t \in [0, T]$. Since m is odd, we again have $\text{sign}(S) = (-1)^N (-1)^{i+j}$. This establishes that, for large $a > 0$ and for all i, j as in (25), $K_{i,N-j}(t) > 0$, $t \in [0, T]$. Similarly, one can conclude the positivity of L . Therefore, Nash equilibrium also exist for sufficiently large values of the attraction parameter as well. \triangle

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