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Brief paper

Non-linear pricing by convex duality[☆]

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ABSTRACT

We consider the pricing problem of a risk-neutral monopolist who produces (at a cost) and offers an infinitely divisible good to a single potential buyer that can be of a finite number of (single dimensional) types. The buyer has a non-linear utility function that is differentiable, strictly concave and strictly increasing. Using a simple reformulation and shortest path problem duality as in Vohra (2011) we transform the initial non-convex pricing problem of the monopolist into an equivalent optimization problem yielding a closed-form pricing formula under a regularity assumption on the probability distribution of buyer types. We examine the solution of the problem when the regularity condition is relaxed in different ways, or when the production function is non-linear and convex. For arbitrary type distributions, we offer a complete solution procedure.

1. The setting

Non-linear pricing is a basic problem of economic mechanism design under asymmetric information. Consider a monopolist who is producing an infinitely divisible good, e.g., sugar, and wishes to sell the good to a potential buyer with unknown valuation for his/her product. The seller's production function is assumed to be linear with a slope equal to $c > 0$. The seller is risk neutral, and therefore, seeks to maximize the expected revenue from the sale. The buyer can be one of a finite number of types t from the index set $\mathcal{T} = \{1, \ldots, m\}$ with $m > 2$. The parameter t for the type of the buyer is assumed to represent the valuation of a potential buyer for the good. The buyer derives a utility equal to $t \cdot u(\mathcal{A}_t) - p_t$ from acquisition of a quantity $\mathcal{A}_t$ (allocation to buyer of type t) of the good, where u is a differentiable, strictly concave, strictly increasing function ($u''(x) < 0, u'(x) > 0$ for every x) with $u(0) = 0$ and a strictly decreasing $(u')^{-1}$, and p_t is the price paid for acquisition of the quantity $\mathcal{A}_t \geq 0$. The crux of the problem is that a potential buyer's type (or valuation of the good) t is private, i.e., unknown to the seller. However, the seller's beliefs about t are given by a probability mass function f on the discrete set $\mathcal{T}$. The problem of the seller is to devise a mechanism that will maximize expected revenue while it elicits a truthful declaration of type by the seller and ensures his/her participation.

The non-linear pricing problem briefly described above occurs in many industries, e.g., wireless communication services, other telecom and technology products, legal plans, fitness clubs, automobile clubs and healthcare plans; see Bagh and Bhargava (2013) for further details. It is part of the general theory of basic static adverse selection problems in economics. The study of the problem was started in Mirrlees (1971) and developed into a mature subject with numerous contributions (a notable one is the paper by Myerson, 1981) that would be impractical to cite in this short note. An authoritative and detailed reference on nonlinear pricing is Wilson (1997).[1] As it is closer to our treatment, we adopt as our desktop reference on non-linear pricing of a single good the book by Bolton and Dewatripont (2004) which contains a list of the main references on the subject up to 2005. One can find in Chapter 2 of Bolton and Dewatripont (2004) discussions of the non-linear pricing problem first with two types, and then with a finite number of different types and then, a continuum of types using methods that are different from that of the present note. In fact, the Ref. Bolton and Dewatripont (2004) does not offer an explicit solution for the case of discrete types while (nor does Wilson, 1997 for that matter) for a continuum of types a closed-form pricing formula (credited to Baron & Myerson, 1982 and Maskin & Riley, 1984) is given under a condition on the utility function and the monotonicity assumption on the probability distribution of types. When the monotonicity assumption is violated, a so-called *ironing* procedure gives the optimal contract with a *bunching/pooling* property (the optimal

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1 Our utility model differs from that of Wilson where the dependence of buyer utility on type is not made explicit.

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allocation remains constant over some interval) using the methods of calculus of variations. Other noteworthy references include Champsaur and Rochet (1989), Figueroa and Skreta (2007), Guesnerie and Seade (1982), Matthews and Moore (1987) and Moore (1984). In Figueroa and Skreta (2007), the auction of multiple goods is considered for the case of a continuum of types where preferences are represented by a non-linear utility function. It is shown that when incentive compatibility constraints bind, a *randomized* mechanism may be optimal as opposed to the deterministic mechanisms considered in the present paper. An interesting application of non-linear pricing where monotonicity assumption may be violated is reported in Crawford and Shum (2007) where the authors explore the degree of quality degradation in cable television markets and the impact of regulation on those choices using empirical data from cable networks. Taking the utility function of consumers to be linear in quality, they utilize two, three or four types of consumers in the model of monopoly choice of Mussa and Rosen (1978) which addresses the problem of a monopolist selling two goods whose qualities varies over a finite interval to consumers that are differentiated by a parameter that can take distinct values where the first type represents consumers who prefer not to purchase any of the cable network products. The empirical consumer type distributions derived from market share data may indeed violate monotonicity (cf. Table 5, p. 201 of Crawford & Shum, 2007).

Against this background, the purpose of the present note is to derive a simple explicit price formula for the case of discrete types using the machinery of convex optimization and duality as advocated by Vohra (2011, 2012) although the mechanism design problem is initially formulated as a non-convex optimization problem. The contribution of the manuscript is to bring to bear the novel analysis technique based on convex duality of Vohra on instances where regularity of the types distribution is violated. The main results are the discrete-types analogs of the continuous types results of the literature. Our first result is obtained under a regularity (monotonicity) assumption of the probability mass f as in Bolton and Dewatripont (2004). The result extends in a straightforward manner to the case of convex production cost function of the monopolist. Then, we relax gradually the regularity assumption and prove further results for the optimal mechanism which mimics the *ironing/bunching(pooling)* solution of the continuous types case. To the best of our knowledge, the present note is one of the few papers that addresses the discrete (single dimensional) multiple types (more than two types) non-linear pricing problem from a mathematical programming perspective along with e.g., Bandi and Bertsimas (2012) and Vohra (2012). This short note may also serve as an entry point for newcomers to the subject as it treats a simpler setting and uses rather basic tools of optimization, compared to e.g., Vohra (2012) which involves optimization over poly-matroids. An important feature of our paper is that any instance of the non-linear pricing problem described in the present paper can be solved explicitly without resorting to a non-linear optimization software. We illustrate our results with examples.

By virtue of the Revelation Principle (Vohra, 2011), the seller is interested in designing a direct mechanism that consists of the two discrete functions p (for price) and $\mathcal{A}$ (for allocation), both functions of type t. In other words, the seller implementing a direct mechanism declares a price p_t and a quantity allocation $\mathcal{A}_t$ for each type t. Against this background, the problem of pricing the indivisible good is formulated as the following optimization problem. We define the decision variables p_t for all $t \in \mathcal{T}$ for the price quoted by the seller to a buyer of type t, in addition to the non-negative allocation variables $\mathcal{A}_t$. The seller wishes to maximize the expected profits from the sale:

$$\sum_{t=1}^{m} f_t(p_t - c\mathcal{A}_t) \tag{1}$$

under the restrictions of Incentive Compatibility (IC) and Individual Rationality (IR) that are, respectively:

$$t(u(\mathcal{A}_t) - u(\mathcal{A}_s)) \geq p_t - p_s, \quad \forall t, s \in \mathcal{T} \tag{2}$$

$$t \cdot u(\mathcal{A}_t) - p_t \geq 0, \quad \forall t \in \mathcal{T}. \tag{3}$$

The constraint (IC) ensures that the utility of the seller that declares his/her type truthfully is at least as large as the utility derived from reporting a different type. The constraint (IR) is to ensure that the minimum (reservation) utility of any buyer of any type is at least zero, which leads to ensuring participation of the buyers into the mechanism.

Therefore, the seller seeks a pair $p_t, \mathcal{A}_t \geq 0$ for each type $t \in \mathcal{T}$ that maximizes (1) under the restrictions (2)–(3). Note that the problem (1)–(2)–(3) is in general non-convex due to the presence of the difference $u(\mathcal{A}_t) - u(\mathcal{A}_s)$ which is not necessarily a concave function. In the next section we prove a simple result departing from hidden convex (more precisely, concave since we are maximizing) structure in the problem.

2. The optimal mechanism under monotonicity

Let $\nu_t = t - \frac{1-F_t}{f_t}$ for all $t \in \mathcal{T}$ where we denote by F the cumulative distribution function associated with the mass function f (ν_t is commonly referred to as the *virtual valuation*). The economic meaning attached to the virtual valuation of the bidder is the marginal revenue obtained by allocating the item to this bidder. As is common to most references, see e.g., Bolton and Dewatripont (2004), Tirole (1990) and the references therein, we assume ν_t to be monotone increasing in t. We call f *regular* if the m-vector ν associated with f is monotone increasing.[2] A way to enforce the above monotonicity is the so-called Monotone Hazard Rate (MHR) condition. A distribution F with density f is said to satisfy the MHR condition if the *hazard rate* $\frac{f(t)}{1-F(t)}$ is non-increasing with t. Most well-known continuous distributions satisfy the MHR condition, e.g., the uniform, the normal, the Pareto, the logistic, the exponential; see Section 3.5 of Tirole (1990). Therefore, one may safely assume that it will hold for their discretized counterparts.

The first result of the note is the following.

Proposition 1. *For regular f there exists an optimal direct mechanism with the allocation*

$$\mathcal{A}_t^* = (u')^{-1}\left(\frac{c}{\nu_t}\right), \tag{4}$$

and the optimal prices

$$p_t^* = t \cdot u(\mathcal{A}_t^*) - \sum_{j=1}^{t-1} u(\mathcal{A}_j^*) \tag{5}$$

for all $t = t^*, \ldots, T$ where t^* is the smallest value of t that satisfies: $\nu(t^*) > 0$, and $\mathcal{A}_t^* = p_t^* = 0$ for all $t = 1, \ldots, t^* - 1$.

Proof. We can always define a dummy type $t = 0$ with $\mathcal{A}_0 = p_0 = 0$ and incorporate constraint (3) into (2); see Vohra (2011). Then, we re-write the problem (1)–(2)–(3) as the following convex optimization problem

$$\max_{p_t, y_t, \mathcal{A}_t \geq 0} \sum_{t=1}^{m} f_t(p_t - c\mathcal{A}_t)$$

subject to

$$t(y_t - y_s) \geq p_t - p_s, \quad \forall t, s \in \mathcal{T}$$

$$u(\mathcal{A}_t) \geq y_t, \quad \forall t \in \mathcal{T}.$$

[2] In fact, it is sufficient that the positive components of ν are monotone increasing.

By the assumptions we made, the previous convex optimization problem is equivalent to (1)–(2)–(3) with the property that at an optimum solution one has $y_t = u(\mathcal{A}_t)$ for all $t \in \mathcal{T}$. Now using the development in Vohra (2011) we have that for fixed $y_t,\ t \in \mathcal{T}$, the inequalities for incentive compatibility

$$t(y_t - y_s) \geq p_t - p_s, \quad \forall t, s \in \mathcal{T}$$

hold for some $p_t,\ t \in \mathcal{T}$ if and only if y_t is monotone non-decreasing in t by the theory of duality applied to the shortest path problem; see Chapters 3 and 4 of Vohra (2011) for an in-depth analysis of shortest path duality.[3] Furthermore, at optimality one has

$$p_t = ty_t - \sum_{j=1}^{t-1} y_j.$$

Thus, replacing p_t by $ty_t - \sum_{j=1}^{t-1} y_j$ in the objective function, after some simple algebraic manipulation, and recalling that one has $y_t = u(\mathcal{A}_t)$ at optimality, we have transformed the problem (1)–(2)–(3) into the equivalent problem of maximization of

$$\sum_{t=1}^{m} f_t\,(\nu_t u(\mathcal{A}_t) - c\mathcal{A}_t)$$

over the non-negative monotone polyhedron

$$\mathcal{A}_m \geq \mathcal{A}_{m-1} \geq \cdots \mathcal{A}_1 \geq 0,$$

(recall that u is strictly increasing). To see the transformation of the objective function, note that we have

$$\sum_{t=1}^{m} f_t p_t = \sum_{t=1}^{m} f_t \left(ty_t - \sum_{j=1}^{t-1} y_j \right) = \sum_{t=1}^{m} f_t t y_t - \sum_{t=1}^{m} f_t \sum_{j=1}^{t-1} y_j.$$

Changing the order of the summation in the second term above we get

$$\sum_{t=1}^{m} f_t \sum_{j=1}^{t-1} y_j = \sum_{t=1}^{m} y_t(1 - F_t).$$

Hence, dividing and multiplying each term by f_t we get the desired expression.

Now, since the above problem is separable in t, and each term is either a (strictly) concave function of $\mathcal{A}_t$ for when $\nu_t \geq 0$ or (strictly) convex when $\nu_t \leq 0$, we ignore momentarily the monotone non-negativity restriction. Then the result follows by calculus and invoking the monotonicity assumption on $\nu(t)$ as follows. The first-order condition

$$\nu_t u'(\mathcal{A}_t) - c = 0,$$

which is necessary and sufficient when the associated term is concave, leads to the candidate optimal point for the unconstrained problem (without the monotone non-negativity restriction)

$$\mathcal{A}_t = \begin{cases} (u')^{-1}\left(\dfrac{c}{\nu_t}\right) & \text{if } \nu_t > 0 \\ 0 & \text{if } \nu_t \leq 0. \end{cases}$$

The above is a non-negative monotone solution as a result of monotonicity of ν and assumptions imposed on u. ■

From the result above, one can immediately deduce the following identity

$$\frac{p^*_{t+1} - p^*_t}{\mathcal{A}^*_{t+1} - \mathcal{A}^*_t} = (t+1)\left[\frac{u(\mathcal{A}^*_{t+1}) - u(\mathcal{A}^*_t)}{\mathcal{A}^*_{t+1} - \mathcal{A}^*_t}\right]$$

for all $t = t^*, t^*+1, m-1$. Combining the previous with the following gradient inequality

$$u(\mathcal{A}^*_{t+1}) \leq u(\mathcal{A}^*_t) + \frac{c}{\nu_t}(\mathcal{A}^*_{t+1} - \mathcal{A}^*_t), \quad \forall\, t = t^*, \ldots, m-1$$

due to concavity of u, and Proposition 1 one obtains

$$\frac{p^*_{t+1} - p^*_t}{\mathcal{A}^*_{t+1} - \mathcal{A}^*_t} \leq (t+1)\frac{c}{\nu_t}, \quad \forall\, t = t^*, \ldots, m-1.$$

Therefore, under the additional condition that

$$\frac{t+2}{t+1} \leq \frac{\nu_{t+1}}{\nu_t}, \quad \forall\, t = t^*, \ldots, m-2 \tag{7}$$

one can conclude that the average price per unit decreases with the quantity $\mathcal{A}$. In this case, one can implement the optimal nonlinear payment schedule by offering a menu of two-part tariffs as discussed in Section 3.5 of Tirole (1990). In a recent paper Bagh and Bhargava (2013) study the efficiency of two and three-part tariffs and show that a relatively small menu of three-part tariffs may be more profitable than a menu of two-part tariffs of any size.

In Section 2.3.1 of Bolton and Dewatripont (2004) the solution of the above problem is investigated under the so-called *Spence-Mirrlees single crossing condition* on the utility function u, which helps simplify the problem by reducing the number of constraints considerably. The simplification consists in replacing the large number of "global" IC constraints by "local" (consecutive types) IC constraints and making sure that a solution thus obtained would satisfy all IC constraints; see Champsaur and Rochet (1989), Guesnerie and Seade (1982), Matthews and Moore (1987), Moore (1984) and Wilson (1997) for related work. On the other hand, the reduced problem is still non-convex. Hence, the KKT conditions are only necessary provided that a suitable constraint qualification holds. In fact, the KKT conditions are not solved in Bolton and Dewatripont (2004). It is shown that the optimal mechanism results in *efficient consumption*, i.e., $t \cdot u'(\mathcal{A}_t) = c$ only for the highest type $t = m$ while $t \cdot u'(\mathcal{A}_t) > c$ for all other types t, i.e., they all underconsume. Our result also possesses this property as $t \cdot u'(\mathcal{A}_t) = t\frac{c}{\nu_t}$ and $\nu_m = m$ while $\nu_t < t$ for all other t.

Example 1. Consider an example with $m = 10$ types, $u(x) = \sqrt{x}$, $c = 3.5$, and

$$f = (0.1, 0.15, 0.15, 0.15, 0.10, 0.10, 0.08, 0.07, 0.04, 0.06)^T.$$

Here, ν is monotone with the critical value $t^* = 4$. The resulting optimal direct mechanism is given by $\mathcal{A}^* = (0, 0, 0, 0.020, 0.046, 0.250, 0.485, 0.881, 1.148, 2.041)^T$ and $p^* = (0, 0, 0, 0.571, 0.929, 2.643, 4.018, 5.957, 7.151, 10.722)^T$. Here, the sufficient condition (7) is satisfied. Hence we have declining price per unit as quantity increases.

2.1. Convex production costs

When the cost of production function c is a strictly convex function with $c(0) = 0$ and $c'(.) > 0$, we can prove the following immediately using the analysis of Proposition 1.

[3] Rewrite the inequalities for incentive compatibility as

$$p_t - p_s \leq w_{ts}, \quad \forall t, s \in \mathcal{T} \tag{6}$$

where $w_{ts} = t(y_t - y_s)$. For fixed y_t's, one may associate a network with the above constraints. Each type is a node; for each pair (t, s) we make an arc from node t to node s with length w_{st}. Then the system of inequalities (6) is feasible if and only if the network contains no negative length cycles by Corollary 3.4.2 of Vohra (2011). Then add the inequalities corresponding to the cycle $t \to t+1 \to t$:

$$p_t - p_{t+1} \leq t(y_t - y_{t+1}),$$
$$p_{t+1} - p_t \leq (t+1)(y_{t+1} - y_t)$$

to see that y_t is monotone if the system (6) is feasible. That this monotonicity condition is equivalent to the absence of negative cycles is proved in Vohra (2011).

Proposition 2. *For regular f and strictly convex production function $c(.)$ (satisfying $c(0) = 0$ and $c'(.) > 0$), there exists an optimal direct mechanism with the allocation satisfying the relation*

$$\frac{c'(\mathcal{A}_t^*)}{u'(\mathcal{A}_t^*)} = \nu_t \tag{8}$$

and the optimal prices

$$p_t^* = t \cdot u(\mathcal{A}_t^*) - \sum_{j=1}^{t-1} u(\mathcal{A}_j^*)$$

for all $t = t^*, \ldots, T$ where t^* is the smallest value of t that satisfies: $\nu(t^*) > 0$, and $\mathcal{A}_t^* = p_t^* = 0$ for all $t = 1, \ldots, t^* - 1$.

Proof. The proof is similar to the proof of Proposition 1 with the necessary changes. Therefore, we omit it. ■

Example 2. Using the data of Example 1 with $c(x) = 2x^2$ we obtain the optimal allocation from the formula $\mathcal{A}_t = \left(\frac{\nu_t}{8}\right)^{2/3}$, for $t = 4, \ldots, 10$. We get $\mathcal{A}^* = (0, 0, 0, 0.25, 0.327, 0.577, 0.719, 0.876, 0.958, 1.16)^T$ and $p^* = (0, 0, 0, 2.002, 2.36, 3.484, 4.105, 4.809, 5.195, 6.174)^T$. Note that with a convex quadratic production cost, the optimal mechanism tends to give allocations and prices that are closer to one another for consecutive types in comparison to the linear cost case.

3. The optimal mechanism under relaxed monotonicity

The above result in Proposition 1 and its proof break down when f is not regular. However, one may still find explicitly the optimal direct mechanism in sufficiently regular cases using the machinery of the previous section. While all such cases are too numerous to treat here, we prove some typical results in this direction which convey the main ideas but keep the exposition relatively simple. We define f to be $2+$ *regular* if there exists $t^* \in \mathcal{T}$ and non-negative integer ℓ (could be equal to zero) such that

1. $\nu_t < 0$ for $t = 1, \ldots, t^* - 1$
2. $\nu_{t^*} > 0$
3. $\nu_{t^*+\ell} \geq \nu_{t^*+\ell+1}$
4. $\nu_{t^*} < \cdots < \nu_{t^*+\ell-1} < \nu_{t^*+\ell+2} < \cdots < \nu_m$.
5. $\nu_{t^*+\ell-1}(f_{t^*+\ell} + f_{t^*+\ell+1}) \leq \nu_{t^*+\ell} f_{t^*+\ell} + \nu_{t^*+\ell+1} f_{t^*+\ell+1}$.

Notice that when f is $2+$ regular, the monotonicity of the positive elements of the sequence ν is only violated by a single component, $\nu_{t^*+\ell}$, or, equivalently, by a subsequence of length equal to 2.

Proposition 3. *For f $2+$ regular, and linear production cost c, there exists an optimal mechanism with the optimal allocations $\mathcal{A}_i^* = 0$ for all $i = 1, \ldots, t^* - 1$,*

$$\mathcal{A}_i^* = (u')^{-1}\left(\frac{c}{\nu_i}\right) \quad \forall i = \{t^*, \ldots, t^* + \ell - 1\} \cup \{t^* + \ell + 2, \ldots, m\},$$

and

$$\mathcal{A}_i = (u')^{-1}\left(\frac{c(f_{t^*+\ell} + f_{t^*+\ell+1})}{\nu_{t^*} f_{t^*+\ell} + \nu_{t^*+\ell+1} f_{t^*+1}}\right) \quad i = t^* + \ell, t^* + \ell + 1,$$

and the optimal prices

$$p_t^* = t \cdot u(\mathcal{A}_t^*) - \sum_{j=1}^{t-1} u(\mathcal{A}_j^*)$$

for all $t = t^*, \ldots, m$, and $p_t^* = 0$ for all others.

Proof. Using the proof of Proposition 1 above, we pose the problem again as

$$\max_{\mathcal{A}_t} \sum_{t=1}^{m} f_t \left(\nu_t u(\mathcal{A}_t) - c\mathcal{A}_t\right)$$

subject to

$$\mathcal{A}_m \geq \mathcal{A}_{m-1} \geq \cdots \mathcal{A}_1 \geq 0,$$

since the proof does not depend on the regularity of f up to that point. However, we can no longer ignore the monotone non-negativity constraints and treat the problem as unconstrained as that method worked due to monotonicity of ν. The problem is unfortunately non-convex due to a sum of convex and concave terms in the objective function. Hence, a direct treatment using optimality conditions is futile. Instead, we shall proceed by examining a "split relaxation" of the problem. I.e., we shall separate the problem into two sub-problems, one for the negative ν_t and the other for the positive ν_t. To avoid unnecessarily complicated notation we do the proof for the case $\ell = 0$. It is a simple exercise to repeat the proof for arbitrary positive integer ℓ. Hence we consider the following sub-problems referred to as P1 and P2, respectively:

$$\max_{\mathcal{A}_t,\ t=1,\ldots,t^*-1} \sum_{t=1}^{t^*-1} f_t \left(\nu_t u(\mathcal{A}_t) - c\mathcal{A}_t\right)$$

subject to

$$\mathcal{A}_{t^*-1} \geq \cdots \mathcal{A}_1 \geq 0,$$

and

$$\max_{\mathcal{A}_t,\ t=t^*,\ldots,m} \sum_{t=t^*}^{m} f_t \left(\nu_t u(\mathcal{A}_t) - c\mathcal{A}_t\right)$$

subject to

$$\mathcal{A}_m \geq \mathcal{A}_{m-1} \geq \cdots \mathcal{A}_{t^*} \geq 0.$$

Now, problem P1, which is a strictly convex maximization problem, is trivially solved by taking $\mathcal{A}_t = 0$ for all $t = 1, \ldots, t^*-1$. For problem P2, we write the Karush–Kuhn–Tucker conditions (they are necessary and sufficient in P2 since the objective function is strictly concave and we have linear constraints) after attaching non-negative multipliers Υ_i to each constraint $\mathcal{A}_i - \mathcal{A}_{i-1} \geq 0$, for $i = t^* + 1, \ldots, m$ and $\mathcal{A}_{t^*} \geq 0$. Namely, we have the first-order conditions (FOC)

$$f_i(\nu_i u'(\mathcal{A}_i) - c) - \Upsilon_{i+1} + \Upsilon_i = 0, \quad \forall i = t^*, \ldots, m-1,$$
$$f_i(\nu_m u'(\mathcal{A}_m) - c) + \Upsilon_m = 0,$$

and complementarity conditions (CC)

$$\Upsilon_i(\mathcal{A}_i - \mathcal{A}_{i-1}) = 0 \quad \forall i = t^* + 1, \ldots, m$$
$$\Upsilon_{t^*}\mathcal{A}_{t^*} = 0.$$

First, let us examine the FOC for $i = t^* + 2, \ldots, m$. Here, taking $\mathcal{A}_i = (u')^{-1}\left(\frac{c}{\nu_i}\right)$ and $y_i = 0$ satisfies the corresponding equations as well as the associated CC by virtue of the $2+$ regularity of f and assumptions imposed on u. For $i = t^*, t^*+1$ we take $\mathcal{A}_{t^*} = \mathcal{A}_{t^*+1}$, $y_{t^*} = 0$ and solve the 2×2 system of linear equations for $u'(\mathcal{A}_{t^*})$ which gives after elimination of y_{t^*+1}:

$$u'(\mathcal{A}_{t^*}) = \frac{c(f_{t^*} + f_{t^*+1})}{\nu_{t^*} f_{t^*} + \nu_{t^*+1} f_{t^*+1}}$$

and we take

$$y_{t^*+1} = f_{t^*}\nu_{t^*} u(\mathcal{A}_{t^*}) - f_{t^*} c. \tag{9}$$

The resulting allocations are monotone by virtue of condition 5 on f. We also have

$$y_{t^*+1} = -f_{t^*+1}\nu_{t^*+1} u(\mathcal{A}_{t^*}) = f_{t^*+1} c.$$

This equation with (9) imply that $y_{t+1} > 0$. Thus the above construction gives an optimal solution to P2. Now, we can concatenate the solutions to P1 and P2 and obtain a monotone non-negative solution for the original problem. Since we obtained a feasible solution to the original problem by solving a relaxation, we have solved the original problem optimally. ■

Notice that the *bunching(pooling)* behavior observed in the continuous case occurs also here for types $t^*+\ell$ and $t^*+\ell+1$.

Example 3. Consider another example with $m=10$ types, $u(x)=\sqrt{x}$, $c=3$, and

$$f = (0.0832, 0.0099, 0.1801, 0.1885, 0.0979, 0.0976, 0.0674, 0.1796, 0.0737, 0.0222)^T.$$

Here, f is $2+$ regular with the critical value $t^*=4$, and $\ell=0$, i.e., we have

$$\nu = (-10.01, -84.84, -1.03, 1.14, 0.50, 2.49, 2.91, 7.47, 8.69, 10.00)^T.$$

The resulting optimal direct mechanism is given by $\mathcal{A}^*=(0, 0, 0.024, 0.024, 0.172, 0.236, 1.549, 2.103, 2.779)^T$ and $p^*=(0, 0, 0, 0.616, 0.616, 2.179, 2.675, 8.747, 10.596, 12.765)^T$.

Such examples can be repeated as we move the location of the monotonicity violating entry $t^*+\ell+1$ in ν.

Example 4. Consider again an example with ten types. With

$$f = (0.25, 0.05, 0.25, 0.15, 0.05, 0.06, 0.07, 0.07, 0.02, 0.03)^T,$$

$c=2$ and $u(x)=\sqrt{x}$ we have an f that is $2+$ regular with $t^*=3$, $\ell=1$ and

$$\nu = (-2, -12, 1.2, 2, 0, 2.83, 5.28, 7.28, 7.5, 10.00)^T.$$

From Proposition 3, the optimal allocation is obtained as

$$\mathcal{A}^* = (0, 0, 0.09, 0.141, 0.141, 0.502, 1.746, 3.318, 3.516, 6.250)^T.$$

A more relaxed regularity definition is the following: we define f to be $n+$ *regular* if there exists $t^*, \ell \in \mathcal{T}$ and integer $n < m - t^*$ such that

1. $\nu_t < 0$ for $t = 1, \ldots, t^*-1$
2. $\nu_{t^*} > 0$
3. $\nu_{t^*+\ell} \ge \nu_{t^*+\ell+1} \ge \cdots \ge \nu_{t^*+\ell+n-1} \ge 0$
4. $\nu_{t^*} < \cdots < \nu_{t^*+\ell-1} < \nu_{t^*+\ell+n} < \cdots < \nu_m$.
5. $\nu_{t^*+\ell-1}(\sum_{j=0}^{n-1} f_{t^*+\ell+j}) \le \sum_{j=0}^{n-1} \nu_{t^*+\ell+j} f_{t^*+\ell+j}$.

Here, there is a subsequence $\nu_{t^*+\ell} \ge \nu_{t^*+\ell+1} \ge \cdots \ge \nu_{t^*+\ell+n-1} \ge 0$ breaking the monotone increasing property of the positive components of ν. The following result can be proved similarly to Proposition 3.

Proposition 4. *For f $n+$ regular, and linear production cost c, there exists an optimal mechanism with the optimal allocations $\mathcal{A}_i^*=0$ for all $i=1,\ldots,t^*-1$,*

$$\mathcal{A}_i = (u')^{-1}\left(\frac{c}{\nu_i}\right) \quad \forall i = \{t^*, \ldots, t^*+\ell-1\} \cup \{t^*+\ell+n, \ldots, m\},$$

and (pooling)

$$\mathcal{A}_i^* = (u')^{-1}\left(\frac{c\left(\sum_{j=0}^{n-1} f_{t^*+\ell+j}\right)}{\sum_{j=0}^{n-1} \nu_{t^*+\ell+j} f_{t^*+\ell+j}}\right)$$

for $i=t^*+\ell, \ldots, t^*+\ell+n-1$, and the optimal prices

$$p_t^* = t \cdot u(\mathcal{A}_t^*) - \sum_{j=1}^{t-1} u(\mathcal{A}_j^*)$$

for all $t=t^*, \ldots, m$, and $p_t^*=0$ for all others.

Example 5. Consider a modification of the previous example with ten types. With

$$f = (0.25, 0.05, 0.25, 0.15, 0.07, 0.04, 0.07, 0.07, 0.02, 0.03)^T,$$

$c=2$ and $u(x)=\sqrt{x}$ we have an f that is $n+$ regular with $t^*=3$, $\ell=1$ and $n=3$ and

$$\nu = (-2, -12, 1.2, 2, 1.714, 1.25, 5.28, 7.28, 7.5, 10.00)^T.$$

From Proposition 4, the optimal allocation is obtained as

$$\mathcal{A}^* = (0, 0, 0.09, 0.204, 0.204, 0.204, 1.746, 3.318, 3.516, 6.25)^T.$$

It is certainly possible that there exist several decreasing subsequences violating monotonicity of the positive part of the ν vector. In such cases, pooling in the optimal mechanism will occur in all such sequences. A general result is quite messy to state in such cases. So, we refrain from it. One can invoke the procedure outlined in Proposition 4 repeatedly, first isolating the decreasing sub-sequences, dealing with the increasing subsequence first, and implement a pooling procedure for each decreasing sub-sequence.

4. Non-separated (arbitrary) distributions

Up to this point we have implicitly assumed that the negative and positive values of ν are separated in our relaxed definitions of regularity. I.e., for all $t \ge t^*$ we have $\nu_t \ge 0$. However, this assumption may fail to hold as the following example shows.

Example 6. Consider an example with ten types. With

$$f = (0.25, 0.05, 0.25, 0.05, 0.10, 0.10, 0.08, 0.07, 0.02, 0.03)^T,$$

$c=2$ and $u(x)=\sqrt{x}$ we have an f that is not $n+$ regular with a non-separated

$$\nu = (-2, -12, 1.2, -4, 2, 4, 5.5, 7.28, 7.5, 10.00)^T.$$

The optimal allocation is obtained as

$$\mathcal{A}^* = (0, 0, 0.007, 0.007, 0.250, 0.998, 1.883, 3.314, 3.507, 6.245)^T.$$

We shall now see how this optimal allocation is obtained from a closed-form formula as in the previous results.

The difficulty with non-separated ν is that the proof of Proposition 3 (or Proposition 4) fails to go through although the result remains true under a slight additional assumption.

We define f to be 2-*regular* if there exists $t^* \in \mathcal{T}$ and non-negative integer ℓ (could be equal to zero) such that

1. $\nu_t < 0$ for $t=1,\ldots,t^*-1$
2. $\nu_{t^*} > 0$
3. $\nu_{t^*+\ell} \ge \nu_{t^*+\ell+1}$ with $\nu_{t^*+\ell+1} < 0$ and $\nu_{t^*+\ell} \ge 0$
4. $\nu_{t^*} < \cdots < \nu_{t^*+\ell-1} < \nu_{t^*+\ell+2} < \cdots < \nu_m$
5. $f_{t^*+\ell}\nu_{t^*+\ell} + f_{t^*+\ell+1}\nu_{t^*+\ell+1} > 0$.

Notice that the probability mass f of Example 6 is 2-*regular*.

Proposition 5. *For f 2-regular, and linear production cost c, there exists an optimal mechanism with the optimal allocations $\mathcal{A}_i^*=0$ for all $i=1,\ldots,t^*-1$,*

$$\mathcal{A}_i^* = (u')^{-1}\left(\frac{c}{\nu_i}\right) \quad \forall i = \{t^*, \ldots, t^*+\ell-1\} \cup \{t^*+\ell+2, \ldots, m\},$$

and

$$\mathcal{A}_i = (u')^{-1}\left(\frac{c(f_{t^*+\ell}+f_{t^*+\ell+1})}{\nu_{t^*}f_{t^*+\ell}+\nu_{t^*+\ell+1}f_{t^*+1}}\right) \quad i = t^*+\ell, t^*+\ell+1,$$

and the optimal prices

$$p_t^* = t\cdot u(\mathcal{A}_t^*) - \sum_{j=1}^{t-1} u(\mathcal{A}_j^*)$$

for all $t = t^*, \ldots, m$, *and* $p_t^* = 0$ *for all others.*

Proof. We may consider a split relaxation of the problem as in the proof of Proposition 3. We may define and treat P1 as before. However, there is an added difficulty with P2 in that the objective function of P2 is no longer necessarily concave due to the presence of a negative coefficient $\nu_{t^*+\ell+1}$ after a positive coefficient $\nu_{t^*+\ell}$. Therefore, we shall proceed in a different manner. Ignoring momentarily the monotonicity constraints on $\mathcal{A}_t$, $t = t^*, \ldots, m$ we observe that the function $f_{t^*+\ell+1}(\nu_{t^*+\ell+1}u(\mathcal{A}_{t^*+\ell+1}) - c\mathcal{A}_{t^*+\ell+1})$ is strictly convex, and therefore would be maximized at $\mathcal{A}_{t^*+\ell+1} = 0$ (recall that allocation variables are restricted to be non-negative). This implies that the monotonicity constraint $\mathcal{A}_{t^*+\ell+1} \geq \mathcal{A}_{t^*+\ell}$ will bind at optimality. Now, isolating the portion of the problem corresponding to $\mathcal{A}_{t^*+\ell+1}$, $\mathcal{A}_{t^*+\ell}$ and ignoring monotonicity restrictions we have the lower negative ν_t part (the left sub-problem) where we have the identically zero allocation as in the proof of Proposition 1, and the upper positive ν_t part (the right sub-problem) where $\mathcal{A}_t = (u')^{-1}\left(\frac{c}{\nu_t}\right)$ for $t = t^*+\ell+1, \ldots, m$. Now, for the problem in two variables corresponding to $\mathcal{A}_{t^*+\ell+1}$, $\mathcal{A}_{t^*+\ell}$, we use the binding property and reduce the problem to maximization of only

$$\begin{aligned} g(\mathcal{A}_{t^*+\ell}) \equiv\ & f_{t^*+\ell}(\nu_{t^*+\ell}u(\mathcal{A}_{t^*+\ell}) - c\mathcal{A}_{t^*+\ell}) \\ & + f_{t^*+\ell+1}(\nu_{t^*+\ell+1}u(\mathcal{A}_{t^*+\ell}) - c\mathcal{A}_{t^*+\ell}) \end{aligned}$$

which is strictly concave (by property 5. of 2-regular f) and maximized at the point

$$u'(\mathcal{A}_{t^*+\ell}) = \frac{c(f_{t^*\ell}+f_{t^*+\ell+1})}{\nu_{t^*+\ell}f_{t^*+\ell}+\nu_{t^*+\ell+1}f_{t^*+\ell+1}}$$

which is positive by our assumption on f and satisfies monotonicity when concatenated with the right sub-problem. Therefore, we have constructed a monotone solution to the original problem splitting the problem objective function into three parts and solving each piece ignoring the monotonicity restriction in the two sub-problems left and right, and pooling in the middle part. ■

Our argumentation in the proof of Proposition 5 implies that all types with a negative ν_t tend to receive an allocation as small as possible, i.e., either zero, or a value dictated by pooling/bunching where the allocation is decided by some lower type to the left.

There are situations where Proposition 5 can be extended in a straightforward fashion. One such extension concerns the case when the decreasing subsequence breaking the monotonicity in Proposition 5 can have more than 2 elements, say q elements. In this case, condition 5 should be modified as

Condition 5b. $\sum_{j=0}^{q-1} f_{t^*+\ell+j}\nu_{t^*+\ell+j} > 0$.

On should also modify the optimal allocation formula as

$$\mathcal{A}_i^* = (u')^{-1}\left(\frac{c\left(\sum_{j=0}^{q-1} f_{t^*+\ell+j}\right)}{\sum_{j=0}^{q-1} f_{t^*+\ell+j}\nu_{t^*+\ell+j}}\right),$$

$$i = t^*+\ell, \ldots, t^*+\ell+q-1. \tag{10}$$

It may also occur that the first decreasing sub-sequence occurs right after the initial negative portion of the vector ν (cf. Example 7 below). In that case, if condition 5 above fails, then one can immediately make null assignment for these types.

In general with arbitrary f one can expect several decreasing sub-sequences in ν with possibly negative entries. In such cases, one should treat separately each such sub-sequence for possible pooling. However, one should be careful in that a sub-sequence (with or without negative elements) may fail to satisfy condition 5 above in the definition of 2-regularity (2 + regularity or n + regularity, respectively). In that case one may have to pool together two consecutive such sub-sequences (or pool with a previous monotone increasing sub-sequence) based on repeated application of Propositions 3–5, and illustrated with examples. I.e., each decreasing sub-sequence which fails to receive an allocation on its own will be a candidate for pooling with a previous sub-sequence, be it a decreasing or an increasing sub-sequence.

Example 7. Consider an example with $m = 20$ types, $u(x) = \sqrt{x}$, $c = 3$, and

$$\begin{aligned} f = (&0.020, 0.080, 0.052, 0.017, 0.038, 0.067, 0.021, 0.082, \\ &0.027, 0.102, 0.030, 0.085, 0.021, 0.032, 0.010, 0.064, \\ &0.076, 0.060, 0.047, 0.071)^T. \end{aligned}$$

Here, we have

$$\begin{aligned} \nu = (&-49.282, -9.287, -13.189, -45.195, -16.053, -4.811, \\ &-26.269, 0.367, -13.182, 5.121, -4.617, 7.505, -4.215, \\ &3.686, -16.475, 12.009, 14.638, 16.048, 17.493, 20.00)^T. \end{aligned}$$

Here $t^* = 8$. The resulting optimal direct mechanism is given by optimal allocations $\mathcal{A}_t^* = 0$, for $t = 1, \ldots, 9$ and

$$\begin{aligned} (&0.235, 0.235, 0.316, 0.316, 0.316, \\ &0.316, 4.004, 5.949, 7.161, 8.508, 11.122)^T \end{aligned}$$

for $t = 11, \ldots, 20$. The particularity of this example is that while we notice four monotonicity breaking sub-sequences with negative ν_t's, namely the pairs indexed $s_0 = (8, 9)$, $s_1 = (10, 11)$, $s_2 = (12, 13)$ and $s_3 = (14, 15)$, pooling occurs for $(8, 9)$, and $(10, 11)$ separately whereas $(12, 13, 14, 15)$ are pooled as a single sub-sequence. The reason for zero allocation to $(8, 9)$ is that $f_8\nu_8 + f_9\nu_9 < 0$, i.e., condition 5 fails and we make a zero allocation for types $(8, 9)$ since it is the first such sub-sequence after the left sub-problem. We move next to s_1, condition 5 holds and we make the allocation assignment equal to 0.235 for both 10 and 11. We move to s_2 where condition 5 holds and we check condition 5 for s_3 where it fails. Then we make a single subsequence $(12, 13, 14, 15)$ where now condition 5b holds. Then we make the allocation equal to 0.316 using the equivalent of formula (10).

Example 8. Consider another example with $m = 20$ types, $u(x) = \sqrt{x}$, $c = 3$, and

$$\begin{aligned} f = (&0.026, 0.030, 0.065, 0.028, 0.086, 0.103, 0.076, 0.036, \\ &0.061, 0.011, 0.095, 0.092, 0.086, 0.027, 0.062, 0.002, \\ &0.045, 0.033, 0.017, 0.019)^T. \end{aligned}$$

Here, we have

$$\begin{aligned} \nu = (&-36.023, -29.030, -10.602, -26.608, -3.860, -0.438, \\ &-0.662, -7.261, 1.008, -32.212, 6.968, 8.846, 10.610, \\ &7.505, 13.150, -30.958, 15.467, 16.914, 17.899, 20.00)^T. \end{aligned}$$

Here $t^* = 9$. According to our results, the optimal direct mechanism is given by optimal allocations $\mathcal{A}_t^* = 0$, for $t = 1, \ldots, 8$ (the lower sub-problem) and

$$\begin{aligned} (&0, 0, 1.350, 2.174, 2.697, 2.697, 3.677, \\ &3.677, 6.642, 7.951, 8.889, 11.119)^T \end{aligned}$$

for $t = 9, \ldots, 20$, obtained as follows. The first decreasing sub-sequence $s_0 = (9, 10)$ has a negative element ν_{10} and condition

5 fails. So, null allocation. Next is (11, 12) which is an increasing sequence so is a candidate for receiving the allocations $\mathcal{A}_t = (u')^{-1}\left(\frac{c}{v_t}\right)$. However, there is a decreasing sub-sequence following it. That sub-sequence (13, 14) has positive elements only and receives a pooling allocation according to Proposition 3. Therefore, the allocation of (11, 12) can be finalized. Finally, $s_1 = (15, 16)$ is another sub-sequence with a negative element, passes condition 5, and receives allocation according to Proposition 5. The solution of the right sub-problem gives optimal allocations for the remaining types.

5. Concluding remarks

In this brief paper, we examined the pricing problem of a risk-neutral monopolist producing an infinitely divisible good at a cost and offering the good to a single potential buyer with a non-linear utility function and a private valuation for the good expressed as a positive integer number. Under the usual assumption of regularity of the type distribution, we gave a closed-form solution for the pricing problem. Then by gradually relaxing the regularity assumption, we showed how a complete solution can be obtained analytically by means of a simple procedure in the absence of any regularity in the type distribution. While a full-fledged application as in Crawford and Shum (2007) is beyond the scope of this paper, it will be interesting to test the results of the paper on a suitable economic application in the future.

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