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"Backward Differential Flow" May Not Converge to a Global Minimizer of Polynomials

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Abstract We provide a simple counter-example to prove and illustrate that the *backward differential flow* approach, proposed by Zhu, Zhao and Liu for finding a global minimizer of coercive even-degree polynomials, can converge to a local minimizer rather than a global minimizer. We provide additional counter-examples to stress that convergence to a local minimum via the *backward differential flow* method is not a rare occurence.

Keywords Polynomial optimization · Global optimization · Trajectory methods

1 Introduction

In their recent article, Zhu et al. [1] provide a method for finding a solution to global minimization of multivariate polynomials of even degree. In this note, we exemplify, and thus prove, that their method does not necessarily yield a global minimizer.

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2 Preliminaries

For simplicity, we focus on the special case of monic quartic univariate polynomials $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = x^4 + a_3\, x^3 + a_2\, x^2 + a_1\, x + a_0,$$

where a_0, a_1, a_2 and a_3 are real numbers. What Zhu et al. propose in [1] can be translated into this setting as one of solving the following initial value problem. With $x : \mathbb{R} \to \mathbb{R}$ as the dependent variable and t as the independent variable,

$$\dot{x}(t) = -\frac{x(t)}{f''(x(t)) + t}, \quad 0 \le t \le t_0\,, \quad x(t_0) = x_0, \tag{1}$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}t$, such that

$$f'(x_0) + t_0\, x_0 = 0 \tag{2}$$

and

$$f''(x) + t_0 > 0\,, \quad \text{for all } x \in \mathbb{R}. \tag{3}$$

Theorem 4.1 in [1], which is the main result for the so-called backward differential flow method, can then be rephrased as follows.

> "If $x(t)$ solves (1) and $f''(x(t)) + t > 0$ for all $t \in]0, t_0]$, then $x(0)$ is a global minimizer of $f(x)$."

We note that, because f is a monic quartic polynomial, and so is coercive, a large enough positive t_0 can always be found so that Condition (3) is satisfied. Zhu et al. provide an estimate of t_0 by restricting the domain of f to a closed ball (in the univariate case, $-a \le x \le a$), in which a global minimizer is contained. In the quartic univariate case, one can even find the smallest t_0 satisfying (3) easily (as illustrated in the counter-example below). Therefore, an estimate for t_0 as proposed in [1] is not needed. Then, by (3), there exists a unique solution x_0 to (2). Finally, the initial value problem (1) is solved from $x(t_0) = x_0$ backward in t, with the resulting solution referred to as *backward differential flow* by Zhu et al., to obtain $x(0)$. The point $x(0)$ is claimed in [1] to be a global minimizer. We will prove, via a counter-example, that $x(0)$ is not necessarily a global minimizer.

Before providing a counter-example to Theorem 4.1 of [1], we will make some remarks in order to view the problem from a slightly different point.

Remark 2.1 Define

$$\varphi(x, t) := f(x) + \frac{t}{2}\, x^2.$$

Then, $\varphi(x, t)$ can be viewed as a quadratic regularization of $f(x)$, with regularization parameter $t > 0$. Note that $\varphi_x(x, t) = f'(x) + t\, x$ and $\varphi_{xx}(x, t) = f''(x) + t$, where the subscripts x and xx stand for $\partial/\partial x$ and $\partial^2/\partial x^2$, respectively. Therefore, (2)–(3) above can be rewritten as

$$\varphi_x(x_0, t_0) = 0,$$

and

$$\varphi_{xx}(x, t_0) > 0\,, \quad \text{for all } x \in \mathbb{R}.$$

We now recall a well-known fact regarding maximal extension of solutions of ODEs.

Remark 2.2 Assume that $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable everywhere. Let $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that

$$f'(x_0) + t_0\, x_0 = 0 \quad \text{and} \quad f''(x_0) + t_0 > 0. \tag{4}$$

The following hold.

(a) There exists $r > 0$ such that there is a unique solution $x(\cdot)$ of (1) in $]t_0 - r, t_0 + r[$.
(b) There exists a maximal interval to the left of t_0, say $]m_0, t_0]$, such that there exists a solution of (1) in $]m_0, t_0]$.
(c) Either $m_0 = -\infty$, or $m_0 \in \mathbb{R}$ and $f''(x(m_0)) + m_0 = 0$.

Part (a) follows from the classical Picard-Lindelöf existence and uniqueness theorem (see [2]), because the right-hand side of the ODE in (1) is Lipschitz continuous in x and continuous in t in a neighborhood of t_0. Part (b) is the classical result on maximal extension of solutions of ODEs. The option $m_0 = -\infty$ of part (c) corresponds to the case in which the right-hand side remains Lipschitz continuous in x for all $t < t_0$. The remaining option happens when the denominator

$$q(t) := f''(x(t)) + t \tag{5}$$

vanishes at $t = m_0$.

In the following simple lemma, we state a straightforward reformulation of the initial value problem in (1).

Lemma 2.1 *Assume that $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable everywhere. Let $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ be chosen as in (4). Let $x(\cdot)$ be the maximally extended solution of (1), and $]m_0, t_0]$ the corresponding maximal interval. Then, we have that*

$$\begin{aligned} \varphi_x(x(t), t) &= f'(x(t)) + t\,x(t) = 0\,, \quad \varphi_{xx}(x(t), t) \\ &= f''(x(t)) + t > 0\,, \ \forall t \in [m_0, t_0]. \end{aligned}$$

Proof Solvability of (1) over $]m_0, t_0]$ implies that the right-hand side of the ODE is continuous on $]m_0, t_0]$. In other words, the denominator of the right-hand side of the ODE is not zero and so it does not change sign on $]m_0, t_0]$. Since $\varphi_{xx}(x(t_0), t_0) > 0$ and the solution exists in $]m_0, t_0]$, we must have

$$\varphi_{xx}(x(t), t) = f''(x(t)) + t > 0, \tag{6}$$

for all $t \in]m_0, t_0]$. Then, for all $t \in]m_0, t_0]$, we can rewrite the ODE in (1) as

$$\dot{x}(t)\,(f''(x(t)) + t) + x(t) = 0,$$

which can be rewritten in terms of φ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\varphi_x(x(t), t) = 0. \tag{7}$$

By (4), we also have

$$\varphi_x(x(t_0), t_0) = f'(x(t_0)) + x(t_0)\,t_0 = 0. \tag{8}$$

Equalities (7) and (8) imply that

$$\varphi_x(x(t), t) = f'(x(t)) + x(t)\,t = 0, \tag{9}$$

for all $t \in]m_0, t_0]$. Equality (9) holds at $t = m_0$ by continuity of f' and $x(\cdot)$. □

Next lemma shows that if we start with a negative initial value at t_0, then the solution of the initial value problem (1) remains negative over its maximal domain of definition.

Lemma 2.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable everywhere. Let $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ be chosen as in* (4)*. Consider the initial value problem* (1)*. Let $x(\cdot)$ be the maximally extended solution of* (1)*, and $]m_0, t_0]$ the corresponding (finite or infinite) maximal interval of definition of $x(\cdot)$. If $x_0 < 0$, then $x(t) < 0$ for all $t \in]m_0, t_0]$. If $m_0 \in \mathbb{R}$, then $x(m_0) < 0$.*

Proof Suppose that for some $t \in]m_0, t_0]$, we have $x(t) \geq 0$. Consider the set $S := \{t \in]m_0, t_0] \;:\; x(t) \geq 0\}$. This set is non-empty and bounded above by t_0. Let

$$t_1 := \sup\, S.$$

Note that $t_1 \in S$ and $t_1 < t_0$. We claim that $x(t_1) \geq 0$. Indeed, if $x(t_1) < 0$, then for some $r > 0$, we have

$$x(t) < 0\,, \;\text{ for all } t \in]t_1 - r, t_1 + r[\,. \tag{10}$$

By definition of t_1 as a supremum of S, there exists $t \in S$ such that $t \in]t_1 - r, t_1]$, which means that $x(t) \geq 0$, contradicting (10). Hence, $x(t_1) \geq 0$ and by definition of t_1, we have

$$x(t) < 0\,, \;\text{ for all } t \in]t_1, t_0]. \tag{11}$$

Using (11) and Lemma 2.1 in the ODE in (1), we conclude that

$$\dot{x}(t) > 0, \;\text{ for all } t \in]t_1, t_0]. \tag{12}$$

By the mean value theorem, there exists $s \in]t_1, t_0]$ such that

$$x(t_1) = x(t_0) + \dot{x}(s)\,(t_1 - t_0) < x_0,$$

where we used (12). The above expression implies that

$$x(t_1) < x_0 < 0\,, \text{ for all } t \in]t_1, t_0], \tag{13}$$

which is a contradiction. Hence, $x(t) < 0$, for all $t \in]m_0, t_0]$. To prove the last assertion of the lemma, assume on the contrary that $x(m_0) \geq 0$. Since $x(t) < 0$, for all $t \in]m_0, t_0]$, use again Lemma 2.1 in the ODE in (1), to obtain (12) with m_0 in the place of t_1. Using the mean value theorem again, we get

$$0 \leq x(m_0) = x(t_0) + \dot{x}(s)\,(m_0 - t_0) < x_0 < 0,$$

for some $s \in]m_0, t_0]$. The above expression entails a contradiction, which implies that $x(m_0) < 0$. □

Lemma 2.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable everywhere. Let $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ be chosen as in* (4)*. Consider the initial value problem* (1) *with $x_0 < 0$. Assume that the system, with the unknown $(x, t) \in \mathbb{R}^2$, given by*

$$f'(x) + t\,x = 0\,, \quad f''(x) + t = 0, \tag{14}$$

has a unique real solution $(\overline{x}, \overline{t})$ with $\overline{x} > 0$ and $\overline{t} > 0$. Then, the solution of (1) *can be infinitely extended to the left; in other words, $m_0 = -\infty$, and so $x(t) < 0$, for all $t \leq t_0$.*

Proof Indeed, assume that, on the contrary, $m_0 \in \mathbb{R}$. By Remark 2.2(c), this can only happen if the right-hand side of (1) becomes discontinuous at $t = m_0$. This implies that

$$f''(x(m_0)) + m_0 = 0. \tag{15}$$

By Lemma 2.1, we have

$$f'(x(t)) + t\,x(t) = 0,$$

for all $t \in [m_0, t_0]$. This fact combined with (4) implies that

$$f'(x(m_0)) + m_0\,x(m_0) = 0. \tag{16}$$

By Lemma 2.2, we have that $x(m_0) < 0$. Equations (15) and (16) imply that there is a pair $(x, t) = (x(m_0), m_0)$ which solves system (14), with $x < 0$. Since system (14) has a unique solution $(\overline{x}, \overline{t})$ with $\bar{x} > 0$, we arrive at a contradiction. Hence, we must have $m_0 = -\infty$. It follows by Lemma 2.2 that $x(t) < 0$, for all $t \leq t_0$. □

3 Counter-Example

Proposition 3.1 *Consider*

$$f(x) = x^4 - 8\,x^3 - 18\,x^2 + 56\,x.$$

Suppose that $x(t)$ solves (1). Then, one has that $f''(x(t)) + t > 0$ for all $t \in]0, t_0]$, but that $x(0)$ is not a global minimizer of $f(x)$.

Proof We will first show that this quartic polynomial function $f(x)$ verifies the hypotheses of Lemma 2.3. Then, we will conclude that there exists t_0 such that the denominator $q(t)$, defined in (5), is positive for all $t \in]-\infty, t_0]$. Hence, $f(x)$ satisfies the assumptions of Theorem 4.1 in [1].

Note that $f(x)$ has local minima at $x = -2$ and $x = 7$ and a local maximum at $x = 1$. We also note that $f(-2) = -104$, $f(7) = -833$ and $f(1) = 31$. Therefore, $x = 7$ is the global minimizer of $f(x)$.

Let us now compute t_0 and x_0. We have

$$\varphi_x(x_0, t_0) = 4\,x_0^3 - 24\,x_0^2 + (t_0 - 36)\,x_0 + 56 = 0 \tag{17}$$

and

$$\varphi_{xx}(x, t_0) = 12\,x^2 - 48\,x + t_0 - 36 > 0, \quad \text{for all } x \in \mathbb{R}.$$

The minimum of the quadratic function $\varphi_{xx}(x, t_0)$ above occurs at $x = 2$. Therefore, one gets $t_0 > 84$, to guarantee that (3) holds. Let $t_0 = 100$. Then we obtain, as the only real solution of (17),

$$x_0 = 2 + \left((\sqrt{18417}/9) - 15\right)^{1/3} - 4 \Bigg/ \left(3\,\left((\sqrt{18417}/9) - 15\right)^{1/3}\right) < 0,$$

by means of some computer algebra package, e.g., MATLAB. Approximately, $x_0 \approx -0.681220$. The initial value problem (1) becomes

$$\dot{x}(t) = -\frac{x(t)}{12\,x^2(t) - 48\,x(t) + t - 36}, \quad 0 \le t \le 100\,, \quad x(100) = x_0. \tag{18}$$

Next, let us show that f verifies the hypotheses of Lemma 2.3. From $\varphi_{xx}(\overline{x}, \overline{t}) = 0$, which is the second equation of (14), we get

$$\overline{t} = -12\,\overline{x}^2 + 48\,\overline{x} + 36.$$

Substitution of this expression for $\overline{t}$ into $\varphi_x(\overline{x}, \overline{t}) = 0$, which is the first equation of (14), yields

$$8\,\overline{x}^3 - 24\,\overline{x}^2 - 56 = 0.$$

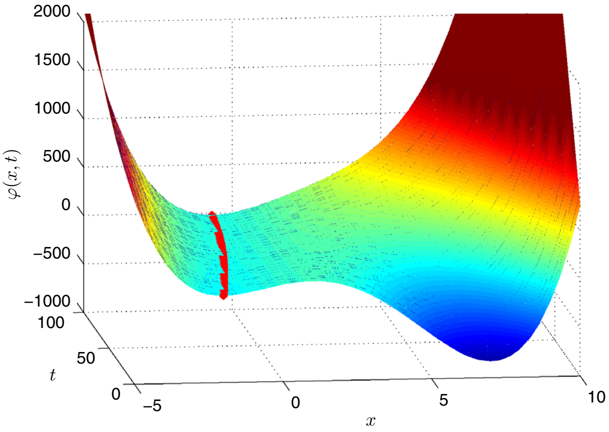

Fig. 1 Backward differential flow for the counter-example, $f(x) = x^4 - 8\,x^3 - 18\,x^2 + 56\,x$

The only real solution of the latter equation is found as

$$\overline{x} = 1 + \left(\frac{9+\sqrt{77}}{2}\right)^{1/3} - \left(\frac{2}{9+\sqrt{77}}\right)^{1/3} > 0,$$

by MATLAB. Approximately, $\overline{x} \approx 3.554149$ and, in turn, $\overline{t} \approx 55.01544$.

Therefore, the hypotheses of Lemma 2.3 are satisfied. Note also that the denominator in (5), $q(t_0) = q(100) > 0$. Since, by Lemma 2.3, the solution of (18) is well-defined on $]-\infty, 100]$, we have that the denominator $q(t) > 0$ for all $t \in [0, 100]$, satisfying the hypotheses of Theorem 4.1 in [1].

Since $x_0 < 0$ and $q(100) > 0$, we have $\dot{x}(100) > 0$, and so, by Lemma 2.3, the unique $x(t)$ which solves (18) is negative for all $t \in [0, 100]$. However, $x(0) < 0$ is not the global minimizer of $f(x)$. $\square$

In Fig. 1, an illustration of the backward differential flow method, as applied to the polynomial in Proposition 3.1, is given. The solution curve of (18) is depicted on a surface plot of the function $\varphi(x, t)$. The curve is generated by solving (18) numerically using the MATLAB function `ode113`, with `RelTol = 1e-06`. It can be clearly observed in the figure that $x(0)$ approximates the local minimizer $x = -2$, rather than the global minimizer $x = 7$.

3.1 Other Counter-Examples

The fact that $x(0)$ is not a global minimizer is not a rare occurence; indeed, it is frequently encountered. In what follows, we provide a few more examples for which $x(0)$ of the backward differential flow is not a global minimizer.

$f(x) = x^4 - (16/3)\,x^3 - 2\,x^2 + 16\,x + 2$ (global minimizer: $x = 4$; local minimizer: $x = -1$)

$f(x) = x^4 + (20/3)\,x^3 - 2\,x^2 - 20\,x + 3$ (global minimizer: $x = -5$; local minimizer: $x = 1$)

4 Conclusions

We have demonstrated, via a counter-example, that the backward differential flow approach presented by Zhu et al. [1] does not necessarily yield a global minimizer of a coercive even-degree polynomial. The counter-example will hopefully help/prompt to determine where the proof of Theorem 4.1 in [1] breaks down. This might in turn help find a correct statement for the theorem.

References

1. Zhu, J., Zhao, S., Liu, G.: Solution to global minimization of polynomials by backward differential flow. J. Optim. Theory Appl. **161**, 828–836 (2014)
2. Arnold, V.I.: Ordinary Differential Equations. The MIT Press, Cambridge (1978)