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Tornehave morphisms III: The reduced Tornehave morphism and the Burnside unit functor

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ABSTRACT

We shall show that a morphism anticipated by Tornehave induces (and helps to explain) Bouc's isomorphism relating a quotient of the Burnside unit functor (measuring a difference between real and rational representations of finite 2-groups) and a quotient of the kernel of linearization (measuring a difference between rhetorical and rational 2-biset functors).

1. Introduction

This paper is the third in a trilogy concerning a map which might have seemed to be merely a remarkable curiosity when it was first introduced by Tornehave [14] in 1984. For a finite group G and a set of primes π, he characterized his map $\overline{\mathrm{torn}}_G^\pi$ in two different ways: by means of a symplectic construction involving real representations and Galois automorphisms; and by means of a π-adic formula involving the sizes of orbits in permutation sets. We call $\overline{\mathrm{torn}}_G^\pi$ the reduced Tornehave map. In his application, he

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made use of the observation that $\overline{\text{torn}}^{\pi}_{G}$ commutes with restriction. In fact, $\overline{\text{torn}}^{\pi}_{G}$ also commutes with induction, inflation and isogation. That observation immediately suggests a reassessment of the significance of his map, since it allows us to reinterpret the ideas in the context of group functors. Recall that inflaky functors – inflation Mackey functors – are equipped with restriction, induction, inflation and isogation maps.

The construction involving real representations is discussed in the first paper [3] of the trilogy. There, it is explained how, as an inflaky morphism, the reduced Tornehave morphism $\overline{\text{torn}}^{\pi}$ can be factorized through the orientation functor $O_{\mathbb{R}}$, which is a quotient of the real representation functor $A_{\mathbb{R}}$. In the second paper [4], the π-adic formula is lifted to the dual of the Burnside functor, and it is shown that, if $p \in \pi$ then, for finite p-groups, the lifted Tornehave morphism torn^{π} is, up to $\mathbb{Q}$-multiples, the unique inflaky morphism with a certain specified domain and codomain.

Still, to reassure ourselves that the morphisms $\overline{\text{torn}}^{\pi}$ and torn^{π} are indeed of fundamental interest, we need to see a substantial application beyond the original one in [14]. That is the purpose of the present paper. Recently, Bouc [9] discovered a link between two lines of research which had previously seemed to be quite separate from each other: the study of rational units and real units of the Burnside ring; the study of rational and rhetorical biset functors. The link is expressed by his result, Theorem 2.4 below, which asserts the existence and uniqueness of an isomorphism of p-biset functors

$$\overline{\text{bouc}}^{p} \ : \ {}_pK/\!\!/^{\mathbb{Q}}_{p}K \to {}_pB^{\times}/\!\!/^{\mathbb{Q}}_{p}B^{\times}.$$

The notation, here, will be introduced in Section 2. For the moment, let us just mention that the presubscript p indicates that we are dealing with p-biset functors rather than biset functors for arbitrary finite groups.

Our main result, Theorem 2.5 below, describes how the isomorphism $\overline{\text{bouc}}^{p}$ is induced by the reduced Tornehave morphism $\overline{\text{torn}}^{p} = \overline{\text{torn}}^{\{p\}}$. Actually, the case of odd p is degenerate. Indeed, when p is odd, $\overline{\text{bouc}}^{p}$ is the zero morphism between two zero functors. For the vital case $p = 2$, the piece-by-piece construction of the isomorphism $\overline{\text{bouc}} = \overline{\text{bouc}}^{2}$, in the proof of [9, 6.5], is quite complicated. We shall show how that construction can be avoided by using the characterization of $\overline{\text{bouc}}$ in terms of the morphism $\overline{\text{torn}} = \overline{\text{torn}}^{2}$.

Thus, the focus of our attention in the present paper is with the 2-biset functor ${}_2B^{\times}/\!\!/^{\mathbb{Q}}_{2}B^{\times} \cong {}_2K/\!\!/^{\mathbb{Q}}_{2}K$, which is evidently of importance in the study of 2-groups, especially the real representations of 2-groups. The inflaky morphisms $\overline{\text{torn}}^{\pi}$ and torn^{π} have not yet seen any application to the representation theory of arbitrary finite groups. However, one of the main motives behind this trilogy of papers has been the hope of subsequently extending some of the ideas to a scenario involving the linearization morphism $\text{lin}^{\mathbb{F}} : B_{\mathbb{F}} \to A_{\mathbb{F}}$, where $B_{\mathbb{F}}$ and $A_{\mathbb{F}}$ are, respectively, the monomial Burnside functor and the character functor associated with a field $\mathbb{F}$. In this monomial scenario, inflaky morphisms and surjectivity properties appear again: Brauer's Induction Theorem says that $\text{lin}^{\mathbb{F}}$ is an inflaky epimorphism; on the downside, it is not a morphism of biset functors

when $\mathbb{F}$ has prime characteristic but, on the upside, the surjectivity of the linearization map holds for all finite groups, not just for finite p-groups.

There will be three main variables: When working with group representations, our highest level of generality will be that where the coefficient ring is a given field $\mathbb{K}$ with characteristic zero. We take G to be given finite group and P to be a given finite 2-group.

2. Summary

In this section, we state the main relevant results from previous work and the main original results. We also outline the flow of deductions, deferring details to later sections.

We shall be concerned with the Burnside functor B, the rational representation functor $A_{\mathbb{Q}}$, the real representation functor $A_{\mathbb{R}}$ and the Burnside unit functor $B^{\times}$. Their coordinate modules at G are the Burnside ring $B(G)$, the ring of $\mathbb{Q}G$-representations $A_{\mathbb{Q}}(G)$, the ring of $\mathbb{R}G$-representations $A_{\mathbb{R}}(G)$ and the unit group $B^{\times}(G)$. All four of them are biset functors; in particular, they are equipped with deflation maps as well as restriction, induction, inflation and isogation maps. The linearization morphism $\mathrm{lin} : B \to A_{\mathbb{R}}$, the reduced tom Dieck morphism $\overline{\mathrm{die}} : A_{\mathbb{R}} \to B^{\times}$ and the reduced exponential morphism $\overline{\exp} = \overline{\mathrm{die}} \circ \mathrm{lin}$ are reviewed in [3] and [4]. These three morphisms of biset functors are also discussed in Bouc and Yalçın [11] and other papers cited therein. The reduced Tornehave morphism is an inflaky morphism $\overline{\mathrm{torn}}^{\pi} : K \to B^{\times}$, where $K = \mathrm{Ker}(\mathrm{lin})$. We shall be invoking some results whose proofs make use of the lifted morphisms die, exp, torn^{π}, which all have codomain B^*, the dual of the Burnside functor. Thankfully, those results have already been established in [3] and [4]; there will be no need to discuss the lifted morphisms in the present paper.

The group $B^{\times}(G)$ is an elementary abelian 2-group, and we can regard $B^{\times}$ as a biset functor over the field $\mathbb{F}_2$ with order 2. As biset subfunctors of $B^{\times}$, we define the **rational unit functor** and the **real unit functor** to be, respectively,

$$ {}^{\mathbb{Q}}B^{\times} = \overline{\mathrm{die}}(A_{\mathbb{Q}}), \qquad {}^{\mathbb{R}}B^{\times} = \overline{\mathrm{die}}(A_{\mathbb{R}}) $$

where $A_{\mathbb{Q}}$ is regarded as a biset subfunctor of $A_{\mathbb{R}}$ in the usual way. The elements of the groups ${}^{\mathbb{Q}}B^{\times}(G)$ and ${}^{\mathbb{R}}B^{\times}(G)$ are called, respectively, the **rational units** and the **real units** of $B^{\times}(G)$.

We shall be dealing with biset functors at two quite different levels of generality: sometimes for arbitrary groups, sometimes for p-groups. The distinction between the two is very important. So, to signal when we are confining our attention to biset functors for p-groups, we shall often call them p-biset functors, we shall often write them in the form ${}_pL$ and, for additional emphasis, we shall often insert a warning clause: *for p-groups*. The statement of Theorem 2.4, below, illustrates all three idioms employed in a single sentence.

Theorem 2.1 *(Ritter–Segal Theorem). If G is a p-group, then $A_{\mathbb{Q}}(G) = \mathrm{lin}_G(B(G))$. That is to say, for p-groups, ${}_pA_{\mathbb{Q}} = \mathrm{lin}({}_pB)$.*

More generally, Rasmussen [13] provided a necessary and sufficient criterion for the equality when G is nilpotent. For such G, the smallest counter-example to the equality is the case $G = Q_8 \times C_3$. Theorem 2.1 has the following immediate corollary.

Corollary 2.2. *If G is a p-group, then ${}^{\mathbb{Q}}B^{\times}(G) = \exp_G(B(G))$. That is to say, for p-groups, ${}^{\mathbb{Q}}_{p}B^{\times} = \overline{\exp}({}_pB)$.*

Tornehave's application of the map $\overline{\mathrm{torn}}_G$ was to prove the next theorem [14]. We shall present an updated rendition of his argument in Sections 3 and 4.

Theorem 2.3 *(Tornehave's Unit Theorem). If G is nilpotent, then $B^{\times}(G) = {}^{\mathbb{R}}B^{\times}(G)$. In particular, for p-groups, ${}_pB^{\times} = {}^{\mathbb{R}}_{p}B^{\times}$.*

In Section 1, we mentioned that the isomorphism $\overline{\mathrm{bouc}}$ provides a link between two different lines of study. One of those two lines of study concerns a comparison between the rational unit functor ${}^{\mathbb{Q}}B^{\times}$ and the real unit functor ${}^{\mathbb{R}}B^{\times}$. As a chain of biset functors,

$$\overline{\exp}(B) \leq {}^{\mathbb{Q}}B^{\times} \leq {}^{\mathbb{R}}B^{\times} \leq B^{\times}.$$

Corollary 2.2 and Theorem 2.3 tell us that, for p-groups,

$$\overline{\exp}({}_pB) = {}^{\mathbb{Q}}_{p}B^{\times} \leq {}^{\mathbb{R}}_{p}B^{\times} = {}_pB^{\times}.$$

Thus, for p-groups, all the units of the Burnside ring are real, and the question arises: for which p-groups are all of the units rational?

The question is trivial when p is odd. Indeed, Yoshida [16, 6.5] implies that, if p is odd and G is a p-group, then $B^{\times}(G) = \{\pm 1\}$, hence ${}^{\mathbb{Q}}B^{\times}(G) = {}^{\mathbb{R}}B^{\times}(G)$ and ${}^{\mathbb{Q}}_{p}B^{\times} = {}^{\mathbb{R}}_{p}B^{\times}$. So the question reduces to the case $p = 2$: for which 2-groups P do we have ${}^{\mathbb{Q}}B^{\times}(P) = {}^{\mathbb{R}}B^{\times}(P)$?

The question was tackled in a succession of papers. Matsuda [12] observed that ${}^{\mathbb{Q}}B^{\times}(G) \neq {}^{\mathbb{R}}B^{\times}(G)$ when $G = D_{2^n}$ with $n \geq 4$ (the dihedral 2-group with order 2^n). Yalçın [15, 7.6] showed that ${}^{\mathbb{Q}}B^{\times}(G) = {}^{\mathbb{R}}B^{\times}(G)$ when G is a 2-group with no subquotient isomorphic to D_{16}. But the converse is false: in [15, 7.7], Yalçın exhibited a group G with order 32 such that ${}^{\mathbb{Q}}B^{\times}(G) = {}^{\mathbb{R}}B^{\times}(G)$ yet G has a subgroup isomorphic to D_{16}. The role of the dihedral groups became clear when Bouc [8, 8.7] gave a necessary and sufficient criterion for the equality: supposing that G is a 2-group, then ${}^{\mathbb{Q}}B^{\times}(G) = {}^{\mathbb{R}}B^{\times}(G)$ if and only if $\mathbb{Q}G$ has no irreducible character with genotype D_{2^n} where $n \geq 4$. This is equivalent to the condition that every absolutely irreducible $\mathbb{R}G$-representation is realizable over $\mathbb{Q}$. In fact, a result of Bouc, recorded below as Theorem 5.3, tells us that the $\mathbb{F}_2$-dimension of the biset functor ${}_2B^{\times}/{}^{\mathbb{Q}}_{2}B^{\times} = {}^{\mathbb{R}}_{2}B^{\times}/{}^{\mathbb{Q}}_{2}B^{\times}$ is equal to the number of Galois conjugacy classes of absolutely irreducible $\mathbb{R}G$-representations that are not realizable over $\mathbb{Q}$.

The other line of study concerns a comparison between the rhetorical biset functors introduced in [2] and the rational p-biset functors introduced in Bouc [6]. It is easily shown that every rhetorical p-biset functor is rational. Bouc [9] showed that the converse holds if and only if $p \neq 2$. Let us say a few words about how Bouc established that result. We define ${}^{\mathbb{Q}}_{p}K$ to be the p-biset subfunctor of ${}_{p}K$ generated by the coordinate module $K(D)$ where D is dihedral with order 8 (when $p = 2$) or extra-special with order p^3 (when p is odd). Let L be a p-biset functor and consider the cross product operation

$$B(P \times Q) \times L(Q) \to L(P)$$

where P and Q are p-groups. By the definition of a rhetorical biset functor, L is rhetorical if and only if, for all P and Q, every element of $K(P \times Q)$ acts as the zero map $L(Q) \to L(P)$. On the other hand, [9, 5.3] asserts that L is rational if and only if every element of ${}^{\mathbb{Q}}_{p}K(P \times Q)$ acts as the zero map $L(Q) \to L(P)$. Now [9, 3.8] says that, if p is odd, then ${}^{\mathbb{Q}}_{p}K = {}_{p}K$, hence every rational p-biset functor is rhetorical. But [9, 6.3] implies that, for 2-groups, ${}^{\mathbb{Q}}_{2}K < {}_{2}K$, and it follows that the 2-biset functor ${}_{2}B/{}^{\mathbb{Q}}_{2}K$ is rational but not rhetorical.

Thus, the p-biset functor ${}_{p}K/{}^{\mathbb{Q}}_{p}K$ can be interpreted as a measure of the difference between the category of rhetorical p-biset functors and the category of rational p-biset functors. Meanwhile, in view of comments above, the p-biset functor ${}_{p}B^{\times}/{}^{\mathbb{Q}}_{p}B^{\times}$ can be interpreted as a measure of the difference between the absolutely irreducible real representations and the absolutely irreducible rational representations. The next result shows that, in some sense, the two differences coincide, indeed, the two measures of difference are isomorphic as p-biset functors.

Theorem 2.4 *(Bouc). For p-groups, there exists a unique isomorphism of p-biset functors*

$$\overline{\text{bouc}}^{\,p} \; : \; {}_{p}K/{}^{\mathbb{Q}}_{p}K \to {}_{p}B^{\times}/{}^{\mathbb{Q}}_{p}B^{\times}.$$

The domain and the codomain are non-zero if and only if $p = 2$.

The version of the theorem in Bouc [9, 6.5] does not mention the uniqueness property of $\overline{\text{bouc}}^{\,p}$ but, in Section 5, we shall explain how the uniqueness follows easily from Bouc's filtration [9, 6.4] of ${}_{2}B^{\times}$. Also in Section 5, we shall give a quick alternative to part of Bouc's proof of Theorem 2.4, and we shall give a proof of the next result. It was conjectured by Yalçın in June 2006, when he (and the author) received a draft copy of [9].

Theorem 2.5. *Let $\pi^{K}_{\mathbb{Q}} : {}_{p}K \to {}_{p}K/{}^{\mathbb{Q}}_{p}K$ and $\pi^{\times}_{\mathbb{Q}} : {}_{p}B^{\times} \to {}_{p}B^{\times}/{}^{\mathbb{Q}}_{p}B^{\times}$ be the canonical epimorphisms of p-biset functors. Then*

$$\overline{\text{bouc}}^{\,p} \circ \pi^{K}_{\mathbb{Q}} = \pi^{\times}_{\mathbb{Q}} \circ \overline{\text{torn}}^{\,p}.$$

In the rest of this paper, we shall be showing how an exploration of the properties of $\overline{\text{torn}}^p$ leads to proofs of Theorems 2.3, 2.4, 2.5. The proof we shall give for Theorem 2.3 will be a review of Tornehave's argument [14], except that we shall be making a note of the functorial features that arise. The proof we shall give for Theorem 2.4 will overlap with Bouc's argument [9], but we shall also be making use of some ideas and intermediate results of Tornehave. The first of those intermediate results will be the following theorem, which is implicit in Tornehave [14, Section 3]. Note that it holds for arbitrary finite groups. We shall review Tornehave's proof of it in Section 3.

Theorem 2.6 *(Tornehave). As a sum of biset subfunctors,* $B^\times = K^\times + {}^{\mathbb{Q}}B^\times$.

In Section 4, we shall complete the proof of Theorem 2.3 by establishing the inequalities $K^\times(P) \leq \overline{\text{torn}}(K(P)) \leq {}^{\mathbb{R}}B^\times(P)$ for all 2-groups P. After reviewing some structural features of the 2-biset functor ${}_2B^\times/{}^{\mathbb{Q}}_2B^\times$ in Section 5, we shall complete the proofs of Theorems 2.4 and 2.5 in Section 6.

3. A short filtration of the unit functor

We shall establish a short exact sequence of inflaky functors for arbitrary finite groups

$$\text{Lin}^\times \,:\, 0 \longrightarrow K^\times \longrightarrow B^\times \xrightarrow{\text{lin}^\times} A^\times_{\mathbb{Q}} \longrightarrow 0.$$

Then, making use of the reduced tom Dieck morphism $\overline{\text{die}}$, we shall prove Theorem 2.6. Our notation will be consistent with [3] and [4], but we shall present it in a self-contained way, beginning with a brief review of the Burnside ring $B(G)$, the $\mathbb{Q}$-representation ring $A_{\mathbb{Q}}(G)$ and the linearization map $\text{lin}_G : B(G) \to A_{\mathbb{Q}}(G)$. It will be convenient to express some of the definitions in the more general context of the $\mathbb{K}$-representation ring $A_{\mathbb{K}}(G)$, where $\mathbb{K}$ is a field of characteristic zero.

The elements of the Burnside ring $B(G)$ have the form $[X]-[Y]$ where $[X]$ and $[Y]$ are the isomorphism classes of (finite) G-sets X and Y. Letting U run over representatives of the conjugacy classes of subgroups of G, the elements $d^G_U = [G/U]$ comprise a $\mathbb{Z}$-basis for $B(G)$. Of course, the elements d^G_U also comprise a $\mathbb{Q}$-basis for the Burnside algebra $\mathbb{Q}B(G)$. The primitive idempotents comprise another $\mathbb{Q}$-basis for $\mathbb{Q}B(G)$. Letting I run over representatives of the conjugacy classes of subgroups of G, the primitive idempotents are the elements e^G_I specified by Gluck's Idempotent Formula

$$e^G_I = \frac{1}{|N_G(I)|} \sum_{U \leq I} |U|\, \mu(U, I)\, d^G_U.$$

Here, μ denotes the Möbius function for the poset of subgroups of G. The species of $\mathbb{Q}B(G)$ (we mean, the algebra maps $\mathbb{Q}B(G) \to \mathbb{Q}$) are the functions ϵ^G_I given by

$$\epsilon^G_I([X] - [Y]) = |X^I| - |Y^I|$$

where X^I denotes the set of I-fixed points in X. The species and the primitive idempotents are related by the condition that $\epsilon_I^G(e_{I'}^G) = 1$ when $I =_G I'$, otherwise $\epsilon_I^G(e_{I'}^G) = 0$. So any element $x \in \mathbb{Q}B(G)$ can be written in the form

$$x = \sum_{I \leq_G G} \epsilon_I^G(x)\, e_I^G$$

where the notation indicates that, again, I runs over representatives of the conjugacy classes of subgroups of G.

The $\mathbb{K}$-representation ring $A_{\mathbb{K}}(G)$, also called the $\mathbb{K}$-character ring, can be described in a similar way. Its elements can be expressed in the form $[L] - [M]$ where $[L]$ and $[M]$ are the isomorphism classes of (finite-dimensional) $\mathbb{K}G$-modules L and M. The elements can also be expressed in the form $\lambda - \mu$, where λ and μ are $\mathbb{K}G$-characters. Writing M_μ to denote a $\mathbb{K}G$-module with character μ, we make the identification $\mu = [M_\mu]$. Thus, as elements of $A_{\mathbb{K}}(G)$, we identify $\mathbb{K}G$-characters with isomorphism classes of $\mathbb{K}G$-modules.

Since character values are algebraic integers, we can embed $A_{\mathbb{K}}(G)$ in the complex representation ring $A_{\mathbb{C}}(G)$ by fixing an embedding from the ring of algebraic integers in $\mathbb{K}$ to the ring of algebraic integers in $\mathbb{C}$. In this way, $\mathbb{C}A_{\mathbb{K}}(G)$ becomes a $\mathbb{C}$-vector subspace of $\mathbb{C}A_{\mathbb{C}}(G)$. We regard the $\mathbb{C}G$-representation algebra $\mathbb{C}A_{\mathbb{C}}(G)$ as the $\mathbb{C}$-vector space of G-invariant functions $G \to \mathbb{C}$. The primitive idempotents e_g^G of $A_{\mathbb{C}}(G)$ are determined by the condition that, for all $\xi \in \mathbb{C}A_{\mathbb{C}}(G)$, we have

$$\xi = \sum_{g \in_G G} \xi(g)\, e_g^G$$

where the notation indicates that g runs over representatives of the conjugacy classes of G.

The linearization map $\mathrm{lin}_G : B(G) \to A_{\mathbb{K}}(G)$ is the ring homomorphism given by $\mathrm{lin}_G[X] = [\mathbb{K}X]$ where $\mathbb{K}X$ denotes the permutation $\mathbb{K}G$-module associated with X. The $\mathbb{Z}$-module $K(G) = \mathrm{Ker}(\mathrm{lin}_G)$ is independent of $\mathbb{K}$ and, except for an ambiguity as to the codomain, lin_G is also independent of $\mathbb{K}$. Indeed, $[\mathbb{Q}X] = [\mathbb{K}X] = [\mathbb{C}X]$, and we can equally well understand the codomain of lin_G to be $A_{\mathbb{Q}}(G)$ or $A_{\mathbb{K}}(G)$ or $A_{\mathbb{C}}(G)$.

By restriction, lin_G restricts to a group homomorphism

$$\mathrm{lin}_G^\times \,:\, B^\times(G) \to A_{\mathbb{K}}^\times(G)$$

between the unit groups $B^\times(G)$ and $A_{\mathbb{K}}^\times(G)$ of $B(G)$ and $A_{\mathbb{K}}(G)$. Again, we can equally well understand the codomain of $\mathrm{lin}_G^\times$ to be $A_{\mathbb{Q}}^\times(G)$ or $A_{\mathbb{K}}^\times(G)$ or $A_{\mathbb{C}}^\times(G)$. We define $K^\times(G) = \mathrm{Ker}(\mathrm{lin}_G^\times)$. Thus, $K^\times(G)$ is the group consisting of those units in $B(G)$ that can be written in the form $1 + \kappa$ with $\kappa \in K(G)$. More concisely,

$$K^\times(G) = B^\times(G) \cap (1 + K(G)). \qquad (*)$$

It is easy to see that the $\mathbb{C}$-linear extension $\mathrm{lin}_G : \mathbb{C}B(G) \to \mathbb{C}A_{\mathbb{C}}(G)$ is given by

$$\mathrm{lin}_G(e_I^G) = \sum_{g \in_G G \,:\, I =_G \langle g \rangle} e_g^G$$

where the notation indicates that g runs over representatives of those conjugacy classes of G such that I is G-conjugate to the cyclic group $\langle g \rangle$ generated by g. The next remark follows easily.

Remark 3.1. Given an element $x \in B(G)$, then:

(1) $x \in B^{\times}(G)$ if and only if $\epsilon_I^G(x) = \pm 1$ for every subgroup I of G,
(2) $x \in K(G)$ if and only if $\epsilon_C^G(x) = 0$ for every cyclic subgroup C of G,
(3) $x \in K^{\times}(G)$ if and only if $\epsilon_I^G(x) = \pm 1$ for every subgroup I of G and $\epsilon_C^G(x) = 1$ for every cyclic subgroup C of G.

We have the following analogous remark for the unit group $A_{\mathbb{Q}}^{\times}(G)$.

Remark 3.2. Given $\xi \in A_{\mathbb{Q}}(G)$, then $\xi \in A_{\mathbb{Q}}^{\times}(G)$ if and only if $\xi(g) = \pm 1$ for all $g \in G$.

Proof. The function $A_{\mathbb{Q}}^{\times}(G) \ni \xi \mapsto \xi(g) \in \mathbb{Q}$ is a group homomorphism. The numbers $\xi(g)$ are rational, yet they are also units in the ring of algebraic integers, so each $\xi(g) = \pm 1$. □

Tornehave [14, Lemma 3.1] gave the following more powerful description of $A_{\mathbb{Q}}^{\times}(G)$.

Lemma 3.3 *(Tornehave). The unit group $A_{\mathbb{Q}}^{\times}(G)$ is the set whose elements have the form $\pm\phi$ where ϕ is a linear $\mathbb{Q}G$-character.*

Proof. Plainly, the elements of $A_{\mathbb{Q}}(G)$ that can be written in the specified form are units. We must prove the converse. Let $\langle - \mid - \rangle$ denote the usual inner product on the $\mathbb{C}$-vector space $\mathbb{C}A_{\mathbb{C}}(G)$. The latest remark implies that, given $\xi \in A_{\mathbb{Q}}^{\times}(G)$, then $\langle \xi \mid \xi \rangle = 1$. It follows that $\xi = \pm\phi$ for some absolutely irreducible $\mathbb{Q}G$-character ϕ. But $0 < \phi(1) = \pm\xi(1) = \pm 1$, so ϕ is a linear $\mathbb{Q}G$-character. □

The linear $\mathbb{Q}G$-characters are easy to classify. The condition $\mathrm{Ker}(\phi) = F$ characterizes a bijective correspondence $\phi \leftrightarrow F$ between the linear $\mathbb{Q}G$-characters ϕ and the subgroups $F \leq G$ with index $|G : F| \leq 2$. Of course, $\phi(g) = 1$ for $g \in F$ while $\phi(g) = -1$ for $g \in G - F$. So $\phi = [\mathbb{Q}G/F] - 1 = \mathrm{lin}_G^{\times}(d_F^G - 1)$ and $-\phi = \mathrm{lin}_G^{\times}(1 - d_F^G)$. Hence, via the latest lemma, we deduce that the group homomorphism $\mathrm{lin}_G^{\times} : B^{\times}(G) \to A_{\mathbb{Q}}^{\times}(G)$ is surjective, and we have a short exact sequence of elementary abelian 2-groups

$$\mathrm{Lin}_G^{\times} \;:\; 0 \longrightarrow K^{\times}(G) \longrightarrow B^{\times}(G) \xrightarrow{\mathrm{lin}_G^{\times}} A_{\mathbb{Q}}^{\times}(G) \longrightarrow 0.$$

Let us emphasize that the exactness of $\mathrm{Lin}_G^\times$ holds for all finite groups G. For the sake of comparison, let us point out that we also have a left-exact sequence of free $\mathbb{Z}$-modules

$$\mathrm{Lin}_G \ : \ 0 \longrightarrow K(G) \longrightarrow B(G) \xrightarrow{\mathrm{lin}_G} A_{\mathbb{Q}}(G).$$

As we already noted in Section 2, the Ritter Segal Theorem asserts that Lin_G is right exact when G is a p-group, but the right exactness can fail for nilpotent G.

The biset functor structures of the Burnside functor B and the representation functor $A_{\mathbb{K}}$ were introduced by Bouc [5], [10, 1.1, 1.2] Let us briefly recall some notation (exactly the same as is used in [4]). Consider a group homomorphism $\phi : G \to F$ and subgroups $H \leq G \trianglerighteq N$. Allowing G to vary, the rings $B(G)$ and $A_{\mathbb{K}}(G)$ give rise to the biset functors B and $A_{\mathbb{K}}$, whose elemental maps are the isogation maps $\mathrm{iso}_{F,G}^{\phi}$, the restriction maps $\mathrm{res}_{H,G}$, the inflation maps $\mathrm{inf}_{G,G/N}$, the induction maps $\mathrm{ind}_{G,H}$ and the deflation maps $\mathrm{def}_{G/N,G}$. The linearization maps lin_G give rise to a morphism of biset functors $\mathrm{lin} : B \to A_{\mathbb{K}}$, and we have a left exact sequence of biset functors

$$\mathrm{Lin} \ : \ 0 \longrightarrow K \longrightarrow B \xrightarrow{\mathrm{lin}_G^\times} A_{\mathbb{K}}.$$

The biset functor structure of the Burnside unit functor $B^\times$ was established by Bouc [8], [10, 11.2.21]. Again, let us recall some notation (again, exactly the same as in [4]). Letting G vary, the rings $B(G)$ admit two kinds of multiplication-preserving functions: the Japanese deflation functions $\mathrm{jef}_{G/N,G}$ (sometimes called multiplicative deflation); the Japanese induction functions $\mathrm{jnd}_{G,H}$ (sometimes called multiplicative induction or tensor induction). The distinction between these and the usual deflation and induction on $B(G)$ is that $\mathrm{jef}_{G/N,G}$ is given by passage to the N-fixed points whereas $\mathrm{def}_{G,G/N}$ is given by passage to N-orbits; $\mathrm{jnd}_{G,H}$ is given by a tensor product construction discussed in, for instance, Yoshida [16, Section 3]. The unit groups $B^\times(G)$ can be regarded as vector spaces over the field $\mathbb{F}_2 = \{0, 1\}$ and they give rise to a biset functor $B^\times$ over $\mathbb{F}_2$ whose isogation, restriction and inflation maps are the same as for the rings $B(G)$, but the induction and deflation maps are $\mathrm{jnd}_{G,H}$ and $\mathrm{jef}_{G/N,G}$.

We now introduce an inflaky functor $A_{\mathbb{Q}}^\times$ whose coordinate modules are the unit groups $A_{\mathbb{Q}}^\times(G)$. The rings $A_{\mathbb{K}}(G)$ admit multiplication-preserving functions $\mathrm{jnd}_{G,H} : A_{\mathbb{K}}(H) \to A_{\mathbb{K}}(G)$ called Japanese induction (or multiplicative induction or tensor induction). These functions are discussed in Yoshida [16, Section 3]. Regarding the unit groups $A_{\mathbb{K}}^\times(G)$ as vector spaces over $\mathbb{F}_2$, then the function $\mathrm{jnd}_{G,H}$ restricts to a linear map $A_{\mathbb{K}}^\times(H) \to A_{\mathbb{K}}^\times(G)$.

Specializing now to the case $\mathbb{K} = \mathbb{Q}$, we claim that the unit groups $A_{\mathbb{Q}}^\times(G)$ give rise to an inflaky functor $A_{\mathbb{Q}}^\times$ over $\mathbb{F}_2$ whose isogation, restriction, inflation and induction maps are $\mathrm{iso}_{F,G}^{\phi}$, $\mathrm{res}_{H,G}$, $\mathrm{inf}_{G,G/N}$, $\mathrm{jnd}_{G,H}$, respectively. Yoshida [16, Section 3d] observed that the functions $\mathrm{jnd}_{G,H} : B(H) \to B(G)$ and $\mathrm{jnd}_{G,H} : A_{\mathbb{K}}(H) \to A_{\mathbb{K}}(G)$ commute with lin_H and lin_G. In particular, the $\mathbb{F}_2$-linear maps $\mathrm{jnd}_{G,H} : B^\times(H) \to B^\times(G)$ and $\mathrm{jnd}_{G,H} : A_{\mathbb{K}}^\times(H) \to A_{\mathbb{K}}^\times(G)$ commute with $\mathrm{lin}_H^\times$ and $\mathrm{lin}_G^\times$. But, as we noted above, the

maps $\mathrm{lin}_G^\times$ are surjective when $\mathbb{K} = \mathbb{Q}$. The claim now follows, and we have also shown that the maps $\mathrm{lin}_G^\times$ give rise to an epimorphism of inflaky functors $\mathrm{lin}^\times : B^\times \to A_{\mathbb{Q}}^\times$. Since $\mathrm{lin}^\times$ is an inflaky morphism, the kernel $K^\times = \mathrm{Ker}(\mathrm{lin}^\times)$ is an inflaky subfunctor of $B^\times$. We have established the short exact sequence $\mathrm{Lin}^\times$ indicated at the beginning of this section.

However, a straightforward calculation in the case $1 < N < G \cong V_4$ shows that $K^\times$ is not a biset subfunctor of $B^\times$. Indeed, for such G and N, we have $1 - 2e_G^G \in K^\times(G)$ but

$$\mathrm{jef}_{G/N,G}(1 - 2e_G^G) = 1 - 2e_{G/N}^{G/N} \notin K^\times(G/N).$$

Perforce, there is no way of imposing deflation maps on $A_{\mathbb{Q}}^\times$ so as to make $\mathrm{lin}^\times$ become a morphism of biset functors. We speculate that, for arbitrary $\mathbb{K}$, the unit groups $A_{\mathbb{K}}^\times(G)$ give rise to an inflaky functor $A_{\mathbb{K}}^\times$ in a similar way. Possibly, the above argument could be adapted by replacing the Burnside functor B with the monomial Burnside functor $B_{\mathbb{K}}$.

We define the **parity** of an integer n to be $\mathrm{par}(n) = (-1)^n$. Recall that the **reduced tom Dieck map** $\overline{\mathrm{die}}_G : A_{\mathbb{R}}(G) \to B^\times(G)$ is defined to be the linear map such that, given an $\mathbb{R}G$-character χ, then

$$\epsilon_I^G(\overline{\mathrm{die}}_G(\chi)) = \mathrm{par}(\dim_{\mathbb{R}}(M_\chi^I)).$$

Here, M_χ is an $\mathbb{R}G$-module affording χ and M_χ^I denotes the subspace of M_χ fixed by the subgroup $I \leq G$. By Yoshida [16, 3.5], the maps die_G give rise to a morphism of biset functors

$$\overline{\mathrm{die}}_G \ : \ A_{\mathbb{R}} \to B^\times.$$

A review of the morphism $\overline{\mathrm{die}}$, in our context of concern, appears in [3, Section 3].

Consider an element $\xi \in B^\times(G)$. By Lemma 3.3, there exists a linear $\mathbb{Q}G$-character ϕ such that $\mathrm{lin}_G(\xi) = \pm\phi$. Let $F = \mathrm{Ker}(\phi)$. The defining formula for die_G yields

$$\epsilon_I^G(\overline{\mathrm{die}}_G(\phi)) = \begin{cases} -1 & \text{if } I \leq F, \\ 1 & \text{if } F \not\geq I \leq G. \end{cases}$$

So the values of the character $\mathrm{lin}_G(\overline{\mathrm{die}}_G(\phi))$ are

$$\mathrm{lin}_G(\overline{\mathrm{die}}_G(\phi))(g) = \begin{cases} -1 & \text{if } g \in F, \\ 1 & \text{if } g \in G - F. \end{cases}$$

In other words, $\mathrm{lin}_G(\overline{\mathrm{die}}_G(\phi)) = -\phi$. Replacing ϕ with the trivial $\mathbb{Q}G$-character, we obtain $\mathrm{lin}_G(\overline{\mathrm{die}}_G(1)) = -1$, hence $\mathrm{lin}_G(\overline{\mathrm{die}}_G(1 + \phi)) = \phi$. If $\mathrm{lin}_G(\xi) = -\phi$, we put $\eta = \overline{\mathrm{die}}_G(\phi)$, while if $\mathrm{lin}_G(\xi) = \phi$, we put $\eta = \overline{\mathrm{die}}_G(1 + \phi)$. Either way, $\eta \in {}^{\mathbb{Q}}B^\times$ and $\mathrm{lin}_G^\times(\eta) = \mathrm{lin}_G^\times(\xi)$. Since $K^\times(G)$ is the kernel of $\mathrm{lin}_G^\times$, we deduce that $B^\times(G) = K^\times(G) + {}^{\mathbb{Q}}B^\times(G)$. This completes the proof of Theorem 2.6.

4. Tornehave's Unit Theorem

Here, we prove Theorem 2.3 and we pave the way towards a new proof of Theorem 2.4 and a proof of Theorem 2.5. The crucial case of Theorem 2.3 is that of a 2-group, and we shall deal with that case first. The generalization to nilpotent groups will be done in an easy little paragraph at the end of this section.

Recall (see Section 2) that the reduced exponential map $\overline{\exp}_G : B(G) \to B^\times(G)$ and the reduced exponential morphism $\overline{\exp} : B \to B^\times$ are the composites $\overline{\exp}_G = \overline{\mathrm{die}}_G \circ \mathrm{lin}_G$ and $\overline{\exp} = \overline{\mathrm{die}} \circ \mathrm{lin}$. Thus

$$\epsilon_I^G(\overline{\exp}_G([X]-[Y])) = \mathrm{par}(|I\backslash X| - |I\backslash Y|) = \mathrm{par}|I\backslash X| \,.\, \mathrm{par}|I\backslash Y|$$

where X and Y are G-sets, I is a subgroup of G and $I\backslash X$ denotes the set of I-orbits in X.

For a set π of prime numbers, we now define the **reduced Tornehave map** $\overline{\mathrm{torn}}_G^\pi$: $K(G) \to B^\times(G)$ and the **reduced Tornehave morphism**

$$\overline{\mathrm{torn}}^{\,\pi} \;:\; K \to B^\times.$$

The definition involves the π-adic valuation $\log_\pi$ on the positive integers, which is given by the conditions $\log_\pi(n_1 n_2) = \log_\pi(n_1) + \log_\pi(n_2)$ and, if $p \in \pi$, then $\log_\pi(p) = 1$, otherwise $\log_\pi(p) = 0$. Observe that the $\mathbb{Z}$-module $K(G)$ consists of those elements of $B(G)$ that can be written in the form $[X]-[Y]$ where $\mathbb{K}X \cong \mathbb{K}Y$ as $\mathbb{K}G$-modules. We define $\overline{\mathrm{torn}}_G^\pi$ such that, assuming $\mathbb{K}X \cong \mathbb{K}Y$, then

$$\begin{aligned}\epsilon_I^G(\overline{\mathrm{torn}}_G^\pi([X]-[Y])) &= \mathrm{par}\Big(\sum_{\mathcal{O}\in I\backslash X} \log_\pi|\mathcal{O}| - \sum_{\mathcal{O}\in I\backslash Y} \log_\pi|\mathcal{O}|\Big)\\ &= \prod_{\mathcal{O}\in I\backslash X} \mathrm{par}(\log_\pi|\mathcal{O}|) \,.\, \prod_{\mathcal{O}\in I\backslash Y} \mathrm{par}(\log_\pi|\mathcal{O}|).\end{aligned}$$

It is shown in [3, Section 7] that the maps $\overline{\mathrm{torn}}_G^\pi$ give rise to an inflaky morphism $\overline{\mathrm{torn}}^{\,\pi}$. In the present paper, we shall be concerned with the morphism of p-biset functors $\overline{\mathrm{torn}}^{\,p} = \overline{\mathrm{torn}}^{\,\{p\}}$, especially the morphism of 2-biset functors $\overline{\mathrm{torn}} = \overline{\mathrm{torn}}^{\,2}$.

Tornehave's proof [14] of the following result is presented also in [3, Section 7].

Lemma 4.1 *(Tornehave). We have* $\overline{\mathrm{torn}}_G^\pi(K(G)) \leq {}^{\mathbb{R}}B^\times(G)$. *In other words,* $\overline{\mathrm{torn}}^{\,\pi}(K)$ *is an inflaky subfunctor of* ${}^{\mathbb{R}}B^\times$.

The next result, again due to Tornehave [14], is recorded in [4, Section 10].

Lemma 4.2 *(Tornehave). Suppose that the* 2*-group* P *is non-cyclic. Then* $2e_P^P \in K(P)$ *and* $\overline{\mathrm{torn}}_P(2e_P^P) = 1 - 2e_P^P$.

Lemma 4.3 *(Tornehave). We have* $K^\times(P) \leq \overline{\mathrm{torn}}_P(K(P))$.

Proof. We define a function $\sigma_P : K^\times(P) \ni \eta \mapsto \eta - 1 \in K(P)$. The definition makes sense in view of the equality $(*)$ in Section 3. We claim that the function $\overline{\mathrm{torn}}_P \circ \sigma_P : K^\times(P) \to K^\times(P)$ is a bijection. Deny, and assume that P is a counter-example with minimal order. Then $\overline{\mathrm{torn}}_P \circ \sigma_P$ is not injective. Perforce, $|K^\times(P)| \geq 2$ so P cannot be cyclic. Let η_1 and η_2 be distinct elements of $K^\times(P)$ such that $\overline{\mathrm{torn}}_P(\sigma_p(\eta_1)) = \overline{\mathrm{torn}}_P(\sigma_p(\eta_2))$. The functions $\overline{\mathrm{torn}}_P$ and σ_P commute with restriction so, by the minimality of P, we have $\mathrm{res}_{I,P}(\eta_1) = \mathrm{res}_{I,P}(\eta_2)$ for all $I < P$. Therefore $\eta_1 - \eta_2 = \pm e_P^P$, in other words, $\sigma_P(\eta_1) - \sigma_P(\eta_2) = \pm e_P^P$. Since $B^\times(G)$ is an elementary abelian 2-group,

$$1_{B(P)} = \overline{\mathrm{torn}}_P(\sigma_P(\eta_1)) \,.\, \overline{\mathrm{torn}}_P(\sigma_P(\eta_2)) = \overline{\mathrm{torn}}_P(\sigma_P(\eta_1 - \eta_2)) = \overline{\mathrm{torn}}_P(2e_P^P).$$

This contradicts Lemma 4.2. The claim is established and the required conclusion follows immediately. □

We can now complete the proof of Theorem 2.3. By Lemmas 4.1 and 4.3, $K^\times(P) \leq {}^{\mathbb{R}}B^\times(P)$. Trivially, ${}^{\mathbb{Q}}B^\times(P) \leq {}^{\mathbb{R}}B^\times(P)$. So, by Theorem 2.6, $B^\times(P) = {}^{\mathbb{R}}B^\times(P)$. We have established Theorem 2.4 in the case of a finite 2-group. Finally, suppose that G is nilpotent, and write $G = P \times Q$ where P is the Sylow 2-subgroup. Bouc [8, 6.3] showed that the map $\mathrm{inf}_{G,G/Q} \circ \mathrm{iso}_{G/Q,P} : B^\times(P) \to B^\times(G)$ is an isomorphism. Plainly, $\mathrm{inf}_{G,G/Q} \circ \mathrm{iso}_{G/Q,P}$ sends ${}^{\mathbb{R}}B^\times(P)$ to ${}^{\mathbb{R}}B^\times(G)$. Therefore Theorem 2.3 holds in general.

5. Genotypes of irreducible representations

As we noted in Section 1, the focus of our attention is the 2-biset functor ${}_2B^\times/{}^{\mathbb{Q}}_2B^\times$. Bouc [8, Section 9], [9, Section 6] determined its structure, interpreting the simple composition factors in terms of irreducible rational representations. However, as we also noted in Section 1, much of the motive for studying the functor $B^\times$ arises from its relevance to the study of irreducible real representations. This brief section is a review, linking Bouc's results to some material [1, Sections 5, 6] concerning irreducible real representations.

The following theorem is due to Kronstein in the special case $\mathbb{K} = \mathbb{C}$ and to Bouc in the special case $\mathbb{K} = \mathbb{Q}$. Citations and a proof for arbitrary $\mathbb{K}$ can be found in [1, 1.1]. Below, we shall be making use of the case $\mathbb{K} = \mathbb{R}$.

Theorem 5.1. *Suppose that G is a p-group. Let ψ be an irreducible $\mathbb{K}G$-character. Then there exists a section $K \trianglelefteq H \leq G$ such that the following three conditions hold: every normal abelian subgroup of H/K is cyclic; $\psi = \mathrm{ind}_{G,H}(\mathrm{inf}_{H,H/K}(\phi))$ for some faithful irreducible $\mathbb{K}H/K$-character ϕ; no Galois conjugate of ϕ occurs in the $\mathbb{K}H/K$-character $\mathrm{def}_{H/K,H}(\mathrm{res}_{H,G}(\psi)) - \phi$. Furthermore, if $K' \trianglelefteq H' \leq G$ is another such subquotient, then $H/K \cong H'/K'$.*

As a group well-defined up to isomorphism, the group $\mathrm{Type}(\psi) = H/K$ is called the **genotype** of ψ. The following theorem, a special case of [1, 3.5], says that the genotype is invariant under change of fields.

Theorem 5.2. *Suppose that G is a p-group. Then there is a bijective correspondence between the irreducible $\mathbb{Q}G$-characters χ and the Galois conjugacy classes of irreducible $\mathbb{K}G$-characters ψ. The correspondence is characterized by the condition that χ is a $\mathbb{Z}$-multiple of the sum of the Galois conjugates of ψ. The genotypes of χ and ψ coincide.*

Via the latest theorem, Bouc [8, 9.5, 9.6] can be expressed as the next result. Recall that the simple biset functors over a field $\mathbb{F}$ are the biset functors denoted $S_{L,V}$ where L is the minimal group such that $S_{L,V}(L) \neq 0$ and, as $\mathbb{F}\mathrm{Out}(L)$-modules, $S_{L,V} \cong V$. We let C_n denote the cyclic group with order $n \geq 1$.

Theorem 5.3 *(Bouc). For finite 2-groups, the 2-biset functor $B^\times$ is uniserial with filtration $0 < {}^{\mathbb{Q}}_{2}B^\times = L_3 < L_4 < \ldots < {}_2B^\times$ where ${}^{\mathbb{Q}}_{2}B^\times \cong S_{C_1,\mathbb{F}_2}$ and $L_n/L_{n-1} \cong S_{D_{2^n},\mathbb{F}_2}$ for $n \geq 4$. Furthermore, $\dim_{\mathbb{F}_2}(S_{C_1,\mathbb{F}_2}(P))$ is the number of Galois conjugacy classes of irreducible $\mathbb{K}P$-characters with genotype C_1 or C_2, while $\dim_{\mathbb{F}_2}(S_{D_{2^n},\mathbb{F}_2}(P))$ is the number of Galois conjugacy classes of irreducible $\mathbb{K}P$-characters with genotype D_{2^n}, for $n \geq 4$.*

In the proof of Theorem 2.4 presented in the next section, the overlap with the arguments in [9] lies in the invocation of the following result, [9, 6.3]. Again, our rendition of the result depends on Theorem 5.2.

Lemma 5.4. *The dimension of $K(P)/{}^{\mathbb{Q}}K(P)$ is equal to the number of Galois conjugacy classes of irreducible $\mathbb{K}P$-characters ψ such that $\mathrm{Type}(\psi) = D_{2^n}$ for some $n \geq 4$.*

The next result is a special case of [1, 5.13].

Theorem 5.5. *Let ψ be an irreducible $\mathbb{R}P$-character. Then:*

(1) $\mathrm{Type}(\psi) \cong C_1$ *if and only if ψ is the trivial character.*
(2) $\mathrm{Type}(\psi) \cong C_2$ *if and only if ψ is non-trivial, absolutely irreducible and realizable over $\mathbb{Q}$.*
(3) $\mathrm{Type}(\psi) \cong D_{2^n}$ *for some $n \geq 4$ if and only if ψ is non-trivial, absolutely irreducible, and not realizable over $\mathbb{Q}$.*
(4) $\mathrm{Type}(\psi)$ *is semidihedral with order at least 16 or generalized quaternion with order at least 8 if and only if ψ is not absolutely irreducible.*

Combining the latest three results, we recover the following corollary, which will be of crucial use in the next section. Via [1, 6.6, 6.7], the corollary is already essentially in Bouc [9, 6.5].

Corollary 5.6 *(Bouc). The dimensions of $B^\times(P)/{}^{\mathbb{Q}}B^\times(P)$ and $K(P)/{}^{\mathbb{Q}}K(P)$ are both equal to the number of Galois conjugacy classes of absolutely irreducible $\mathbb{R}P$-characters that are not realizable over $\mathbb{Q}$.*

6. Bouc's isomorphism

We shall prove Theorems 2.4 and 2.5 simultaneously. Most of that task will be showing that there exists an isomorphism of biset functors $\overline{\text{bouc}}^{\,p}$ such that the following square commutes.

$$\begin{array}{ccc} {}_pK & \xrightarrow{\overline{\text{torn}}^{\,p}} & {}_pB^\times \\ \downarrow{\scriptstyle \pi_{\mathbb{Q}}^K} & & \downarrow{\scriptstyle \pi_{\mathbb{Q}}^\times} \\ {}_pK/{}_p^{\mathbb{Q}}K & \xrightarrow{\overline{\text{bouc}}^{\,p}} & {}_pB^\times/{}_p^{\mathbb{Q}}B^\times \end{array}$$

First suppose that p is odd. Bouc [7, 6.12] asserts that ${}_p^{\mathbb{Q}}K = {}_pK$. Another proof of that equality appears in Bouc [9, 3.8]. As we observed in Section 2, ${}_p^{\mathbb{Q}}B^\times = {}_pB^\times$. Theorems 2.4 and 2.5 are now clear for odd p. It remains to deal with the case where $p = 2$.

Proposition 6.1. *The composite $\pi_{\mathbb{Q}}^\times \circ \overline{\text{torn}} : {}_2K \to {}_2B^\times/{}_2^{\mathbb{Q}}B^\times$ is an epimorphism of 2-biset functors.*

Proof. It was shown in [4, Section 10] that $\pi_{\mathbb{Q}}^\times \circ \overline{\text{torn}}$ is a morphism of 2-biset functors. By Theorem 2.6 and Lemma 4.3, $\pi_{\mathbb{Q}}^\times \circ \overline{\text{torn}}$ is an epimorphism. □

Recall that ${}_2^{\mathbb{Q}}K$ is defined to be the 2-biset functor generated by the coordinate module ${}_2^{\mathbb{Q}}K(D_8)$. Every $\mathbb{R}D_8$ character is realizable over $\mathbb{Q}$ so, by Corollary 5.6, ${}^{\mathbb{Q}}B^\times(D_8) = B^\times(D_8)$. Hence $\text{Ker}(\pi_{\mathbb{Q}}^K) \leq \text{Ker}(\pi_{\mathbb{Q}}^\times \circ \overline{\text{torn}})$. Since $\pi_{\mathbb{Q}}^K$ is an epimorphism, Proposition 6.1 implies that there exists a unique morphism of 2-biset functors $\overline{\text{bouc}}$ such that the diagram above commutes. The proposition also implies that $\overline{\text{bouc}}$ is an epimorphism. By Corollary 5.6 again, $\overline{\text{bouc}}$ is an isomorphism.

We complete the proof of Theorem 2.4 with the following slightly stronger version for the case $p = 2$. Theorem 2.5 will then follow because of the way we constructed $\overline{\text{bouc}}$.

Theorem 6.2. *For 2-groups, there is a unique non-zero morphism of 2-biset functors*

$$\overline{\text{bouc}} : {}_2K/{}_2^{\mathbb{Q}}K \to {}_2B^\times/{}_2^{\mathbb{Q}}B^\times.$$

Furthermore, $\overline{\text{bouc}}$ is an isomorphism.

Proof. All that remains is to establish the uniqueness. It suffices to show that the endomorphism algebra of ${}_2B^\times/{}_2^{\mathbb{Q}}B^\times$ is isomorphic to $\mathbb{F}_2$. Theorem 5.3 tells us that ${}_2B^\times/{}_2^{\mathbb{Q}}B^\times$

has a unique filtration $0 = L_2 < L_3 < \dots$ such that each L_n/L_{n-1} is simple. Furthermore, the terms L_n/L_{n-1} are mutually non-isomorphic and they all have endomorphism algebra isomorphic to $\mathbb{F}_2$. Any endomorphism θ of ${}_2B^\times/{}^{\mathbb{Q}}_2B^\times$ must act on L_n/L_{n-1} either as zero or as the identity. So θ or $\theta - 1$ sends L_n to L_{n-1}. But L_n/L_{n-1} does not occur as a simple composition factor of L_{n-1}, so θ or $\theta - 1$ must annihilate L_n. In other words, θ must act on L_n either as zero or as the identity. For distinct n and m, it is impossible for θ to act as zero L_n and as the identity on L_m. So θ must act either as zero on all the L_n or else as the identity on all the L_n. □

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