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Blocks of Mackey categories

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ABSTRACT

For a suitable small category $\mathcal{F}$ of homomorphisms between finite groups, we introduce two subcategories of the biset category, namely, the deflation Mackey category $\mathcal{M}_{\mathcal{F}}^{\leftarrow}$ and the inflation Mackey category $\mathcal{M}_{\mathcal{F}}^{\rightarrow}$. Let $\mathcal{G}$ be the subcategory of $\mathcal{F}$ consisting of the injective homomorphisms. We shall show that, for a field $\mathbb{K}$ of characteristic zero, the $\mathbb{K}$-linear category $\mathbb{K}\mathcal{M}_{\mathcal{G}} = \mathbb{K}\mathcal{M}_{\mathcal{G}}^{\leftarrow} = \mathbb{K}\mathcal{M}_{\mathcal{G}}^{\rightarrow}$ has a semisimplicity property and, in particular, every block of $\mathbb{K}\mathcal{M}_{\mathcal{G}}$ owns a unique simple functor up to isomorphism. On the other hand, we shall show that, when $\mathcal{F}$ is equivalent to the category of finite groups, the $\mathbb{K}$-linear categories $\mathbb{K}\mathcal{M}_{\mathcal{F}}^{\leftarrow}$ and $\mathbb{K}\mathcal{M}_{\mathcal{F}}^{\rightarrow}$ each have a unique block.

1. Introduction

Mackey functors are characterized by induction and restriction maps associated with some group homomorphisms. For example, the groups involved can be the subgroups of a fixed finite group and the homomorphisms can be the composites of inclusions and

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conjugations. As another example, the groups can be arbitrary finite groups and the homomorphisms can be arbitrary.

We shall use Bouc's theory of bisets [4] to recast the theory of Mackey functors in the following way. Let $\mathfrak{K}$ be a set of finite groups that is closed under taking subgroups. (In applications, $\mathfrak{K}$ can play the role of a proper class. For instance, if $\mathfrak{K}$ owns an isomorphic copy of every finite group, then $\mathfrak{K}$ can play the role of the class of all finite groups.) Generalizing the notion of a fusion system on a finite p-group, we shall introduce the notion of a *Mackey system* on $\mathfrak{K}$, which is a category $\mathcal{F}$ such that the set of objects is $\mathrm{Obj}(\mathcal{F}) = \mathfrak{K}$ and the morphisms in $\mathcal{F}$ are group homomorphisms subject to certain axioms. In the case where all the homomorphisms in $\mathcal{F}$ are injective, we call $\mathcal{F}$ an *ordinary Mackey system*.

For any Mackey system $\mathcal{F}$ on $\mathfrak{K}$, we shall define two subcategories of the biset category, namely, the *deflation Mackey category* $\mathcal{M}_{\mathcal{F}}^{\leftarrow}$ and the *inflation Mackey category* $\mathcal{M}_{\mathcal{F}}^{\rightarrow}$. The category $\mathcal{M}_{\mathcal{F}}^{\leftarrow}$ is generated by inductions via homomorphisms in $\mathcal{F}$ and restrictions via inclusions. The category $\mathcal{M}_{\mathcal{F}}^{\rightarrow}$ is generated by inductions via inclusions and restrictions via homomorphisms in $\mathcal{F}$. When $\mathcal{F}$ is an ordinary Mackey system, $\mathcal{M}_{\mathcal{F}}^{\leftarrow}$ and $\mathcal{M}_{\mathcal{F}}^{\rightarrow}$ coincide, and we write it as $\mathcal{M}_{\mathcal{F}}$, calling it an *ordinary Mackey category*.

Let R be a commutative unital ring and let $R\mathcal{M}_{\mathcal{F}}^{\leftarrow}$ be the R-linear extension of $\mathcal{M}_{\mathcal{F}}^{\leftarrow}$. The notion of a Mackey functor over R will be replaced by the notion of an $R\mathcal{M}_{\mathcal{F}}^{\leftarrow}$*-functor*, which is a functor from $R\mathcal{M}_{\mathcal{F}}^{\leftarrow}$ to the category of R-modules. Our approach to the study of $R\mathcal{M}_{\mathcal{F}}^{\leftarrow}$-functors will be ring-theoretic. We shall introduce an algebra ${}^{\Pi}R\mathcal{M}_{\mathcal{F}}^{\leftarrow}$ over R, called the *extended quiver algebra* of $R\mathcal{M}_{\mathcal{F}}^{\leftarrow}$, which has the feature that every $R\mathcal{M}_{\mathcal{F}}^{\leftarrow}$-functor is a ${}^{\Pi}R\mathcal{M}_{\mathcal{F}}^{\leftarrow}$-module. We define a **block** of $R\mathcal{M}_{\mathcal{F}}^{\leftarrow}$ to be a block of ${}^{\Pi}R\mathcal{M}_{\mathcal{F}}^{\leftarrow}$. As in the block theory of suitable rings, every indecomposable $R\mathcal{M}_{\mathcal{F}}^{\leftarrow}$-functor belongs to a unique block of $R\mathcal{M}_{\mathcal{F}}^{\leftarrow}$. Similar constructions can be made for the inflation Mackey category $\mathcal{M}_{\mathcal{F}}^{\rightarrow}$.

Let $\mathbb{K}$ be a field of characteristic zero. Regarding the blocks of $\mathbb{K}\mathcal{M}_{\mathcal{F}}^{\leftarrow}$ as a partitioning of the simple $\mathbb{K}\mathcal{M}_{\mathcal{F}}^{\leftarrow}$-functors, the blocks sometimes partition the simple functors very finely. Corollary 4.7 says that, for any ordinary Mackey system $\mathcal{G}$, each block of $\mathbb{K}\mathcal{M}_{\mathcal{G}}$ owns a unique simple $\mathbb{K}\mathcal{M}_{\mathcal{G}}$-functor. But the blocks can also partition the simple functors very coarsely. Our main result, Theorem 7.1, asserts that if $\mathfrak{K}$ owns an isomorphic copy of every finite group and $\mathcal{F}$ owns every homomorphism between groups in $\mathfrak{K}$, then $\mathbb{K}\mathcal{M}_{\mathcal{F}}^{\leftarrow}$ and $\mathbb{K}\mathcal{M}_{\mathcal{F}}^{\rightarrow}$ each have a unique block.

We shall be needing two theorems whose conclusions have been obtained before under different hypotheses. Theorem 4.6 asserts that the category $\mathbb{K}\mathcal{M}_{\mathcal{G}}$, though sometimes infinite-dimensional, has a semisimplicity property. This result was obtained by Webb [10, 9.5] in the special case where $\mathcal{G}$ is equivalent to the category of injective group homomorphisms. The same conclusion was established by Thévenaz–Webb [8], [9, 3.5] in a different scenario where the group isomorphisms that come into consideration are conjugations within a fixed finite group. Their result is not a special case of ours because their relations [9, page 1868] on the conjugation maps are weaker than ours. Theorem 5.2 asserts that, taking $\mathcal{G}$ to be the largest ordinary Mackey system that is a subcategory

of $\mathcal{F}$, restriction and inflation yield mutually inverse bijective correspondences between the simple $\mathbb{K}\mathcal{M}^{\leftarrow}_{\mathcal{F}}$-functors and the simple $\mathbb{K}\mathcal{M}_{\mathcal{G}}$-functors. A similar result holds for the simple $\mathbb{K}\mathcal{M}^{\rightarrow}_{\mathcal{F}}$-functors. A version of this result was obtained by Yaraneri [11, 3.10] in the scenario where the isomorphisms are conjugations within a fixed finite group and, again, the relations on the conjugation maps are as in [9, page 1868].

A scenario similar to ours was studied in Boltje–Danz [2]. We shall make much use of their techniques. They considered some subalgebras of the double Burnside algebra that can be identified with endomorphism algebras of objects of Mackey categories. Boltje and Danz obtained analogues [2, 5.8, 6.5] of Theorems 4.6 and 5.2 for the endomorphism algebras. Those analogues can be recovered from Theorems 4.6 and 5.2 by cutting by idempotents.

The material is organized as follows. Section 2 is an account of the general notion of a block of an R-linear category. In Section 3, we classify the simple functors of the R-linear extension of a Mackey category. In Section 4, we prove that the $\mathbb{K}$-linear extension of an ordinary Mackey category has a semisimplicity property. In Section 5, we compare the $\mathbb{K}$-linear extension of a deflation Mackey category with the $\mathbb{K}$-linear extension of an ordinary Mackey category. Section 6 concerns the unique non-ordinary deflation Mackey category in the case where $\mathfrak{K}$ consists only of a trivial group and a group with prime order. Section 7 proves a theorem on the uniqueness of the block of a deflation Mackey category that is, in some sense, maximal among all deflation Mackey categories.

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2. Blocks of linear categories

An **R-linear category** (also called an R-preadditive category) is defined to be a category whose morphism sets are R-modules and whose composition is R-bilinear. An **R-linear functor** between R-linear categories is defined to be a functor which acts on morphism sets as R-linear maps. We shall define the notion of a block of an R-linear category, and we shall establish some of its fundamental properties. It will be necessary to give a brief review of some material from [1] on quiver algebras and extended quiver algebras of R-linear categories.

Let $\mathcal{L}$ be a small R-linear category. Consider the direct product $\Pi = \prod_{F,G\in \mathrm{Obj}(\mathcal{L})} \mathcal{L}(F,G)$ where $\mathrm{Obj}(\mathcal{L})$ denotes the set of objects of $\mathcal{L}$ and $\mathcal{L}(F,G)$ denotes the R-module of morphisms $F \leftarrow G$ in $\mathcal{L}$. Given $x \in \Pi$, we write $x = ({}_Fx_G)$ where ${}_Fx_G \in \mathcal{L}(F,G)$. Let ${}^{\Pi}\mathcal{L}$ be the R-submodule of Π consisting of those elements x such that, for each $F \in \mathrm{Obj}(\mathcal{L})$, there exist only finitely many $G \in \mathrm{Obj}(\mathcal{L})$ satisfying ${}_Fx_G \neq 0$ or ${}_Gx_F \neq 0$. We make ${}^{\Pi}\mathcal{L}$ become a unital algebra with multiplication operation such that

$$ {}_F(xy)_G = \sum_{G\in \mathrm{Obj}(\mathcal{L})} {}_Fx_G y_H $$

where $F, H \in \mathrm{Obj}(\mathcal{L})$ and $x, y \in {}^{\Pi}\mathcal{L}$ and ${}_F x_G y_H = {}_F x_G \cdot {}_G y_H$. The sum makes sense because only finitely many of the terms are non-zero. We call ${}^{\Pi}\mathcal{L}$ the **extended quiver algebra** of $\mathcal{L}$. The rationale for the term will become apparent later in this section.

A family $(x_i : i \in I)$ of elements $x_i \in {}^{\Pi}\mathcal{L}$ is said to be **summable** provided, for each $F \in \mathrm{Obj}(\mathcal{L})$, there are only finitely many $i \in I$ and $G \in \mathrm{Obj}(\mathcal{L})$ such that ${}_F(x_i)_G \neq 0$ or ${}_G(x_i)_F \neq 0$. In that case, we define the **sum** $\sum_i x_i \in {}^{\Pi}\mathcal{L}$ to be such that its (F, G)-coordinate is ${}_F(\sum_i x_i)_G = \sum_i {}_F(x_i)_G$. Any element $x \in {}^{\Pi}\mathcal{L}$ can be written as a sum

$$x = \sum_{F,G \in \mathrm{Obj}(\mathcal{L})} {}_F x_G \, .$$

The unity element of ${}^{\Pi}\mathcal{L}$ is the sum

$$1_{\mathcal{L}} = \sum_{G \in \mathrm{Obj}(\mathcal{L})} \mathrm{id}_G \, .$$

Proof of the next remark is straightforward.

Remark 2.1. Any element z of the centre $Z({}^{\Pi}\mathcal{L})$ can be expressed as a sum

$$z = \sum_{G \in \mathrm{Obj}(\mathcal{L})} z_G$$

where $z_G \in \mathcal{L}(G, G)$. Conversely, given elements $z_G \in \mathcal{L}(G, G)$ defined for each $G \in \mathrm{Obj}(\mathcal{L})$, then we can form the sum $z \in {}^{\Pi}\mathcal{L}$ as above, whereupon $z \in Z({}^{\Pi}\mathcal{L})$ if and only if, for all $F, G \in \mathrm{Obj}(\mathcal{L})$ and $x \in \mathcal{L}(F, G)$, we have $z_F x = x z_G$.

We define a **block** of a unital ring Λ to be a primitive idempotent of $Z(\Lambda)$. Let $\mathrm{blk}(\Lambda)$ denote the set of blocks of Λ. It is easy to see that $Z(\Lambda)$ has finitely many idempotents if and only if Λ has finitely many blocks and the sum of the blocks is the unity element 1_Λ. In that case, we say that Λ has a **finite block decomposition**. We define a **block** of $\mathcal{L}$ to be a block of ${}^{\Pi}\mathcal{L}$.

Theorem 2.2. *If the algebra $\mathcal{L}(G, G) = \mathrm{End}_{\mathcal{L}}(G)$ has a finite block decomposition for all $G \in \mathrm{Obj}(\mathcal{L})$, then*

$$1_{\mathcal{L}} = \sum_{b \in \mathrm{blk}(\mathcal{L})} b \, .$$

Proof. We adapt the proof of Boltje–Külshammer [3, 5.4]. Let

$$\mathcal{E} = \bigcup_{G \in \mathrm{Obj}(\mathcal{L})} \mathrm{blk}(\mathcal{L}(G, G)) \, .$$

Let $\sim$ be the reflexive symmetric relation on $\mathcal{E}$ such that, given $F, G \in \mathrm{Obj}(\mathcal{L})$ and $d \in \mathrm{blk}(\mathcal{L}(F,F))$ and $e \in \mathrm{blk}(\mathcal{L}(G,G))$, then $d \sim e$ provided $d\mathcal{L}(F,G)e \neq \{0\}$ or $e\mathcal{L}(G,F)f \neq \{0\}$. Let $\equiv$ be the transitive closure of $\sim$. We mean to say, $\equiv$ is the equivalence relation such that $d \equiv e$ if and only if there exist elements $f_0, \ldots, f_n \in \mathcal{E}$ such that $f_0 = d$ and $f_n = e$ and each $f_{i-1} \sim f_i$. The hypothesis on the algebra $\mathcal{L}(G,G)$ implies that every subset of $\mathcal{E}$ is summable. Plainly, $1_{\mathcal{L}} = \sum_{e \in \mathcal{E}} e$. It suffices to show that there is a bijective correspondence between the equivalence classes E under $\equiv$ and the blocks b of $\mathcal{L}$ such that $E \leftrightarrow b$ provided $b = \sum_{e \in E} e$.

Let E be an equivalence class under $\equiv$ and let $b = \sum_{e \in E} e$. We must show that b is a block of $\mathcal{L}$. Plainly, b is an idempotent of ${}^{\Pi}\mathcal{L}$. Given $F, G \in \mathrm{Obj}(\mathcal{L})$ and $x \in \mathcal{L}(F,G)$, then

$$b_F x = b_F x 1_{\mathcal{L}} = \big(\sum_{d \in E_F} d\big)\, x \,\big(\sum_{e \in \mathrm{blk}(\mathcal{L}(G,G))} e\big) = \sum_{d \in E_F,\, e \in E_G} dxe = x b_G$$

where $E_F = E \cap \mathrm{blk}(\mathcal{L}(F,F))$. So, by Remark 2.1, $b \in Z({}^{\Pi}\mathcal{L})$. Suppose that $b = b_1 + b_2$ as a sum of orthogonal idempotents of $Z({}^{\Pi}\mathcal{L})$ with $b_1 \neq 0$. Since $bb_1 \neq 0$, there exist $F \in \mathrm{Obj}(\mathcal{L})$ and $d \in E_F$ such that $db_1 \neq 0$. We have $db_1 = d(b_1)_F = d$ because $(b_1)_F$ is a central idempotent of $\mathcal{L}(F,F)$. For all $G \in \mathrm{Obj}(\mathcal{L})$ and $e \in E_G$, we have

$$d\mathcal{L}(F,G)b_1 e = db_1\mathcal{L}(F,G)e = d\mathcal{L}(F,G)e\ .$$

So, if $d\mathcal{L}(F,G)e \neq \{0\}$, then $b_1 e \neq 0$, whereupon, by an argument above, $b_1 e = e$. Similarly, the condition $e\mathcal{L}(G,F)d \neq \{0\}$ implies that $b_1 e = e$. We deduce that $b_1 e = e$ for all $e \in E$. Therefore, $b_1 = b$ and $b_2 = 0$. We have shown that b is a block of $\mathcal{L}$.

Conversely, given a block b of $\mathcal{L}$, letting $f \in \mathcal{E}$ such that $bf \neq 0$ and letting E be the equivalence class of f, then $b\sum_{e \in E} e \neq 0$, hence b coincides with the block $\sum_{e \in E} e$. We have established the bijective correspondence $E \leftrightarrow b$, as required. □

As a subalgebra of ${}^{\Pi}\mathcal{L}$, we define

$${}^{\oplus}\mathcal{L} = \bigoplus_{F,G \in \mathrm{Obj}(\mathcal{L})} \mathcal{L}(F,G)\ .$$

We call ${}^{\oplus}\mathcal{L}$ the **quiver algebra** of $\mathcal{L}$. When no ambiguity can arise, we write $\mathcal{L} = {}^{\oplus}\mathcal{L}$. Plainly, the following three conditions are equivalent: $\mathrm{Obj}(\mathcal{L})$ is finite; the algebra $\mathcal{L}$ is unital; we have an equality of algebras $\mathcal{L} = {}^{\Pi}\mathcal{L}$.

We define an **$\mathcal{L}$-functor** to be an R-linear functor $\mathcal{L} \to R$–Mod. Given an $\mathcal{L}$-functor M, we can form a ${}^{\Pi}\mathcal{L}$-module $M_{\Pi} = \bigoplus_G M(G)$ where an element $x \in \mathcal{L}(F,G)$ acts on M_{Π} as $M(x)$, annihilating $M(G')$ for all objects G' distinct from G. By restriction, we obtain an $\mathcal{L}$-module $M_{\oplus}$. Note that $\mathcal{L}M_{\Pi} = M_{\Pi}$, in other words, $\mathcal{L}M_{\oplus} = M_{\oplus}$. Given another $\mathcal{L}$-functor M', then each natural transformation $M \to M'$ gives rise, in an evident way, to a ${}^{\Pi}\mathcal{L}$-map $M_{\Pi} \to M'_{\Pi}$ which is also an $\mathcal{L}$-map $M_{\oplus} \to M'_{\oplus}$. Conversely,

the $\mathcal{L}$-maps $M_\oplus \to M'_\oplus$ coincide with the ${}^\Pi\mathcal{L}$-maps $M_\Pi \to M'_\Pi$ and give rise to natural transformations $M \to M'$. Putting the constructions in reverse, given an $\mathcal{L}$-module $M_\oplus$ such that $\mathcal{L}M_\oplus = M_\oplus$, we can extend $M_\oplus$ to a ${}^\Pi\mathcal{L}$-module M_Π and we can also form an $\mathcal{L}$-functor M such that $M(G) = \mathrm{id}_G M_\oplus = \mathrm{id}_G M_\Pi$. Henceforth, we shall neglect to distinguish between M and M_Π and $M_\oplus$. That is to say, we identify the category of $\mathcal{L}$ functors with the category of ${}^\Pi\mathcal{L}$-modules M satisfying $\mathcal{L}M = M$ and with the category of $\mathcal{L}$-modules M satisfying $\mathcal{L}M = M$.

An $\mathcal{L}$-functor M is said to **belong to** a block b of $\mathcal{L}$ provided $bM = M$. In that case, we also say that b **owns** M. Theorem 2.2 has the following immediate corollary.

Corollary 2.3. *If $\mathcal{L}(G,G)$ has a finite block decomposition for all $G \in \mathrm{Obj}(\mathcal{L})$, then every indecomposable $\mathcal{L}$-functor belongs to a unique block of $\mathcal{L}$.*

Proof. Let M be an indecomposable $\mathcal{L}$-functor. Choose an object G of $\mathcal{L}$ such that $M(G) \neq 0$. We have $\mathrm{id}_G = \sum_{b \in \mathrm{blk}(\mathcal{L})} b_G$ as a sum with only finitely many non-zero terms. So $b_G M(G) \neq 0$ for some b. In particular, $bM \neq 0$. But $M = bM \oplus (1-b)M$ and M is indecomposable, so $M = bM$. □

The next three results describe how the simple $\mathcal{L}$-functors and the blocks of $\mathcal{L}$ are related to the simple functors and blocks of a full subcategory of $\mathcal{L}$.

Proposition 2.4. *Let $\mathcal{K}$ be a full subcategory of $\mathcal{L}$. Then there is a bijective correspondence between the isomorphism classes of simple $\mathcal{K}$-functors S and the isomorphism classes of simple $\mathcal{L}$ functors T such that $1_\mathcal{K} T \neq 0$. The correspondence is such that $S \leftrightarrow T$ provided $S \cong 1_\mathcal{K} T$.*

Proof. We have ${}^\Pi\mathcal{K} = 1_\mathcal{K} .{}^\Pi\mathcal{L} . 1_\mathcal{K}$. So the assertion is a special case of Green [6, 6.2] which says that, given an idempotent i of a unital ring Λ, then the condition $S \cong iT$ characterizes a bijective correspondence between the isomorphism classes of simple $i\Lambda i$-modules S and the isomorphism classes of simple Λ-modules T satisfying $iT \neq 0$. □

Proposition 2.5. *Suppose that $\mathcal{L}(G,G)$ has a finite block decomposition for all $G \in \mathrm{Obj}(\mathcal{L})$. Let $\mathcal{K}$ be a full subcategory of $\mathcal{L}$ and let S and S' be simple $\mathcal{K}$-functors. Let T and T' be the isomorphically unique simple $\mathcal{L}$-functors such that $S \cong 1_\mathcal{K} T$ and $S' \cong 1_\mathcal{K} T'$. If S and S' belong to the same block of $\mathcal{K}$, then T and T' belong to the same block of $\mathcal{L}$.*

Proof. Let a and a' be the blocks of $\mathcal{K}$ owning S and S', respectively. Let b and b' be the blocks of $\mathcal{L}$ owning T and T', respectively. The central idempotent $b1_\mathcal{K}$ of ${}^\Pi\mathcal{K}$ acts as the identity on S, so $ab = a$. Similarly, $a'b' = a'$. If $a = a'$ then $abb' = a \neq 0$, hence $bb' \neq 0$, which implies that $b = b'$. □

Proposition 2.6. *Suppose that $\mathcal{L}(G,G)$ has a finite block decomposition for all $G \in \mathrm{Obj}(\mathcal{L})$. Let T and T' be simple $\mathcal{L}$-functors. Then T and T' belong to the same block*

of $\mathcal{L}$ if and only if there exists a full R-linear subcategory $\mathcal{K}$ of $\mathcal{L}$ such that $\mathrm{Obj}(\mathcal{K})$ *is finite and the simple $\mathcal{K}$-functors $1_{\mathcal{K}}T$ and $1_{\mathcal{K}}T'$ are non-zero and belong to the same block of $\mathcal{K}$.*

Proof. In one direction, this is immediate from the previous proposition. Conversely, suppose that T and T' belong to the same block b of $\mathcal{L}$. Let $G, G' \in \mathrm{Obj}(\mathcal{L})$ such that $T(G) \neq 0$ and $T'(G') \neq 0$. Let $e \in \mathrm{blk}(\mathcal{L}(G,G))$ and $e' \in \mathrm{blk}(\mathcal{L}(G',G'))$ be such that $eT(G) \neq 0$ and $e'T'(G') \neq 0$. Since $ebT(G) = eT(G)$, we have $eb \neq 0$. Similarly, $e'b \neq 0$. Therefore $e \equiv e'$ where $\equiv$ is the equivalence relation in the proof of Theorem 2.2. So there exist $G_0, \ldots, G_n \in \mathrm{Obj}(\mathcal{L})$ and $f_i \in \mathrm{blk}(\mathcal{L}(G_i, G_i))$ such that $G_0 = G$, $f_0 = e$, $G_n = G'$, $f_n = e'$ and each $f_{i-1} \sim f_i$. Let $\mathcal{K}$ be the full subcategory of $\mathcal{L}$ such that $\mathrm{Obj}(\mathcal{K}) = \{G_0, \ldots, G_n\}$. Then e and e' are still equivalent under the equivalence relation associated with $\mathcal{K}$. By the proof of Theorem 2.2, there exists a block a of $\mathcal{K}$ such that $ae = e$ and $ae' = e'$. We have $ea1_{\mathcal{K}}T = e1_{\mathcal{K}}T = eT(G) \neq 0$, hence $a1_{\mathcal{K}}T \neq 0$ and, similarly, $a1_{\mathcal{K}}T' \neq 0$. Therefore $1_{\mathcal{K}}T$ and $1_{\mathcal{K}}T'$ both belong to a. □

3. Mackey categories and their simple functors

We shall introduce the notions of a Mackey system and a Mackey category. We shall also classify the simple functors of the R-linear extension of a given Mackey category.

First, let us briefly recall some features of the biset category $\mathcal{C}$. Details can be found in Bouc [4, Chapters 2, 3]. Let F, G, H be finite groups. The biset category $\mathcal{C}$ is a $\mathbb{Z}$-linear category whose class of objects is the class of finite groups. The $\mathbb{Z}$-module of morphisms $F \leftarrow G$ in $\mathcal{C}$ is

$$\mathcal{C}(F,G) = B(F \times G) = \bigoplus_{A \leq_G F \times G} \mathbb{Z}[(F \times G)/A]$$

where B indicates the Burnside ring, the index A runs over representatives of the conjugacy classes of subgroups of $F \times G$ and $[(F \times G)/A]$ denotes the isomorphism class of the F–G-biset $(F \times G)/A$. The morphisms having the form $[(F \times G)/A]$ are called **transitive morphisms**. The composition operation for $\mathcal{C}$ is defined in [4, 2.3.11, 3.1.1]. A useful formula for the composition operation is

$$\left[\frac{F \times G}{A}\right]\left[\frac{G \times H}{B}\right] = \sum_{p_2(A)gp_1(B) \subseteq G} \left[\frac{F \times H}{A *^{(g,1)} B}\right] .$$

Here, the notation indicates that g runs over representatives of the double cosets of $p_2(A)$ and $p_1(B)$ in G. For an account of the formula and for specification of the rest of the notation appearing in it, see [4, 2.3.24].

Given a group homomorphism $\alpha : F \leftarrow G$, we define transitive morphisms

$${}_F\mathrm{ind}_G^\alpha = [(F \times G)/\{(\alpha(g), g) : g \in G\}] , \qquad {}_G\mathrm{res}_F^\alpha = [(G \times F)/\{(g, \alpha(g)) : g \in G\}]$$

called **induction** and **restriction**. The composite of two inductions is an induction and the composite of two restrictions is a restriction. Indeed, using the above formula for the composition operation, it is easy to see that, given a group homomorphism $\beta : G \leftarrow H$ then,

$$ {}_F\mathrm{ind}_G^\alpha \mathrm{ind}_H^\beta = {}_F\mathrm{ind}_H^{\alpha\beta} \ , \qquad {}_H\mathrm{res}_G^\beta \mathrm{res}_F^\alpha = {}_H\mathrm{res}_F^{\alpha\beta} \ . $$

When α is injective, we call ${}_F\mathrm{ind}_G^\alpha$ an **ordinary induction** and we call ${}_G\mathrm{res}_F^\alpha$ an **ordinary restriction**. When α is an inclusion $F \leftarrow\!\shortmid G$, we omit the symbol α from the notation, just writing ${}_F\mathrm{ind}_G$ and ${}_G\mathrm{res}_F$. When α is surjective, we write

$$ {}_F\mathrm{def}_G^\alpha = {}_F\mathrm{ind}_G^\alpha \ , \qquad {}_G\mathrm{inf}_F^\alpha = {}_G\mathrm{res}_F^\alpha $$

which we call **deflation** and **inflation**. Note that, for arbitrary α, we have factorizations

$$ {}_F\mathrm{ind}_G^\alpha = {}_F\mathrm{ind}_{\alpha(G)}\mathrm{def}_G^\alpha \ , \qquad {}_G\mathrm{res}_F^\alpha = {}_G\mathrm{inf}_{\alpha(G)}^\alpha \mathrm{res}_F \ . $$

When α is an isomorphism, we write

$$ {}_F\mathrm{iso}_G^\alpha = {}_F\mathrm{ind}_G^\alpha = {}_F\mathrm{res}_G^{\alpha^{-1}} $$

which we call **isogation**. In $\mathcal{C}$, the identity morphism on G is the isogation $\mathrm{iso}_G = {}_G\mathrm{iso}_G^1$. Given $g \in G$, we let $c(g)$ denote left-conjugation by g. Let $V, V' \leq G$. Again using the above formula for composition, we recover the familiar Mackey relation

$$ {}_V\mathrm{res}_G \mathrm{ind}_{V'} = \sum_{VgV' \subseteq G} {}_V\mathrm{ind}_{V \cap {}^gV'} \mathrm{iso}_{V^g \cap V'}^{c(g)} \mathrm{res}_{V'} \ . $$

A transitive morphism $\tau : F \leftarrow G$ is said to be **left-free** provided τ is the isomorphism class of an F-free F–G-biset. The left-free transitive morphisms $F \leftarrow G$ are the morphisms that can be expressed in the form

$$ {}_F\mathrm{ind}_V^\alpha \mathrm{res}_G = {}_F\mathrm{ind}_{\alpha(V)} \mathrm{def}_V^\alpha \mathrm{res}_G = \left[\frac{F \times G}{\mathfrak{S}(\alpha, V)} \right] $$

where $V \leq G$ and $\alpha : F \leftarrow V$ and

$$ \mathfrak{S}(\alpha, V) = \{ (\alpha(v), v) : v \in V \} \ . $$

Evidently, the left-free transitive morphisms are those transitive morphism which can be expressed as the composite of an ordinary induction, a deflation and an ordinary restriction. The right-free transitive morphisms, defined similarly, are those transitive morphisms which can be expressed as the composite of an ordinary induction, an inflation and an ordinary restriction.

Proposition 3.1 *(Mackey relation for left-free transitive morphisms). Let F and $V \leq G$ and $W \leq H$ be finite groups. Let $\alpha : F \leftarrow V$ and $\beta : G \leftarrow W$ be group homomorphisms. Then*

$$ {}_F\mathrm{ind}_V^\alpha \mathrm{res}_G \mathrm{ind}_W^\beta \mathrm{res}_H = \sum_{VG\beta(W)\subseteq G} {}_F\mathrm{ind}_{\beta^{-1}(V^g)}^{\alpha_g c(g) \beta_g} \mathrm{res}_H $$

where $\alpha_g : F \leftarrow V \cap {}^g\beta(W)$ and $\beta_g : V^g \cap \beta(W) \leftarrow \beta^{-1}(V^g)$ are restrictions of α and β.

Proof. Using the star-product notation of Bouc [4, 2.3.19],

$$ \{(v, v) : v \in V\} * {}^{(g,1)}\mathfrak{S}(\beta, W) = \mathfrak{S}(c(g)\beta_g, \beta^{-1}(V^g)) \, . $$

Hence ${}_V\mathrm{res}_G \mathrm{ind}_W^\beta = \sum_{VG\beta(W)} {}_V\mathrm{ind}_{\beta^{-1}(V^g)}^{c(g)\beta_g} \mathrm{res}_W$. □

As in Section 1, let $\mathfrak{K}$ be a set of finite groups that is closed under taking subgroups. We define a **Mackey system** on $\mathfrak{K}$ to be a category $\mathcal{F}$ such that the objects of $\mathcal{F}$ are the groups in $\mathfrak{K}$, every morphism in $\mathcal{F}$ is a group homomorphism, composition is the usual composition of homomorphisms, and the following four axioms hold:

MS1: For all $V \leq G \in \mathfrak{K}$, the inclusion $G \hookleftarrow V$ is in $\mathcal{F}$.
MS2: For all $V \leq G \in \mathfrak{K}$ and $g \in G$, the conjugation map ${}^gV \ni {}^gv \hookleftarrow v \in V$ is in $\mathcal{F}$.
MS3: For any morphism $\alpha : F \leftarrow G$ in $\mathcal{F}$, the associated homomorphism $\alpha(G) \leftarrow G$ is in $\mathcal{F}$.
MS4: For any morphism α in $\mathcal{F}$ such that α is a group isomorphism, α^{-1} is in $\mathcal{F}$.

We call $\mathcal{F}$ an **ordinary Mackey system** provided all the morphisms in $\mathcal{F}$ are injective. As an example, a fusion system on a finite p-group P is precisely the same thing as an ordinary Mackey system on the set of subgroups of P.

Remark 3.2. Given a Mackey system $\mathcal{F}$ on $\mathfrak{K}$, then:
(1) There exists a linear subcategory $\mathcal{M}_{\mathcal{F}}^{\leftarrow}$ of $\mathcal{C}$ such that $\mathrm{Obj}(\mathcal{M}_{\mathcal{F}}^{\leftarrow}) = \mathfrak{K}$ and, for $F, G \in \mathfrak{K}$, the morphisms $F \leftarrow G$ in $\mathcal{M}_{\mathcal{F}}^{\leftarrow}$ are the linear combinations of the left-free transitive morphisms ${}_F\mathrm{ind}_V^\alpha \mathrm{res}_G$ where $V \leq G$ and $\alpha : F \leftarrow V$ is a morphism in $\mathcal{F}$.
(2) There exists a linear subcategory $\mathcal{M}_{\mathcal{F}}^{\rightarrow}$ of $\mathcal{C}$ such that $\mathrm{Obj}(\mathcal{M}_{\mathcal{F}}^{\rightarrow}) = \mathfrak{K}$ and, for $F, G \in \mathfrak{K}$, the morphisms $F \leftarrow G$ in $\mathcal{M}_{\mathcal{F}}^{\rightarrow}$ are the linear combinations of the right-free transitive morphisms ${}_F\mathrm{ind}_U \mathrm{res}_G^\beta$ where $U \leq F$ and $\beta : U \to G$ is a morphism in $\mathcal{F}$.

Proof. In the notation of Proposition 3.1, supposing that $F, G, H \in \mathfrak{K}$ and that α and β are morphisms in $\mathcal{F}$ then, by axioms MS1 and MS3, each α_g and β_g are in $\mathcal{F}$ and, by axiom MS2, each $c(g)$ is in $\mathcal{F}$. Part (1) is established. Part (2) can be demonstrated similarly or by considering duality. □

We call $\mathcal{M}_{\mathcal{F}}^{\leftarrow}$ the **deflation Mackey category** of $\mathcal{F}$. The rationale for the terminology is that $\mathcal{M}_{\mathcal{F}}^{\leftarrow}$ is generated by inductions from subgroups, restrictions to subgroups and

deflations coming from surjections in $\mathcal{F}$. We call $\mathcal{M}^{\rightarrow}_{\mathcal{F}}$ the **inflation Mackey category** of $\mathcal{F}$.

Remark 3.3. Given an ordinary Mackey system $\mathcal{G}$, then $\mathcal{M}^{\leftarrow}_{\mathcal{G}} = \mathcal{M}^{\rightarrow}_{\mathcal{G}}$.

Proof. This follows from axiom MS4. □

The category $\mathcal{M}_{\mathcal{G}} = \mathcal{M}^{\leftarrow}_{\mathcal{G}} = \mathcal{M}^{\rightarrow}_{\mathcal{G}}$ is called an **ordinary Mackey category**.

For the rest of this section, we focus on the deflation Mackey category $\mathcal{M}^{\leftarrow}_{\mathcal{F}}$. Similar constructions and arguments yield similar results for the inflation Mackey category $\mathcal{M}^{\rightarrow}_{\mathcal{F}}$. We shall need some notation for extension to coefficients in R. Given a $\mathbb{Z}$-module A, we write $RA = R \otimes_{\mathbb{Z}} A$. Given a $\mathbb{Z}$-map $\theta : A \to A'$, we abuse notation, writing the R-linear extension as $\theta : RA \to RA'$. Given a $\mathbb{Z}$-linear category $\mathcal{L}$, we write $R\mathcal{L}$ to denote the R-linear category such that $(R\mathcal{L})(F, G) = R(\mathcal{L}(F, G))$ for $F, G \in \mathrm{Obj}(\mathcal{L})$.

Remark 3.4. Given a Mackey system $\mathcal{F}$ on $\mathfrak{K}$ and $F, G \in \mathfrak{K}$, then the following three conditions are equivalent: that F and G are isomorphic in $\mathcal{F}$; that F and G are isomorphic in $\mathcal{M}^{\leftarrow}_{\mathcal{F}}$; that F and G are isomorphic in $R\mathcal{M}^{\leftarrow}_{\mathcal{F}}$.

Proof. Given an isomorphism $\gamma : F \leftarrow G$ in $\mathcal{F}$, then ${}_F\mathrm{iso}^{\gamma}_G : F \leftarrow G$ is an isomorphism in $\mathcal{M}^{\leftarrow}_{\mathcal{F}}$. So the first condition implies the second. Trivially, the second condition implies the third. Assume the third condition. Let $\theta : F \leftarrow G$ and $\phi : G \leftarrow F$ be mutually inverse isomorphisms in $R\mathcal{M}^{\leftarrow}_{\mathcal{F}}$. Writing $\theta = \sum_i \lambda_i \theta_i$ and $\phi = \sum_j \mu_j \phi_j$ as linear combinations of transitive morphisms θ_i and ϕ_j, then $\mathrm{iso}_F = \theta\phi = \sum_{i,j} \lambda_i \mu_j \theta_i \phi_j$. An argument in Bouc [4, 4.3.2], making use of [4, 2.3.22], implies that θ_i and ϕ_j are isogations for some i and j. We have deduced the first condition. □

For $F, G \in \mathfrak{K}$, we write $\mathcal{F}(F, G)$ to denote the set of morphisms $F \leftarrow G$ in $\mathcal{F}$. We make $\mathcal{F}(F, G)$ become an $F \times G$-set such that

$$ {}^{(f,g)}\alpha = c(f)\, \alpha\, c(g^{-1}) $$

for $(f, g) \in F \times G$ and $\alpha \in \mathcal{F}(F, G)$. Since $\alpha\, c(g^{-1}) = c(\alpha(g^{-1}))\, \alpha$, the $F \times G$-orbits of $\mathcal{F}(F, G)$ coincide with the F-orbits. Let $\overline{\alpha}$ denote the F-orbit of α. We have $\overline{\alpha}\,\overline{\beta} = \overline{\alpha\beta}$ for $H \in \mathfrak{K}$ and $\beta \in \mathcal{F}(G, H)$. So we can form a quotient category $\overline{\mathcal{F}}$ of $\mathcal{F}$ such that the set of morphisms $F \leftarrow G$ in $\overline{\mathcal{F}}$ is $\overline{\mathcal{F}}(F, G) = \{\overline{\alpha} : \alpha \in \mathcal{F}(F, G)\}$. In $\overline{\mathcal{F}}$, the automorphism group of G is

$$ \mathrm{Out}_{\mathcal{F}}(G) = \mathrm{Aut}_{\mathcal{F}}(G)/\mathrm{Inn}(G) $$

where $\mathrm{Inn}(G)$ denotes the group of inner automorphisms of G.

Remark 3.5. Let $\mathcal{F}$ be a Mackey system on $\mathfrak{K}$. Given $F, G \in \mathfrak{K}$ and $\alpha, \alpha' \in \mathcal{F}(F,G)$, then the following three conditions are equivalent: that ${}_F\mathrm{ind}_G^\alpha = {}_F\mathrm{ind}_G^{\alpha'}$; that ${}_G\mathrm{res}_F^\alpha = {}_G\mathrm{res}_F^{\alpha'}$; that $\overline{\alpha} = \overline{\alpha'}$.

Proof. Another equivalent condition is $\mathfrak{S}(\alpha, G) =_{F\times G} \mathfrak{S}(\alpha', G)$. □

Let $\mathcal{P}^{\mathcal{F}}_{F,G}$ denote the set of pairs (α, V) where $V \leq G$ and $\alpha \in \mathcal{F}(F,V)$. We allow $F \times G$ to act on $\mathcal{P}^{\mathcal{F}}_{F,G}$ such that

$$^{(f,g)}(\alpha, V) = ({}_{(f,g)}\alpha, {}^gV)$$

for $f \in F$ and $g \in G$. Let $\overline{\mathcal{P}}^{\mathcal{F}}_{F,G}$ denote the set of $F\times G$-orbits in $\mathcal{P}^{\mathcal{F}}_{F,G}$. Let $[\alpha, V]$ denote the $F\times G$-orbit of (α, V).

Proposition 3.6. *Let $\mathcal{F}$ be a Mackey system on $\mathfrak{K}$. Then, for $F, G \in \mathfrak{K}$, the R-module of morphisms $F \leftarrow G$ in $R\mathcal{M}^{\leftarrow}_{\mathcal{F}}$ is*

$$R\mathcal{M}^{\leftarrow}_{\mathcal{F}}(F,G) = \bigoplus_{[\alpha,V]\in\overline{\mathcal{P}}^{\mathcal{F}}_{F,G}} R \cdot {}_F\mathrm{ind}_V^\alpha \mathrm{res}_G \, .$$

Proof. For $V, V' \leq G$ and $\overline{\alpha} \in \overline{\mathcal{F}}(F,V)$ and $\overline{\alpha'} \in \overline{\mathcal{F}}(F,V')$, we have ${}_F\mathrm{ind}_V^\alpha \mathrm{res}_G = {}_F\mathrm{ind}_{V'}^{\alpha'} \mathrm{res}_G$ if and only if $\mathfrak{S}(\alpha, V) = \mathfrak{S}(\alpha', V')$, in other words, $[\alpha, V] = [\alpha', V']$. □

We define a **seed** for $\mathcal{F}$ over R to be a pair (G, V) where $G \in \mathfrak{K}$ and V is a simple $R\mathrm{Out}_{\mathcal{F}}(G)$-module. Two seeds (F, U) and (G, V) for $\mathcal{F}$ over R are said to be **equivalent** provided there exist an $\mathcal{F}$-isomorphism $\gamma : F \leftarrow G$ and an R-isomorphism $\phi : U \leftarrow V$ such that, given $\overline{\eta} \in \mathrm{Out}_{\mathcal{F}}(G)$, then $\overline{\gamma\eta\gamma^{-1}} \circ \phi = \phi \circ \overline{\eta}$.

The next result is different in context but similar in form to the classifications of simple functors in Thévenaz–Webb [9, Section 2], Bouc [4, 4.3.10], Díaz–Park [5, 3.2]. It can be proved by similar methods. It is also a special case of [1, 3.7]. Observe that, given $G \in \mathfrak{K}$ and an $R\mathcal{M}^{\leftarrow}_{\mathcal{F}}$-functor M, then $M(G)$ becomes an $R\mathrm{Out}_{\mathcal{F}}(G)$-module such that an element $\overline{\eta} \in \mathrm{Out}_{\mathcal{F}}(G)$ acts as ${}_G\mathrm{iso}_G^\eta$. We call G a **minimal group** for M provided $M(G) \neq 0$ and $M(F) = 0$ for all $F \in \mathfrak{K}$ with $|F| < |G|$.

Theorem 3.7. *Let $\mathcal{F}$ be a Mackey system on $\mathfrak{K}$ and let $\mathcal{M} = \mathcal{M}^{\leftarrow}_{\mathfrak{K}}$. Given a seed (G, V) for $\mathcal{F}$ over R, then there is a simple $R\mathcal{M}$-functor $S^{R\mathcal{M}}_{G,V}$ determined up to isomorphism by the condition that G is a minimal group for $S^{R\mathcal{M}}_{G,V}$ and $S^{R\mathcal{M}}_{G,V}(G) \cong V$ as $R\mathrm{Out}_{\mathcal{F}}(G)$-modules. The equivalence classes of seeds (G, V) for $\mathcal{F}$ over R are in a bijective correspondence with the isomorphism classes of simple $R\mathcal{M}$-functors S such that $(G, V) \leftrightarrow S$ provided $S \cong S^{R\mathcal{M}}_{G,V}$.*

4. Ordinary Mackey categories and semisimplicity

Throughout this section, we let $\mathcal{G}$ be an ordinary Mackey system on $\mathfrak{K}$. We shall consider the ordinary Mackey category $\mathcal{N} = \mathcal{M}_{\mathcal{G}}$. Recall, from Section 1, that $\mathbb{K}$ is a field of characteristic zero. We shall prove that the $\mathbb{K}$-linear category $\mathbb{K}\mathcal{N}$ has a semisimplicity property. As mentioned in Section 1, this conclusion was obtained by Webb [10, 9.5] in a special case and by Thévenaz–Webb [8], [9, 3.5] in scenario involving a fixed finite group. Another related result, with a different conclusion but in a similar scenario, is Boltje–Danz [2, 5.8], which says that the algebra $\mathbb{K}\mathcal{N}(G,G)$ is semisimple for all $G \in \mathfrak{K}$.

Let us discuss, in abstract, the semisimplicity property that we shall be establishing.

Remark 4.1. Given an R-linear category $\mathcal{L}$, then the following two conditions are equivalent:
(a) For every full linear subcategory $\mathcal{L}_0$ of $\mathcal{L}$ with only finitely many objects, the quiver algebra $\mathcal{L}_0$ is semisimple.
(b) The algebra $i\mathcal{L}i$ is semisimple for every idempotent i of the quiver algebra $\mathcal{L}$.

Proof. If each $i\mathcal{L}i$ is semisimple then, given $\mathcal{L}_0$, we have $\mathcal{L}_0 = 1_{\mathcal{L}_0} \,.\, \mathcal{L} \,.\, 1_{\mathcal{L}_0}$, which is semisimple. Conversely, suppose that each $\mathcal{L}_0$ is semisimple. Given i, let $\mathcal{L}_0$ be a subcategory of $\mathcal{L}$ such that $\mathrm{Obj}(\mathcal{L}_0)$ is finite and i has the form $i = \sum_{F,G \in \mathrm{Obj}(\mathcal{L}_0)} {}_F i_G$ with each ${}_F i_G \in \mathcal{L}(F,G)$. Then $1_{\mathcal{L}_0} i = i = i 1_{\mathcal{L}_0}$. Since the algebra $\mathcal{L}_0 = 1_{\mathcal{L}_0} \,.\, \mathcal{L} \,.\, 1_{\mathcal{L}_0}$ is semisimple, the algebra $i\mathcal{L}i = i1_{\mathcal{L}_0} \,.\, \mathcal{L} \,.\, 1_{\mathcal{L}_0} i$ is semisimple. □

When the equivalent conditions in the remark hold, we say that $\mathcal{L}$ is **locally semisimple**. In Theorem 4.6, we shall prove that the $\mathbb{K}$-linear category $\mathbb{K}\mathcal{N}$ is locally semisimple.

For $G, H \in \mathfrak{K}$, let $L(G,H)$ be the $\mathbb{Z}$-module freely generated by the formal symbols ${}_G\overline{\mathrm{ind}}_H^{\beta}$ where $\overline{\beta}$ runs over the elements of $\overline{\mathcal{G}}(G,H)$. It is to be understood that ${}_G\overline{\mathrm{ind}}_H^{\beta} = {}_G\overline{\mathrm{ind}}_H^{\beta'}$ if and only if $\overline{\beta} = \overline{\beta'}$. Thus

$$L(G,H) = \bigoplus_{\overline{\beta} \in \overline{\mathcal{G}}(G,H)} \mathbb{Z}\, {}_G\overline{\mathrm{ind}}_H^{\beta} \,.$$

We define a $\mathbb{Z}$-module

$$L = \bigoplus_{G,H \in \mathfrak{K}} L(G,H) \,.$$

We define a $\mathbb{Z}$-epimorphism $\pi : \mathcal{N} \to L$ such that, given $W \leq H$ and $\beta \in \mathcal{G}(G,W)$, then

$$\pi({}_G\mathrm{ind}_W^{\beta}\mathrm{res}_H) = \begin{cases} {}_G\overline{\mathrm{ind}}_W^{\beta} & \text{if } W = H, \\ 0 & \text{if } W < H. \end{cases}$$

By Proposition 3.1, $\ker(\pi)$ is a left ideal of $\mathcal{N}$. We make L become an $\mathcal{N}$-module with representation $\sigma : \mathcal{N} \to \mathrm{End}_{\mathbb{Z}}(L)$ such that $\sigma(m)\pi(x) = \pi(mx)$ for $m, x \in \mathcal{N}$. The next lemma expresses the action of $\mathcal{N}$ more explicitly.

Lemma 4.2. *For $F, G, H \in \mathfrak{K}$, $V \leq G$, $\alpha \in \mathcal{G}(F, V)$, $\beta \in \mathcal{G}(G, H)$, we have*

$$\sigma({}_F\mathrm{ind}_V^{\alpha}\mathrm{res}_G){}_G\overline{\mathrm{ind}}_H^{\beta} = \sum_{Vg\beta(H) \subseteq G \,:\, V \geq {}^g\beta(H)} {}_F\overline{\mathrm{ind}}_H^{\alpha c(g)\beta} .$$

Proof. This follows from Proposition 3.1. □

Let $\mathcal{I}$ be the linear subcategory of $\mathcal{N}$ generated by the isogations. That is to say, the quiver ring $\mathcal{I}$ is the subring of $\mathcal{M}$ generated by the isogations. In fact, $\mathcal{I}$ is the $\mathbb{Z}$-span of the isogations and

$$\mathcal{I}(J, K) = \bigoplus_{\overline{\delta}} \mathbb{Z}\ {}_J\mathrm{iso}_K^{\delta}$$

where $J, K \in \mathfrak{K}$ and $\overline{\delta}$ runs over the $\overline{\mathcal{G}}$-isomorphisms $J \leftarrow K$. Note that, via the correspondence ${}_H\mathrm{iso}_H^{\gamma} \leftrightarrow \overline{\gamma}$, we have an algebra isomorphism

$$\mathcal{I}(H, H) \cong \mathbb{Z}\mathrm{Out}_{\mathcal{G}}(H) .$$

We make L become an $\mathcal{I}$-module with representation $\tau : \mathcal{I} \to \mathrm{End}_{\mathbb{Z}}(L)$ such that

$$\tau({}_K\mathrm{iso}_J^{\gamma}){}_G\overline{\mathrm{ind}}_H^{\beta} = \begin{cases} {}_G\overline{\mathrm{ind}}_K^{\beta\gamma^{-1}} & \text{if } J = H, \\ 0 & \text{otherwise.} \end{cases}$$

Since the actions of $\mathcal{N}$ and $\mathcal{I}$ commute with each other, σ and τ are ring homomorphisms

$$\sigma : \mathcal{N} \to \mathrm{End}_{\mathcal{I}}(L) , \qquad \tau : \mathcal{I} \to \mathrm{End}_{\mathcal{N}}(L) .$$

As an $\mathcal{N}$-submodule of L, we define

$$L(-, H) = \tau(\mathrm{iso}_H)L = \bigoplus_{G \in \mathfrak{K}} L(G, H) .$$

Each $L(-, H)$ is an $\mathcal{I}(H, H)$-module and becomes a permutation $\mathbb{Z}\mathrm{Out}_{\mathcal{G}}(H)$-module via the isomorphism $\mathcal{I}(H, H) \cong \mathbb{Z}\mathrm{Out}_{\mathcal{G}}(H)$. The action of $\mathbb{Z}\mathrm{Out}_{\mathcal{G}}(H)$ on $L(-, H)$ is such that an element $\overline{\gamma} \in \mathrm{Out}_{\mathcal{G}}(H)$ sends the basis element ${}_G\overline{\mathrm{ind}}_H^{\beta}$ to the basis element ${}_G\overline{\mathrm{ind}}_H^{\beta\gamma^{-1}}$.

Let us recall the notion of a suborbit map on a permutation module. Let Γ be a finite group and Ω a finite Γ-set. For $\omega_1, \omega_2 \in \Omega$, let $\epsilon(\omega_1, \omega_2)$ be the $\mathbb{Z}$-linear endomorphism of $\mathbb{Z}\Omega$ such that, given $\omega \in \Omega$, then

$$\epsilon(\omega_1,\omega_2)\omega = \begin{cases} \omega_1 & \text{if } \omega = \omega_2, \\ 0 & \text{if } \omega \neq \omega_2. \end{cases}$$

The endomorphism ring $\mathrm{End}_{\mathbb{Z}\Gamma}(\mathbb{Z}\Omega)$ has a $\mathbb{Z}$-basis consisting of the maps

$$\$(\omega_1,\omega_2) = \sum_{(\omega_1',\omega_2')\in\Omega\times\Omega\,:\,(\omega_1',\omega_2')=\Gamma(\omega_1,\omega_2)} \epsilon(\omega_1',\omega_2') \ .$$

We call $\$(\omega_1,\omega_2)$ a **suborbit map** on $\mathbb{Z}\Omega$. Since $\$(\omega_1,\omega_2) = \$(\omega_1',\omega_2')$ if and only if $(\omega_1,\omega_2) =_\Gamma (\omega_1',\omega_2')$, we have

$$\mathrm{End}_{\mathbb{Z}\Gamma}(\mathbb{Z}\Omega) = \bigoplus_{(\omega_1,\omega_2)\in_\Gamma\Omega\times\Omega} \mathbb{Z}\,\$(\omega_1,\omega_2)$$

where the notation indicates that (ω_1,ω_2) runs over representatives of the Γ-orbits of $\Omega\times\Omega$.

Proposition 4.3. *Let $H \in \mathfrak{K}$. Then there is a bijective correspondence between:*
(a) *the transitive morphisms ${}_F\mathrm{ind}_V^\alpha\mathrm{res}_G$ in $\mathcal{N}$ such that $V \cong_{\mathcal{G}} H$,*
(b) *the suborbit maps $\$$ on the permutation $\mathrm{Out}_{\mathcal{G}}(H)$-module $L(-,H)$.*

The correspondence ${}_F\mathrm{ind}_V^\alpha\mathrm{res}_G \leftrightarrow \$$ is characterized by the condition that ${}_F\mathrm{ind}_V^\alpha\mathrm{res}_G$ acts on $L(-,H)$ as a positive integer multiple of $\$$.

Proof. Fix $F,G \in \mathfrak{K}$. Two transitive morphisms ${}_F\mathrm{ind}_V^\alpha\mathrm{res}_G$ and ${}_F\mathrm{ind}_{V'}^{\alpha'}\mathrm{res}_G$ coincide provided $[\alpha,V] = [\alpha',V']$, in other words, there exist $f \in F$ and $g \in G$ such that $V' = {}^gV$ and $\alpha' = c(f)\alpha c(g^{-1})$. Two suborbit maps $\$({}_F\overline{\mathrm{ind}}_H^\mu, {}_G\overline{\mathrm{ind}}_H^\nu)$ and $\$({}_F\overline{\mathrm{ind}}_H^{\mu'}, {}_G\overline{\mathrm{ind}}_H^{\nu'})$ coincide provided there exists $\overline{\gamma} \in \mathrm{Out}_{\mathcal{G}}(H)$ such that $\overline{\mu'} = \overline{\mu\gamma^{-1}}$ and $\overline{\nu'} = \overline{\nu\gamma^{-1}}$, in other words, there exist $f \in F$ and $g \in G$ and $\gamma \in \mathrm{Aut}_{\mathcal{G}}(H)$ such that $\mu' = c(f)\mu\gamma^{-1}$ and $\nu' = c(g)\nu\gamma^{-1}$.

Given $\ell = {}_F\mathrm{ind}_V^\alpha\mathrm{res}_G$, we define a suborbit map $\$ = \$({}_F\overline{\mathrm{ind}}_H^\mu, {}_G\overline{\mathrm{ind}}_H^\nu)$ as follows. We choose a $\mathcal{G}$-isomorphism $\nu_0 : V \leftarrow H$ and extend ν_0 to a homomorphism $\nu : G \leftarrow H$ by composing with the inclusion $G \hookleftarrow V$. We define $\mu = \alpha\nu$. The suborbit map $\$$ does not depend on the choice of ν_0 because, if we replace ν_0 with $\nu_0\gamma^{-1}$ for some $\gamma \in \mathrm{Aut}_{\mathcal{F}}(H)$, then $\overline{\mu}$ and $\overline{\nu}$ are replaced by $\overline{\mu\gamma^{-1}}$ and $\overline{\nu\gamma^{-1}}$. To complete the demonstration that $\$$ depends only on ℓ, we must show independence from the choice of α and V. Suppose that $\ell = {}_F\mathrm{ind}_{V'}^{\alpha'}\mathrm{res}_G$. Let f and g be such that $V' = {}^gV$ and $\alpha' = c(f)\alpha c(g^{-1})$. Let $\nu_0' = c(g)\nu_0$. Extending ν_0' to a homomorphism $\nu' : G \leftarrow H$ and defining $\mu' = \alpha'\nu'$, then $\nu' = c(g)\nu$ and $\mu' = c(f)\mu$. So $\$({}_F\overline{\mathrm{ind}}_H^{\mu'}, {}_G\overline{\mathrm{ind}}_H^{\nu'}) = \$$. We have established that $\$$ depends only on ℓ.

Conversely, given a suborbit map $\$ = \$({}_F\overline{\mathrm{ind}}_H^\mu, {}_G\overline{\mathrm{ind}}_H^\nu)$, we define a transitive morphism $\ell = {}_F\mathrm{ind}_V^\alpha\mathrm{res}_G$ as follows. Let $V = \nu(H)$, let $\nu_0 : V \leftarrow H$ be the isomorphism restricted from ν and let $\alpha = \mu\nu_0^{-1} : F \leftarrow V$. We must show that ℓ depends only on $\$$ and not on the choice of μ and ν. Suppose that $\$ = \$({}_F\overline{\mathrm{ind}}_H^{\mu'}, {}_G\overline{\mathrm{ind}}_H^{\nu'})$. Let f, g, γ be

such that $\mu' = c(f)\mu\gamma^{-1}$ and $\nu' = c(g)\nu\gamma^{-1}$. Letting $V' = \nu'(H)$, then $V' = {}^gV$. The isomorphism $V' \leftarrow H$ restricted from ν' is $\nu'_0 = c(g)\nu_0\gamma^{-1}$. Defining $\alpha' = \mu'{\nu'_0}^{-1}$, then

$$\alpha' = c(f)\mu\gamma^{-1}\gamma{\nu_0}^{-1}c(g^{-1}) = c(f)\alpha c(g^{-1}) \ .$$

So ${}_F\mathrm{ind}_{V'}^{\alpha'}\mathrm{res}_G = \ell$. We have established that ℓ depends only on \$.

It is easy to check that the above functions $\ell \mapsto \$$ and $\$ \mapsto \ell$ are mutual inverses. Now suppose that $\ell \leftrightarrow \$$. It remains only to show that the action of ℓ is a positive integer multiple of \$. Since the action of $\mathcal{N}$ on $L(-,H)$ commutes with the action of $\mathbb{Z}\mathrm{Out}_{\mathcal{G}}(H)$, the action of ℓ is a $\mathbb{Z}$-linear combination of suborbit maps. By Lemma 4.2, any suborbit map with non-zero coefficient has a positive integer coefficient. Let $\$_1 = \$({}_F\overline{\mathrm{ind}}_H^{\mu_1}, {}_G\overline{\mathrm{ind}}_H^{\nu_1})$ be a suborbit map with non-zero coefficient. We are to show that $\$_1 = \$$. Since $\sigma(\ell)_G\overline{\mathrm{ind}}_H^{\nu_1} \neq 0$, Lemma 4.2 implies that $V = {}^x\nu_1(H)$ for some $x \in G$. Replacing ν_1 with $c(x)\nu_1$ does not change ${}_G\overline{\mathrm{ind}}_H^{\nu_1}$, so we may assume that $V = \nu_1(H)$. Then $\nu_1 = \nu\gamma^{-1}$ for some $\gamma \in \mathrm{Out}_{\mathcal{G}}(H)$. That is to say, ${}_G\overline{\mathrm{ind}}_H^{\nu_1}$ belongs to the same $\mathrm{Out}_{\mathcal{G}}(H)$-orbit as ${}_G\overline{\mathrm{ind}}_H^{\nu}$. So we may assume that ${}_G\overline{\mathrm{ind}}_H^{\nu_1} = {}_G\overline{\mathrm{ind}}_H^{\nu}$. By Lemma 4.2 again, ${}_F\overline{\mathrm{ind}}_H^{\mu_1} = {}_F\overline{\mathrm{ind}}_H^{\alpha c(g)\nu}$ for some $g \in N_G(V)$. The proof of the well-definedness of the function $\ell \mapsto \$$ now shows that $\ell \mapsto \$_1$, in other words, $\$_1 = \$$. □

Proposition 4.4. *The representation $\sigma : \mathcal{N} \to \mathrm{End}_{\mathcal{I}}(L)$ is injective.*

Proof. Let $\kappa \in \mathcal{N}$. Recall that $\kappa = \sum_{F,G} {}_F\kappa_G$ as a sum with only finitely many non-zero terms. Each term ${}_F\kappa_G \in \mathcal{N}(F,G)$ acts on L as a map

$$\sigma({}_F\kappa_G) \ : \ \bigoplus_{H\in\mathfrak{K}} L(F,H) \leftarrow \bigoplus_{H\in\mathfrak{K}} L(G,H) \ .$$

Suppose that $\kappa \neq 0$. We must show that $\sigma(\kappa) \neq 0$. We may assume that $\kappa = {}_F\kappa_G$ for some $F, G \in \mathfrak{K}$. Write

$$\kappa = \sum_{j=1}^{n} \lambda_j \cdot {}_F\mathrm{ind}_{V_j}^{\alpha_j}\mathrm{res}_G$$

as a $\mathbb{Z}$-linear combination of distinct transitive morphisms in $\mathcal{N}$ with each $\lambda_j \neq 0$. Let V be maximal among the V_j. Replacing some of the V_j with G-conjugates if necessary, we can choose the enumeration such that $V_j = V$ for $j \leq m$ and $V_j \not\geq_G V$ for $j > m$. Invoking Proposition 4.3, let $\$_j$ be the suborbit map corresponding to ${}_F\mathrm{ind}_V^{\alpha_j}\mathrm{res}_G$ for $j \leq m$. Note that the $\$_j$ are mutually distinct. By Lemma 4.2, $\sigma({}_F\mathrm{ind}_{V_j}^{\alpha_j}\mathrm{res}_G)$ annihilates $L(-,V)$ for $j > m$. So, by Proposition 4.3, there exist non-zero integers $z_1, \ldots, z_m$ such that the restriction of $\sigma(\kappa)$ to $L(-,V)$ is $\sum_{j=1}^{m} \lambda_j z_j \$_j$. Perforce, $\sigma(\kappa) \neq 0$. □

Proposition 4.5. *If $\mathfrak{K}$ is finite then the representation $\sigma : \mathbb{K}\mathcal{N} \to \mathrm{End}_{\mathbb{K}\mathcal{I}}(\mathbb{K}L)$ is bijective.*

Proof. By the previous proposition, the $\mathbb{K}$-linear map σ is injective. We argue by comparison of dimensions. Summing over representatives H of the $\mathcal{G}$-isomorphism classes in $\mathfrak{K}$, we have

$$\mathbb{K}\mathcal{I} = \bigoplus_H \mathbb{K}\mathcal{I}_H \ , \qquad \mathbb{K}\mathcal{I}_H = \bigoplus_{H_1,H_2\in\mathfrak{K}\,:\,H_1\cong_{\mathcal{G}}H_2\cong_{\mathcal{G}}H} \mathbb{K}\mathcal{I}(H_1,H_2) \ ,$$
$$\mathbb{K}L = \bigoplus_H \mathbb{K}L_H \ , \qquad \mathbb{K}L_H = \bigoplus_{H_1\in\mathfrak{K}\,:\,H_1\cong_{\mathcal{G}}H} \mathbb{K}L(-,H_1) \ .$$

The subalgebra $\mathbb{K}\mathcal{I}_H$ is isomorphic to a full matrix algebra over $\mathbb{K}\mathcal{I}(H,H) \cong \mathbb{K}\mathrm{Out}_{\mathcal{G}}(H)$. So

$$\mathrm{End}_{\mathbb{K}\mathcal{I}}(\mathbb{K}L) \cong \mathrm{End}_{\bigoplus_H \mathbb{K}\mathrm{Out}_{\mathcal{G}}(H)}\big(\bigoplus_H \mathbb{K}L(-,H)\big) \cong \bigoplus_H \mathrm{End}_{\mathbb{K}\mathrm{Out}_{\mathcal{G}}(H)}(\mathbb{K}L(-,H)) \ .$$

The number of suborbit maps on the permutation $\mathbb{Z}\mathrm{Out}_{\mathcal{G}}(H)$-module $L(-,H)$ is

$$\dim_{\mathbb{K}}(\mathrm{End}_{\mathbb{K}\mathrm{Out}_{\mathcal{G}}(H)}(\mathbb{K}L(-,H))) = \sum_{F,G\in\mathfrak{K}} n_H^{F,G}$$

where $n_H^{F,G}$ is the number of suborbit maps $L(F,H) \leftarrow L(G,H)$. By Proposition 4.3, the number of transitive morphisms $F \leftarrow G$ in $\mathcal{N}$ is

$$\dim_{\mathbb{K}}(\mathbb{K}\mathcal{N}(F,G)) = \sum_H n_H^{F,G} \ .$$

So $\dim_{\mathbb{K}}(\mathbb{K}\mathcal{N}) = \sum_{F,G\in\mathfrak{K}} \big(\sum_H n_H^{F,G}\big) = \sum_H \big(\sum_{F,G\in\mathfrak{K}} n_H^{F,G}\big) = \dim_{\mathbb{K}}(\mathrm{End}_{\mathbb{K}\mathcal{I}}(\mathbb{K}L))$. □

Theorem 4.6. *The $\mathbb{K}$-linear category $\mathbb{K}\mathcal{N}$ is locally semisimple. In particular, if $\mathfrak{K}$ is finite, then the quiver algebra $\mathbb{K}\mathcal{N}$ is semisimple.*

Proof. First suppose that $\mathfrak{K}$ is finite. As we saw in the proof of Proposition 4.5, each algebra $\mathbb{K}\mathcal{I}_H$ is isomorphic to a full matrix algebra over the semisimple algebra $\mathbb{K}\mathrm{Out}_{\mathcal{G}}(H)$. So $\mathbb{K}\mathcal{I}$ is semisimple. Therefore, $\mathrm{End}_{\mathbb{K}\mathcal{I}}(\mathbb{K}L)$ is semisimple. An appeal to Proposition 4.5 now completes the argument in this case.

Now let $\mathfrak{K}$ be arbitrary. Let i be an idempotent of $\mathbb{K}\mathcal{N}$. Let $\mathfrak{K}_0$ be a finite subset of $\mathfrak{K}$ such that $\mathfrak{K}_0$ is closed under taking subgroups and i can be expressed in the form $i = \sum_{F,G\in\mathfrak{K}_0} {}_Fi_G$ with ${}_Fi_G \in \mathbb{K}\mathcal{N}(F,G)$. Let $\mathcal{N}_0$ be the full subcategory of $\mathcal{N}$ such that $\mathrm{Obj}(\mathcal{N}_0) = \mathfrak{K}_0$. Since $\mathfrak{K}_0$ is finite, the algebra $1_{\mathcal{N}_0}\,.\,\mathbb{K}\mathcal{N}\,.\,1_{\mathcal{N}_0} = \mathbb{K}\mathcal{N}_0$ is semisimple. Arguing as in the proof of Remark 4.1, we deduce that $i\mathbb{K}\mathcal{N}i$ is semisimple. □

Corollary 4.7. *There is a bijective correspondence between the isomorphism classes of simple $\mathbb{K}\mathcal{N}$-functors S and the blocks b of $\mathbb{K}\mathcal{N}$ such that $S \leftrightarrow b$ provided S belongs to b.*

Proof. We are to show that, given simple $\Bbbk\mathcal{N}$-functors S and S' belonging to the same block b of $\Bbbk\mathcal{N}$, then $S \cong S'$. By Theorem 4.6, this is already clear when $\mathfrak{K}$ is finite. Generally, by Proposition 2.6, there exists a full subcategory $\mathcal{K}$ of $\Bbbk\mathcal{N}$ such that $\mathrm{Obj}(\mathcal{K})$ is finite and the $\mathcal{K}$-functors $1_{\mathcal{K}}S$ and $1_{\mathcal{K}}S'$ are non-zero and belong to the same block of $\mathcal{K}$. Let $\mathfrak{K}_0$ be the set of subgroups of elements of $\mathrm{Obj}(\mathcal{K})$. Let $\mathcal{N}_0$ be the full subcategory of $\mathcal{N}$ with $\mathrm{Obj}(\mathcal{N}_0) = \mathfrak{K}_0$. Proposition 2.5, applied to the subcategory $\mathcal{K}$ of $\Bbbk\mathcal{N}_0$, tells us that $1_{\mathcal{N}_0}S$ and $1_{\mathcal{N}_0}S'$ belong to the same block of $\Bbbk\mathcal{N}_0$. But $\mathfrak{K}_0$ is finite, so $1_{\mathcal{N}_0}S \cong 1_{\mathcal{N}_0}S'$. By Proposition 2.4, $S \cong S'$. □

5. Simple functors of deflation Mackey categories

Let $\mathcal{F}$ be a Mackey system on $\mathfrak{K}$. Let $\mathcal{G}$ be the ordinary Mackey system such that the morphisms in $\mathcal{G}$ are the injective morphisms in $\mathcal{F}$. Consider the deflation Mackey category $\mathcal{M} = \mathcal{M}^{\leftarrow}_{\mathcal{F}}$ and the ordinary Mackey category $\mathcal{N} = \mathcal{M}_{\mathcal{G}}$. We shall show that the simple $\Bbbk\mathcal{M}$-functors restrict to and are inflated from the simple $\Bbbk\mathcal{N}$-functors. By similar arguments, a similar result holds for the $\Bbbk$-linear extension $\Bbbk\mathcal{M}^{\rightarrow}_{\mathcal{F}}$ of the inflation Mackey category $\mathcal{M}^{\rightarrow}_{\mathcal{F}}$. A variant of this result, in a different scenario, appears in Yaraneri [11, 3.10]. Another related result is Boltje–Danz [2, 6.5], which asserts that, for $G \in \mathfrak{K}$, the simple $\Bbbk\mathcal{M}(G,G)$-modules restrict to and are inflated from the simple $\Bbbk\mathcal{N}(G,G)$-modules.

For $F, G \in \mathfrak{K}$, let $V \leq G$ and let $\alpha : F \leftarrow V$ be a morphism in $\mathcal{F}$. Following Boltje–Danz [2, 4.2], we define a $\Bbbk$-linear map

$$\rho^{F,G}_{\alpha,V} \ : \ \Bbbk\mathcal{M}(F,G) \to \Bbbk$$

such that, given an F–G-biset X whose isomorphism class $[X]$ belongs to $\mathcal{M}(F,G)$, then

$$\rho^{F,G}_{\alpha,V}[X] = |X^{\mathfrak{S}(\alpha,V)}|/|C_G(V)|$$

where $X^{\mathfrak{S}(\alpha,V)}$ denotes the set of elements of X fixed by $\mathfrak{S}(\alpha,V)$. Let $\Bbbk\mathcal{J}(F,G)$ be the $\Bbbk$-submodule of $\Bbbk\mathcal{M}(F,G)$ consisting of those elements $x \in \Bbbk\mathcal{M}(F,G)$ such that $\rho^{F,G}_{\alpha,V}(x) = 0$ whenever α is injective. As a $\Bbbk$-submodule of $\Bbbk\mathcal{M}$, we define

$$\Bbbk\mathcal{J} = \bigoplus_{F,G \in \mathfrak{K}} \Bbbk\mathcal{J}(F,G) \ .$$

Proposition 5.1. *We have $\Bbbk\mathcal{M} = \Bbbk\mathcal{N} \oplus \Bbbk\mathcal{J}$, furthermore, $\Bbbk\mathcal{J}$ is an ideal of $\Bbbk\mathcal{M}$. If $\mathfrak{K}$ is finite, then $\Bbbk\mathcal{J} = J(\Bbbk\mathcal{M})$, the Jacobson radical.*

Proof. Following [2, Section 4], we shall construct an isomorphic copy $\widetilde{\Bbbk\mathcal{M}}$ of the algebra $\Bbbk\mathcal{M}$. For $F, G \in \mathfrak{K}$, we introduce a $\Bbbk$-module $\widetilde{\Bbbk\mathcal{M}}(F,G)$ with a basis consisting of the symbols $(\alpha, V)_{F,G}$ where $(\alpha, V) \in \mathcal{P}^{\mathcal{F}}_{F,G}$. We make the direct sum

$$\widehat{\mathbb{K}\mathcal{M}} = \bigoplus_{F,G\in\mathfrak{K}} \widehat{\mathbb{K}\mathcal{M}}(F,G)$$

become an algebra with multiplication given by

$$(\alpha,V)_{F,G}\,(\beta,W)_{G',H} = \begin{cases} (\alpha\beta,W)_{F,H}\,|C_G(V)|/|G| & \text{if } G=G' \text{ and } V=\beta(W),\\ 0 & \text{otherwise.}\end{cases}$$

The action of $F\times G$ on $\mathcal{P}^{\mathcal{F}}_{F,G}$ gives rise to a permutation action of $F\times G$ on $\widehat{\mathbb{K}\mathcal{M}}(F,G)$. Let

$$\widetilde{\mathbb{K}\mathcal{M}}(F,G) = \widehat{\mathbb{K}\mathcal{M}}(F,G)^{F\times G}\,.$$

As an element of $\widetilde{\mathbb{K}\mathcal{M}}(F,G)$, let $[\alpha,V]^+_{F,G}$ denote the sum of the $F\times G$-conjugates of $(\alpha,V)_{F,G}$. The orbit sums $[\alpha,V]^+_{F,G}$ comprise a basis for $\widetilde{\mathbb{K}\mathcal{M}}(F,G)$, indeed,

$$\widetilde{\mathbb{K}\mathcal{M}}(F,G) = \bigoplus_{[\alpha,V]\in\overline{\mathcal{P}}^{\mathcal{F}}_{F,G}} \mathbb{K}\,.\,[\alpha,V]^+_{F,G}\,.$$

As a subalgebra of $\widehat{\mathbb{K}\mathcal{M}}$, we define

$$\widetilde{\mathbb{K}\mathcal{M}} = \bigoplus_{F,G\in\mathfrak{K}} \widetilde{\mathbb{K}\mathcal{M}}(F,G)\,.$$

It is shown in [2, 4.7] that there is an algebra isomorphism $\rho:\mathbb{K}\mathcal{M}\to\widetilde{\mathbb{K}\mathcal{M}}$ given by the maps $\rho^{F,G}:\mathbb{K}\mathcal{M}(F,G)\to\widetilde{\mathbb{K}\mathcal{M}}(F,G)$ such that, for $x\in\mathbb{K}\mathcal{M}(F,G)$, we have

$$\rho^{F,G}(x) = \sum_{(\alpha,V)\in\mathcal{P}^{\mathcal{F}}_{F,G}} \rho^{F,G}_{\alpha,V}(x)(\alpha,V)_{F,G} = \sum_{[\alpha,V]\in\overline{\mathcal{P}}^{\mathcal{F}}_{F,G}} \rho^{F,G}_{\alpha,V}(x)[\alpha,V]^+_{F,G}\,.$$

Let $\widehat{\mathbb{K}\mathcal{J}}$ be the ideal of $\widehat{\mathbb{K}\mathcal{M}}$ spanned by those elements $(\alpha,V)_{F,G}$ such that α is non-injective. Let $\widetilde{\mathbb{K}\mathcal{J}} = \widetilde{\mathbb{K}\mathcal{M}}\cap\widehat{\mathbb{K}\mathcal{J}}$, which is an ideal of $\widetilde{\mathbb{K}\mathcal{M}}$. Thus, $\widetilde{\mathbb{K}\mathcal{J}}$ is spanned by those orbit sums $[\alpha,V]^+_{F,G}$ such that α is non-injective. By the definitions of $\mathbb{K}\mathcal{J}$ and $\widetilde{\mathbb{K}\mathcal{J}}$, we have $\widetilde{\mathbb{K}\mathcal{J}} = \rho(^{\oplus}\mathbb{K}\mathcal{J})$. Therefore $\mathbb{K}\mathcal{J}$ is an ideal of $\mathbb{K}\mathcal{M}$.

Given $(\alpha,V)\in\mathcal{P}^{\mathcal{F}}_{F,G}$ with α non-injective then, for all (F,G)-bisets X such that $[X]\in\mathcal{N}(F,G)$, we have $X^{\mathfrak{S}(\alpha,V)}=\emptyset$, hence $\rho^{F,G}_{\alpha,V}[X]=0$. So $\rho^{F,G}_{\alpha,V}(\mathbb{K}\mathcal{N}(F,G))=\{0\}$. It follows that $\rho(\mathbb{K}\mathcal{N}(F,G))\cap\widetilde{\mathbb{K}\mathcal{J}}(F,G)=\{0\}$. By considering dimensions, $\widetilde{\mathbb{K}\mathcal{M}}(F,G) = \rho(\mathbb{K}\mathcal{N}(F,G))\oplus\widetilde{\mathbb{K}\mathcal{J}}(F,G)$. So $\widetilde{\mathbb{K}\mathcal{M}} = \rho(\mathbb{K}\mathcal{N})\oplus\widetilde{\mathbb{K}\mathcal{J}} = \rho(\mathbb{K}\mathcal{N})\oplus\rho(\mathbb{K}\mathcal{J})$. Therefore, $\mathbb{K}\mathcal{M} = \mathbb{K}\mathcal{N}\oplus\mathbb{K}\mathcal{J}$.

Now suppose that $\mathfrak{K}$ is finite. Given a non-zero product $(\alpha_1,V_1)_{F_1,G_1}\dots(\alpha_n,V_n)_{F_n,G_n}$ of basis elements of $\widehat{\mathbb{K}\mathcal{J}}$, then each $V_j=\alpha_{j+1}(V_{j+1})$, which is smaller than V_{j+1} because α_{j+1} is non-injective. So $n\leq|\mathfrak{K}|$. In particular, $\widehat{\mathbb{K}\mathcal{J}}$ is nilpotent. It follows that

$\widetilde{\mathbb{K}\mathcal{J}}$ is nilpotent. Therefore $\mathbb{K}\mathcal{J}$ is nilpotent, in other words, $\mathbb{K}\mathcal{J} \leq J(\mathbb{K}\mathcal{M})$. But Theorem 4.6 implies that $\mathbb{K}\mathcal{N}$ is semisimple. So $\mathbb{K}\mathcal{N} \cap J(\mathbb{K}\mathcal{M}) = \{0\}$. We deduce that $\mathbb{K}\mathcal{J} = J(\mathbb{K}\mathcal{M})$. □

Theorem 5.2. *Let (G, V) be a seed for $\mathcal{F}$ over $\mathbb{K}$. Then the simple $\mathbb{K}\mathcal{M}$-functor $S^{\mathbb{K}\mathcal{M}}_{G,V}$ and the simple $\mathbb{K}\mathcal{N}$-functor $S^{\mathbb{K}\mathcal{N}}_{G,V}$ are related by*

$$S^{\mathbb{K}\mathcal{N}}_{G,V} \cong {}_{\mathbb{K}\mathcal{N}}\mathrm{Res}_{\mathbb{K}\mathcal{M}}(S^{\mathbb{K}\mathcal{M}}_{G,V}) , \qquad S^{\mathbb{K}\mathcal{M}}_{G,V} \cong {}_{\mathbb{K}\mathcal{M}}\mathrm{Inf}_{\mathbb{K}\mathcal{N}}(S^{\mathbb{K}\mathcal{N}}_{G,V})$$

where the inflation is via the canonical algebra epimorphism $\mathbb{K}\mathcal{M} \to \mathbb{K}\mathcal{N}$ with kernel $\mathbb{K}\mathcal{J}$.

Proof. By the latest proposition, the description of the inflation functor ${}_{\mathbb{K}\mathcal{M}}\mathrm{Inf}_{\mathbb{K}\mathcal{N}}$ makes sense. The $\mathbb{K}\mathcal{M}$-functor $S = {}_{\mathbb{K}\mathcal{M}}\mathrm{Inf}_{\mathbb{K}\mathcal{N}}(S^{\mathbb{K}\mathcal{N}}_{G,V})$ is simple and $S(G) \cong S^{\mathbb{K}\mathcal{N}}_{G,V}(G) \cong V$ as $\mathbb{F}\mathrm{Out}_{\mathcal{G}}(G)$-modules. By Theorem 3.7, $S \cong S^{\mathbb{K}\mathcal{M}}_{G,V}$. It follows that, $S^{\mathbb{K}\mathcal{N}}_{G,V} \cong {}_{\mathbb{K}\mathcal{N}}\mathrm{Res}_{\mathbb{K}\mathcal{M}}(S)$. □

Theorem 5.3. *Every idempotent of $Z({}^{\Pi}\mathbb{K}\mathcal{M})$ belongs to $Z({}^{\Pi}\mathbb{K}\mathcal{N})$. In particular, every block of $\mathbb{K}\mathcal{M}$ is a central idempotent of ${}^{\Pi}\mathbb{K}\mathcal{N}$.*

Proof. A lemma in Boltje–Külshammer [3, 5.2] asserts that, given a subring Γ of a ring Λ such that $\Lambda = \Gamma \oplus J(\Lambda)$, then every idempotent of $Z(\Lambda)$ belongs to $Z(\Gamma)$. This, together with Proposition 5.1, immediately implies the required conclusion in the case where $\mathfrak{K}$ is finite. For arbitrary $\mathfrak{K}$, let e be an idempotent of $Z({}^{\Pi}\mathbb{K}\mathcal{M})$. Let $G \in \mathfrak{K}$ and let $\mathfrak{K}_G$ be the set of subgroups of G. Let $\mathcal{M}_G$ and $\mathcal{N}_G$ be the full subcategories of $\mathcal{M}$ and $\mathcal{N}$, respectively, such that $\mathrm{Obj}(\mathcal{M}_G) = \mathrm{Obj}(\mathcal{N}_G) = \mathfrak{K}_G$. Since $\mathfrak{K}_G$ is finite, the idempotent $1_{\mathcal{M}_G}e$ of $Z({}^{\Pi}\mathbb{K}\mathcal{M}_G)$ must belong to $Z({}^{\Pi}\mathbb{K}\mathcal{N}_G)$. The (G, G)-coordinate e_G of e coincides with the (G, G)-coordinate of $1_{\mathcal{M}_G}e$. So $e_G \in \mathcal{N}(G, G)$. By Remark 2.1, $e = \sum_{G\in\mathfrak{K}} e_G$, hence $e \in {}^{\Pi}\mathbb{K}\mathcal{N}$. But e is central in ${}^{\Pi}\mathbb{K}\mathcal{M}$, so e is central in ${}^{\Pi}\mathbb{K}\mathcal{N}$. □

6. Multiple blocks

In Corollary 4.7, we found that, for an ordinary Mackey category $\mathcal{N}$, each block of $\mathbb{K}\mathcal{N}$ owns a unique isomorphism class of simple $\mathbb{K}\mathcal{N}$-functors. In this section, we shall give an example of a non-ordinary Mackey category such that most of the blocks of the $\mathbb{K}$-linear extension still own a unique isomorphism class of simple functors.

Let $\mathcal{F}^{\lhd}(\mathfrak{K})$ denote the Mackey system on $\mathfrak{K}$ such that the morphisms in $\mathcal{F}^{\lhd}(\mathfrak{K})$ are the homomorphisms between groups in $\mathfrak{K}$. The deflation Mackey category $\mathcal{M}^{\lhd}_{\mathfrak{K}} = \mathcal{M}^{\leftarrow}_{\mathcal{F}^{\lhd}(\mathfrak{K})}$ is called the **complete deflation Mackey category** on $\mathfrak{K}$. Let $\mathcal{F}^{\Delta}(\mathfrak{K})$ denote the ordinary Mackey system on $\mathfrak{K}$ such that the morphisms in $\mathcal{F}^{\lhd}(\mathfrak{K})$ are the injective homomorphisms between groups in $\mathfrak{K}$. The ordinary Mackey category $\mathcal{M}^{\Delta}_{\mathfrak{K}} = \mathcal{M}_{\mathcal{F}^{\lhd}(\mathfrak{K})}$ is called the **complete ordinary Mackey category** on $\mathfrak{K}$. We shall give an example of a complete deflation Mackey category whose $\mathbb{K}$-linear extension has $p-1$ blocks and p isomorphism classes of simple functors, where p is a given prime.

Lemma 6.1. *Consider the complete ordinary Mackey category $\mathcal{M}_p^\Delta = \mathcal{M}_{\{1,C_p\}}^\Delta$. There are exactly p isomorphism classes of simple $\mathbb{C}\mathcal{M}_p^\Delta$-functors. The category $\mathbb{C}\mathcal{M}_p^\Delta$ has exactly p blocks.*

Proof. The first part follows from Theorem 3.7. The second part then follows from Corollary 4.7. □

As a step towards finding the blocks of $\mathbb{C}\mathcal{M}_p^\triangleleft$, we shall first find the blocks of $\mathbb{C}\mathcal{M}_p^\Delta$. Write $\langle c\rangle = C = C_p$. For $1 \leq j \leq p-1$, let σ_j be the automorphism of C such that $c \mapsto c^j$. Let

$$\alpha = {}_1\mathrm{iso}_1 \,, \qquad \tau = {}_C\mathrm{ind}_1 \,, \qquad \rho = {}_1\mathrm{res}_C \,, \qquad \alpha_j = {}_C\mathrm{iso}_C^{\sigma_j} \,.$$

Observe that $\mathbb{C}\mathcal{M}_p^\Delta$ has a $\mathbb{C}$-basis consisting of the elements α, τ, ρ, $\tau\rho$, $\alpha_1, \ldots, \alpha_{p-1}$. Let

$$e_{1,1} = \alpha + \tau\rho/p \,.$$

We identify $\mathrm{Out}(C)$ with $\mathrm{Aut}(C)$. We also identify $\mathrm{Out}(C)$ with the unit group $(\mathbb{Z}/p)^\times$ of the ring $\mathbb{Z}/p$ of integers modulo p. Let $\mathrm{Irr}(\mathbb{C}\mathrm{Out}(C))$ denote the set of irreducible $\mathbb{C}\mathrm{Out}(C)$-characters. For $\chi \in \mathrm{Irr}(\mathbb{C}\mathrm{Out}(C))$, we define $e_{C,\chi}$ such that, writing 1 to denote the trivial character,

$$e_{C,1} = -\tau\rho/p + \frac{1}{p-1}\sum_{j=1}^{p-1}\alpha_j$$

and, when χ is non-trivial,

$$e_{C,\chi} = \frac{1}{p-1}\sum_{j=1}^{p-1}\chi(j^{-1})\alpha_j \,.$$

Lemma 6.2. *The blocks of $\mathbb{C}M_p^\Delta$ are $e_{1,1}$ and $e_{C,\chi}$ with $\chi \in \mathrm{Irr}(\mathbb{C}\mathrm{Out}(C))$.*

Proof. For $G \in \{1, C\}$, let $A_\mathbb{C}(G)$ denote the character ring of $\mathbb{C}G$. Since G is abelian, the character algebra $\mathbb{C}A_\mathbb{C}(G)$ can be identified with the $\mathbb{C}$-module of functions $G \to \mathbb{C}$. Let e^1 be the element of $\mathbb{C}A_\mathbb{C}(1)$ such that $e^1(1) = 1$. Let $e_0^C, \ldots, e_{p-1}^C$ be the elements of $\mathbb{C}A_\mathbb{C}(C)$ such that $e_i^C(c^i) = 1$ and e_i^C vanishes off $\{c^i\}$. Then $\{e^1\}$ and $\{e_0^C, \ldots, e_{p-1}^C\}$ are bases for $\mathbb{C}A_\mathbb{C}(1)$ and $\mathbb{C}A_\mathbb{C}(C)$, respectively.

We shall make use of the representation $\mathbb{C}\mathcal{M}_p^\Delta \to \mathrm{End}_\mathbb{C}(\mathbb{C}A_\mathbb{C})$ of the $\mathbb{C}\mathcal{M}_p^\Delta$-functor $\mathbb{C}A_\mathbb{C}$. The $\mathbb{C}$-module $\mathbb{C}A_\mathbb{C} = \mathbb{C}A_\mathbb{C}(1) \oplus \mathbb{C}A_\mathbb{C}(C)$ has a basis consisting of the elements e^1 and e_i^C for $0 \leq i \leq p-1$. We have

$$\alpha(e^1) = e^1 \,, \qquad \tau(e^1) = pe_0^C \,, \qquad \rho(e_0^C) = e^1 \,, \qquad \alpha_j(e_i^C) = e_{ij}^C$$

and α, τ, ρ, α_j annihilate the other basis elements of $\mathbb{C}A_{\mathbb{C}}$. Letting

$$s_{C,\chi} = \sum_{i=1}^{p-1} \chi(i^{-1}) e_i^C$$

then $\alpha_j(s_{C,\chi}) = \chi(j) s_{C,\chi}$ and $\rho(s_{C,\chi}) = \tau(s_{C,\chi}) = \alpha(s_{C,\chi}) = 0$. It is now easy to check that, as a direct sum of simple $\mathbb{C}\mathcal{M}_p^{\Delta}$-functors,

$$\mathbb{C}A_{\mathbb{C}} = S_{1,1} \oplus \bigoplus_{\chi \in \mathrm{Irr}(\mathbb{C}\mathrm{Out}(C))} S_{C,\chi}$$

where $S_{1,1} = \mathrm{span}_{\mathbb{C}}\{e^1, e_0^C\}$ and $S_{C,\chi} = \mathrm{span}_{\mathbb{C}}\{s_{C,\chi}\}$. This is a direct sum of p mutually distinct simple $\mathbb{C}\mathcal{M}_p^{\Delta}$-functors. (It is also easy to check that the notation here is compatible with that which appeared in the classification of simple functors in Theorem 3.7, but we shall not be making use of that fact.) By Lemma 6.1, every isomorphism class of simple $\mathbb{C}M_p^{\Delta}$-functor occurs exactly once in $\mathbb{C}A_{\mathbb{C}}$. So the blocks of $\mathbb{C}\mathcal{M}_p^{\Delta}$ are precisely the elements of $\mathbb{C}\mathcal{M}_p^{\Delta}$ that act as the projections to the simple summands. By direct calculation, $e_{1,1}$ acts as the projection to $S_{1,1}$, while $e_{C,\chi}$ acts as the projection to $S_{C,\chi}$. □

Proposition 6.3. *The blocks of $\mathbb{C}\mathcal{M}_p^{\triangleleft}$ are $e_{1,1}+e_{C,1}$ and $e_{C,\chi}$ with $\chi \in \mathrm{Irr}(\mathbb{C}\mathrm{Out}(C))-\{1\}$. The block $e_{1,1}+e_{C,1}$ owns exactly 2 isomorphism classes of simple $\mathbb{C}\mathcal{M}_p^{\triangleleft}$-functors. Each of the other $p-1$ blocks owns a unique isomorphism class of simple $\mathbb{C}\mathcal{M}_p^{\triangleleft}$-functors.*

Proof. By Theorem 5.3, every central idempotent of the algebra $\mathbb{C}\mathcal{M}_p^{\triangleleft} = {}^{\Pi}\mathbb{C}\mathcal{M}_p^{\triangleleft}$ is a central idempotent of the algebra $\mathbb{C}\mathcal{M}_p^{\Delta} = {}^{\Pi}\mathbb{C}\mathcal{M}_p^{\Delta}$. We have $\mathbb{C}\mathcal{M}_p^{\triangleleft} = \mathbb{C}\mathcal{M}_p^{\Delta} \oplus \mathbb{C}\delta$ where $\delta = {}_1\mathrm{def}_C$. So the central idempotents of $\mathbb{C}\mathcal{M}_p^{\triangleleft}$ are precisely those central idempotents of $\mathbb{C}\mathcal{M}_p^{\Delta}$ which commute with δ. Using a formula for composition in Section 3, we obtain the commutation relations

$$\delta\alpha = \alpha_j\delta = \tau\rho\delta = 0\,, \qquad \alpha\delta = \delta\alpha_j = \delta\,, \qquad \delta\tau\rho = \rho\,.$$

We find that δ does not commute with $e_{1,1}$ nor with $e_{C,1}$, but δ does commute with $e_{1,1}+e_{C,1}$ and with $e_{C,\chi}$ for $\chi \neq 1$. So the blocks of $\mathbb{C}\mathcal{M}_p^{\triangleleft}$ are as asserted.

By Theorem 5.2 and the proof of Lemma 6.2, there exist simple $\mathbb{C}\mathcal{M}_p^{\triangleleft}$-functors $S_{1,1}^{\triangleleft}$ and $S_{C,\chi}^{\triangleleft}$ that restrict to the simple $\mathbb{C}\mathcal{M}_p^{\Delta}$ functors $S_{1,1}$ and $S_{C,\chi}$, respectively, where $\chi \in \mathrm{Irr}(\mathbb{C}\mathrm{Out}(C))$. Furthermore, every simple $\mathbb{C}\mathcal{M}_p^{\triangleleft}$-functor is isomorphic to $S_{1,1}^{\triangleleft}$ or one of the $S_{C,\chi}^{\triangleleft}$. Since $e_{1,1}+e_{C,1}$ acts as the identity on $S_{1,1}$ and $S_{C,1}$, the $\mathbb{C}\mathcal{M}_p^{\triangleleft}$-functors $S_{1,1}^{\triangleleft}$ and $S_{C,1}^{\triangleleft}$ belong to $e_{1,1}+e_{C,1}$. Similarly, $S_{C,\chi}^{\triangleleft}$ belongs to $e_{C,\chi}$ for $\chi \neq 1$. □

7. A unique block

Throughout this section, we shall assume that every finite group is isomorphic to a group in $\mathfrak{K}$. We shall prove the following theorem.

Theorem 7.1. *Consider the complete deflation Mackey category $\mathcal{M} = \mathcal{M}_{\mathfrak{K}}^{\triangleleft}$. The $\mathbb{K}$-linear extension $\mathbb{K}\mathcal{M}$ has a unique block.*

We shall make use of the theorem of Hartley–Robinson [7], which implies that, given a finite group G and a prime p not dividing $|G|$, then there exists a finite p-group P and a semidirect product $F = G \ltimes P$ such that $\mathrm{Out}(F) = 1$. In particular, every finite group is a quotient of a finite group with a trivial outer automorphism group.

Let b be the block of $\mathbb{K}\mathcal{M}$ owning the simple $\mathbb{K}\mathcal{M}$-functor $S_{1,1}^{\mathbb{K}\mathcal{M}}$. To prove Theorem 7.1, we must show that $b = 1_{\mathcal{M}}$. Consider the complete ordinary Mackey category $\mathcal{N} = \mathcal{M}_{\mathfrak{K}}^{\Delta}$. By Theorem 5.3, $b \in Z(^{\Pi}\mathbb{K}\mathcal{N})$. By Remark 2.1, we can write $b = \sum_{G \in \mathfrak{K}} b_G$ with each $b_G \in \mathbb{K}\mathcal{N}(G,G)$. Since b owns $S_{1,1}^{\mathbb{K}\mathcal{M}}$, the $(1,1)$-coordinate of b is $b_1 = \mathrm{iso}_1$.

Let $\mathcal{P}_{G,G}^{\triangleleft}$ denote the set of pairs (α, V) such that $V \leq G$ and $\alpha : F \leftarrow V$ is a homomorphism. Let $\mathcal{P}_{G,G}^{\Delta}$ denote the subset of $\mathcal{P}_{G,G}^{\triangleleft}$ consisting of those pairs (α, V) such that α is injective. In the notation of the proof of Proposition 5.1, $\rho(b_G)$ is a linear combination of elements $(\alpha, V)_{G,G} \in \widehat{\mathbb{K}\mathcal{M}}(G,G)$ where (α, V) runs over the elements of $\mathcal{P}_{G,G}^{\triangleleft}$. As we saw in the proof of Proposition 5.1, when α is non-injective, $\rho_{\alpha,V}^{G,G}(\mathbb{K}\mathcal{N}(G,G)) = \{0\}$. In particular, when α is non-injective, $\rho_{\alpha,V}^{G,G}(b_G) = 0$. Therefore,

$$\rho(b_G) = \sum_{(\alpha,V) \in \mathcal{P}_{G,G}^{\Delta}} \rho_{\alpha,V}^{G,G}(b)\,(\alpha, V)_{G,G}\ .$$

Lemma 7.2. *Let $H, K \in \mathfrak{K}$ and let $\pi : H \leftarrow K$ be a surjective homomorphism. Then, in the notation of the proof of Proposition 5.1,*

$$[\pi, K]_{H,K}^{+} = \sum_{hZ(H) \subseteq H} (c(h)\pi, K)_{H,K}\ .$$

Proof. Every $H \times K$-conjugate of (π, K) has the form $(c(h)\pi, K)$ for some $h \in H$. □

Lemma 7.3. *For all $G \in \mathfrak{K}$, we have* $\sum_{\overline{\alpha} \in \mathrm{Out}(G)} \rho_{\alpha,G}^{G,G}(b) = 1.$

Proof. Let π be the homomorphism $1 \leftarrow G$. By Lemma 7.2, $[\pi, G]_{1,G}^{+} = (\pi, G)_{1,G}$. In particular, $(\pi, G)_{1,G}$ belongs to $\widehat{\mathbb{K}\mathcal{M}}$ and commutes with $\rho(b)$. Therefore

$$(\pi, G)_{1,G} = \rho(b_1)(\pi, G)_{1,G} = (\pi, G)_{1,G}\,\rho(b_G) = \sum_{\overline{\alpha} \in \mathrm{Out}(G)} \rho_{\alpha,G}^{G,G}(b)\,(\pi, G)_{1,G}\ . \qquad \square$$

Lemma 7.4. *For all $G \in \mathfrak{K}$ and $\overline{\alpha} \in \mathrm{Out}(G)$, we have* $\rho^{G,G}_{\alpha,G}(b_G) = \begin{cases} 1 & \text{if } \alpha = 1, \\ 0 & \text{otherwise.} \end{cases}$

Proof. By the theorem of Hartley and Robinson mentioned at the beginning of this section, there exists a group $F \in \mathfrak{K}$ such that $\mathrm{Out}(F) = 1$ and G is isomorphic to a quotient of F. Let $\pi : G \leftarrow F$ be a surjective homomorphism. We have

$$[\pi, F]^+_{G,F}\, \rho(b_F) = [\pi, F]^+_{G,F}\, \rho(b) = \rho(b)[\pi, F]^+_{G,F} = \rho(b_G)[\pi, F]^+_{G,F}\ .$$

Using Lemma 7.2,

$$\begin{aligned} [\pi, F]^+_{G,F}\, \rho(b_F) &= \sum_{(\beta,W)\in\mathcal{P}^{\Delta}_{F,F},\ gZ(G)\subseteq G} \rho^{F,F}_{\beta,W}(b_F)(c(g)\pi, F)_{G,F}\, (\beta, W)_{F,F} \\ &= \rho^{F,F}_{1,F}(b_F) \sum_{gZ(G)\subseteq G} (c(g)\pi, F)_{G,F}\ . \end{aligned}$$

On the other hand, using Lemma 7.2 again,

$$\begin{aligned} \rho(b_G)[\pi, F]^+_{G,F} &= \sum_{(\alpha,V)\in\mathcal{P}^{\Delta}_{G,G},\ gZ(G)\subseteq G} \rho^{G,G}_{\alpha,V}(b_G)(\alpha, V)_{G,G}\, (c(g)\pi, F)_{G,F} \\ &= \sum_{\overline{\alpha}\in\mathrm{Out}(G),\ gZ(G)\subseteq G} \rho^{G,G}_{\alpha,G}(b_G)(\alpha c(g)\pi, F)_{G,F}\ . \end{aligned}$$

Comparing coefficients, we deduce that $\rho^{G,G}_{\alpha,G}(b_G) \neq 0$ when $\overline{\alpha} \neq 1$. Lemma 7.3 now yields $\rho^{G,G}_{1,G}(b_G) = 1$. □

Lemma 7.5. *For $G \in \mathfrak{K}$, let $\mathbb{K}\mathcal{N}_<(G,G)$ be the ideal of $\mathbb{K}\mathcal{N}(G,G)$ spanned by the transitive morphisms that have the form ${}_G\mathrm{ind}^{\beta}_W\mathrm{res}_G$ with $W < G$. Then $b_G \equiv 1$ modulo $\mathbb{K}\mathcal{N}_<(G,G)$.*

Proof. By Proposition 3.1, $\mathbb{K}\mathcal{N}_<(G,G)$ is indeed an ideal of $\mathbb{K}\mathcal{N}(G,G)$. We can write

$$b_G = c_G + \sum_{\overline{\alpha}\in\mathrm{Out}(G)} b_{\alpha,G} \cdot {}_G\mathrm{iso}^{\alpha}_G$$

where $c_G \in \mathbb{K}\mathcal{N}_<(G,G)$ and each $b_{\alpha,G} \in \mathbb{K}$. Since $\rho^{G,G}_{\alpha,G}(b_G) = b_{\alpha,G}$, the required conclusion follows from Lemma 7.4. □

The latest lemma implies that, for every seed (G,V) of $\mathbb{K}\mathcal{M}$, the idempotent b_G acts as the identity on $S^{\mathbb{K}\mathcal{M}}_{G,V}$. So b owns $S^{\mathbb{K}\mathcal{M}}_{G,V}$. By Theorem 3.7, b owns every simple $\mathbb{K}\mathcal{M}$-functor. Therefore $b = 1_{\mathcal{M}}$. The proof of Theorem 7.1 is complete.

We mention that, if we were to assume that the isomorphism classes in $\mathfrak{K}$ are those of the finite solvable groups, then the conclusion of Theorem 7.1 would still hold because,

in the proof of Lemma 7.4, we could take F to be solvable. We do not know whether the conclusion of the theorem still holds when the isomorphism classes in $\mathfrak{K}$ are those of the finite p-groups.

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