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A refinement of Alperin's Conjecture for blocks of the endomorphism algebra of the Sylow permutation module

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Abstract. We present a refinement of Alperin's Conjecture involving the blocks of the endomorphism algebra of the permutation module formed by the cosets of a p-subgroup. We prove the conjecture in two special cases where every weight module has a simple socle.

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1. Statement of the Conjecture. Shortly after proposing his weight conjecture [2], Alperin suggested, in seminars, that one approach towards tackling the conjecture would be to examine the endomorphism algebra $\mathrm{End}_{kG}(kG/S)$ of the permutation kG-module kG/S. Here, k is an algebraically closed field of prime characteristic p and S is a Sylow p-subgroup of a finite group G. Naehrig [10] has supplied some empirical evidence to suggest that the simple socle constituents of the regular module of $\mathrm{End}_{kG}(kG/S)$ may serve as an intermediate tool to relate the simple kG-modules with the weight kG-modules.

Recall, a **weight kG -module** is defined to be an indecomposable kG-module W such that, letting P be a vertex of W, then the $kN_G(P)$-module in Green correspondence with W is the inflation of a simple projective $kN_G(P)/P$-module. The weak form of Alperin's Weight Conjecture [2] asserts that the number of isomorphism classes of simple kG-modules is equal to the number of isomorphism classes of weight kG-modules. The block form of Alperin's Conjecture asserts that, given a block b of kG, then the number of isomorphism classes of simple kGb-modules is equal to the number of isomorphism classes of weight kGb-modules.

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By an easy application of Frobenius Reciprocity, every simple kG-module occurs in both the socle and the head of kG/S. The rationale for the study of $\mathrm{End}_{kG}(kG/S)$ arises from the following observation of Alperin [2, Lemma 1], which tells us that, in particular, every weight kG-module occurs in both the socle and the head of kG/S.

Lemma 1.1. (Alperin) *Every weight kG-module occurs as a direct summand of the Sylow permutation kG-module kG/S.*

We deem all kG-modules to be finite-dimensional. A kG-module L is said to be **connected** provided $\mathrm{End}_{kG}(L)$ has a unique block. It is easy to see that a direct summand L of a kG-module M is maximal among the connected direct summands of M if and only if $L = eM$ for some block e of $\mathrm{End}_{kG}(M)$. When these equivalent conditions hold, we call L a **proper component** of M. Plainly, any kG-module is the direct sum of its proper components.

We say that a kG-module L **lies in** a kG-module M, written $L \dashv M$, provided that L is isomorphic to the image of a kG-endomorphism of a direct sum of finitely many copies of M. This is equivalent to the condition that there exists a direct sum M' of finitely many copies of M such that L is isomorphic to a submodule of M' and L is isomorphic to a quotient module of M'. We say that M is **accordant** provided the number of isomorphism classes of simple kG-modules lying in M is equal to the number of isomorphism classes of weight kG-modules lying in M.

Using Lemma 1.1, it is not hard to see that, for any p-subgroup P of G, the weak form of Alperin's Conjecture holds for kG if and only if the permutation kG-module kG/P is accordant.

Conjecture 1.2. *For any p-subgroup P of G, every proper component of kG/P is accordant.*

The next three remarks are very easy and we omit the proofs.

Remark 1.3. Given a connected kG-module L lying in a kG-module M, then L lies in a unique proper component of M.

Remark 1.4. Let U and V be connected kG-modules lying in a kG-module M. Then U and V lie in the same proper component of M if and only if there exist connected kG-modules $W_0, \ldots, W_r$ lying in M such that $W_0 \cong U$ and $W_r \cong V$ and for each $1 \leq i \leq r$, there exists a non-zero kG-map $W_{i-1} \to W_i$ or $W_{i-1} \leftarrow W_i$.

Remark 1.5. Let L and M be kG-modules such that $L \dashv M$. Let U and V be connected kG-modules lying in L. Then U and V lie in M. If U and V lie in the same proper component of L, then U and V lie in the same proper component of M.

In the special case where P is trivial, Conjecture 1.2 is equivalent to the block form of Alperin's Conjecture. So the next result can be interpreted as saying that Conjecture 1.2 is a refinement of Alperin's Conjecture.

Proposition 1.6. *Let P and Q be p-subgroups of G with $P \leq Q$. If every proper component of kG/Q is accordant, then every proper component of kG/P is accordant.*

Proof. By Frobenius Reciprocity, every simple kG-module lies in kG/Q. By Lemma 1.1, every weight kG-module lies in kG/S. But $kG/S \dashv kG/Q$, so every weight kG-module lies in kG/Q. Since $kG/Q \dashv kG/P$, the required conclusion now follows from Remark 1.5. □

Therefore, if Conjecture 1.2 holds when $P = S$, then it holds for all p-subgroups P of G and, in particular, the block form of Alperin's Conjecture holds for kG.

Let us point out a connection with Naehrig [10]. When two indecomposable direct summands U and V of kG/S are equivalent in the sense of [10, 4.1(b)], the corresponding principal indecomposable modules of $\mathrm{End}_{kG}(kG/S)$ lie in the same block of $\mathrm{End}_{kG}(kG/S)$, hence U and V lie in the same connected component of kG/S.

In Sect. 2, we shall illustrate the conjecture with some examples. In Sect. 3, we shall deal with two special cases. We shall show that, when G has a split BN-pair of characteristic p, the Cabanes–Sawada Theorem immediately implies that the conjecture holds for the Sylow permutation kG-module. We shall also show that, letting T be a Sylow p-subgroup of the normalizer of a cyclic defect group of a block b of kG, then the conjecture holds for the proper components of bkG/T.

The conjecture originates in [3]. Though not mentioned in [4], it was one of the motives for the defect theory, in [4], for blocks of endomorphism algebras.

2. Some examples. In this section, to illustrate Conjecture 1.2, we present the structure of the Sylow permutation module in two particular cases.

First put $p = 2$ and $G = A_7$. Using the MAGMA source code in Zimmermann's thesis [12], it can be shown that, over the field $\mathbb{F}_2$ of order 2, the 2-Sylow permutation module has the depicted structure, where n denotes an n-dimensional simple $\mathbb{F}_2G$-module and n^* denotes its dual.

$$(1) \oplus (14) \oplus \begin{pmatrix} 14 & 20 \\ 1 & 1 \\ 14 & 20 \end{pmatrix} \oplus 2 \begin{pmatrix} 20 \\ 1 \\ 14 \\ 1 \\ 20 \end{pmatrix} \oplus \begin{pmatrix} & 14 & \\ & 1 & \\ 14 & & 20 \\ & 1 & \\ & 14 & \end{pmatrix}$$
$$\oplus (6) \oplus \begin{pmatrix} 4^* \\ 6 \\ 4 \end{pmatrix} \oplus \begin{pmatrix} 4 \\ 6 \\ 4^* \end{pmatrix} \oplus \begin{pmatrix} & 6 & \\ 4 & & 4^* \\ & 6 & \end{pmatrix}.$$

Using Zimmermann's MAGMA routines or, alternatively, using data in Benson [5, Appendix], it can be shown that all 6 of the simple $\mathbb{F}_2G$-modules are absolutely simple.

Again using MAGMA or [5, Appendix], it can be shown that the indecomposable summands with Loewy length 5 are projective and therefore cannot be weight modules. The non-simple indecomposable summand with socle 6 has vertex V_4 and has a 4-dimensional non-simple Green correspondent, so this summand is not a weight module. But the simple summand 6 and the indecomposable summands with socles 4 and 4^* all have vertex V_4 and Green correspondents that are 2-dimensional, absolutely simple, and inflated from projective modules. So those three summands are weight modules. Similarly, the simple summands 1 and 14 and the indecomposable summand with socle $14 + 20$ are weight modules. Evidently, the proper components of the Sylow permutation module have dimensions $1, 260, 54$ with $1, 2, 3$ isomorphism classes of simple modules and $1, 2, 3$ isomorphism classes of weight modules lying in them.

Let us give an example where the partitioning of simple modules and weight modules into blocks of $\mathrm{End}_{kG}(kG/S)$ is much finer than the partitioning into blocks of kG. Using MAGMA or [5, Appendix], it is not hard to show that, for $p = 3$ and $G = M_{10}$, the 3-Sylow permutation module has the structure

$$(1) \oplus (1_-) \oplus \begin{pmatrix} & 4 & \\ 1 & & 1_- \\ & 4^* & \end{pmatrix} \oplus \begin{pmatrix} & 4^* & \\ 1 & & 1_- \\ & 4 & \end{pmatrix} \oplus 2 \begin{pmatrix} & 6 & \\ 4 & & 4^* \\ & 6 & \end{pmatrix} \oplus (9_1) \oplus (9_2)\,.$$

In this case, the principal block of kG contains 4 of the proper components.

The authors have also confirmed that Conjecture 1.2 holds for the groups $S_6, A_7, L_2(25), M_{11}, J_1$ in characteristic 2, for $S_6, S_7, A_8, L_3(4), L_2(25), M_{11}$ in characteristic 3, and for McL in characteristic 5. Using data in Lempken–Staszewski [9], it can be shown that, in the principal 5-block of McL, three of the weight modules have socles of the form $2.250 + 896_2$ and $2.560 + 3038 + 3245_1 + 3245_2$ and $896_1 + 3.3038$.

3. Proof in two special cases. Let us first show that the conjecture holds in the scenario of the Cabanes–Sawada Theorem.

Theorem 3.1. (Cabanes–Sawada) *Suppose that G has a split BN-pair of characteristic p. Let S be a Sylow p-subgroup of G. Then:*

1. *Every indecomposable direct summand of kG/S is a weight kG-module. Every weight kG-module occurs with multiplicity 1 in kG/S.*
2. *There is a bijective correspondence between the isomorphism classes of simple kG-modules U and the isomorphism classes of weight kG-modules W such that the isomorphism classes of U and W correspond provided $U \cong \mathrm{soc}(W)$.*

In particular, every proper component of kG/S is accordant.

Proof. This follows from Cabanes [6, Proposition 8], which says that the weak form of Alperin's Conjecture holds for kG, and Sawada [11, 2.8], which says that every simple kG-module has multiplicity 1 in $\mathrm{soc}(kG/S)$. □

For another approach towards simultaneously refining Alperin's Conjecture and generalizing the Cabanes-Sawada Theorem, see [10, Section 3]. We now turn to the case of a block with a cyclic defect group.

Theorem 3.2. *Let b be a block of kG with a cyclic defect group D. Let T be a Sylow p-subgroup of $N_G(D)$. Then every proper component of bkG/T is accordant.*

Proof. Erdmann's Theorem [7] asserts that, given a simple kG-module V with cyclic vertex Q, then Q is the defect group of the block of kG containing V. Hence, using the compatibility of the Green correspondence and the Brauer correspondence, as recorded in Alperin [1, 14.4], it is easy to show that every simple kGb-module and every weight kGb-module has vertex D.

We may assume that D is non-trivial. Let E be the smallest non-trivial subgroup of D. Suppose that $E \trianglelefteq G$. Given a subgroup L of G containing E, we write $\overline{L} = L/E$. Let $\overline{b}$ be the image of b under the canonical epimorphism $kG \to k\overline{G}$. The simple kGb-modules, all of which have vertex D, are the inflations of the simple $k\overline{G}$-modules, all of which have vertex $\overline{D}$. Writing $\overline{b} = \sum_i b_i$ as a sum of blocks b_i of $k\overline{G}$, then all the blocks b_i have defect group $\overline{D}$. Since $\overline{T}$ is a Sylow p-subgroup of the group $N_{\overline{G}}(\overline{D}) = \overline{N_G(D)}$, an inductive argument on $|D|$ allows us to assume that every proper component of $\overline{b}k\overline{G}/\overline{T}$ is accordant. Observing that $\overline{b}k\overline{G}/\overline{T}$ inflates to bkG/T, we deduce that bkG/T is accordant in the case $E \trianglelefteq G$.

Now suppose that E is not normal in G. Let $H = N_G(E)$. Since D is cyclic, $N_G(D) \leq H$. Let c be the block of kH with defect group P such that c is in Brauer correspondence with b. By Erdmann's Theorem combined with the compatibility of the Green correspondence and the Brauer correspondence again, the Green correspondence, with respect to vertex D, restricts to a bijective correspondence between the isomorphism classes of weight kHc-modules and the isomorphism classes of weight kGb-modules. Green [8, Theorem 1(ii)] says that the isomorphism classes of simple kHc-modules V are in a bijective correspondence with the isomorphism classes of simple kGb-modules U whereby $V \leftrightarrow U$ provided U is isomorphic to the socle of the Green correspondent of V.

Let W be a weight kHc-module, and let V be a simple kHc-module. Let $\mathcal{G}(W)$ and $\mathcal{G}(V)$ denote the kGb-modules in Green correspondence with W and V, respectively. By the previous paragraph, $\mathcal{G}(W)$ is a weight kGb-module and $\mathcal{G}(V)$ is an indecomposable kGb-module with a unique simple submodule V_G. Supposing that W and V lie in the same proper component of the kH-module kH/T then, by [4, Corollary 5.7(b)], $\mathcal{G}(W)$ and $\mathcal{G}(V)$ lie in the same proper component of the kG-module $kG/T \cong {}_G\mathrm{Ind}_H(kH/T)$. Plainly, $\mathcal{G}(W)$ and V_G lie in the same proper component of kG/T. We have shown that, given a weight kHc-module and a simple kHc-module lying in the same proper component of kH/T, then the corresponding weight kGb-module and simple kGb-module lie in the same proper component of kG/T. The required conclusion for bkG/T now follows because, by an inductive argument on $|G|$, we may assume that the required conclusion holds for ckH/T. □

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