

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/278679658>

Orthogonal polynomials for the weakly equilibrium Cantor sets

ARTICLE *in* PROCEEDINGS OF THE AMERICAN MATHEMATICAL SOCIETY · DECEMBER 2015

Impact Factor: 0.68 · DOI: 10.1090/proc/13025

CITATION

1

READS

7

2 AUTHORS, INCLUDING:



Gökulp Alpan

Bilkent University

5 PUBLICATIONS 6 CITATIONS

SEE PROFILE

ORTHOGONAL POLYNOMIALS FOR THE WEAKLY EQUILIBRIUM CANTOR SETS

GÖKALP ALPAN AND ALEXANDER GONCHAROV

ABSTRACT. Let $K(\gamma)$ be the weakly equilibrium Cantor type set introduced in [10]. It is proven that the monic orthogonal polynomials Q_{2^s} with respect to the equilibrium measure of $K(\gamma)$ coincide with the Chebyshev polynomials of the set. Procedures are suggested to find Q_n of all degrees and the corresponding Jacobi parameters. It is shown that the sequence of the Widom factors is bounded below.

1. INTRODUCTION

Cantor sets appear as supports of spectral measures for the Jacobi operators in some natural situations. For example, the Schrödinger operator with an almost periodic potential (for example, the almost Mathieu operator) typically has such spectrum (see e.g. the review [17]). On the other hand, we can meet Julia sets K of the Cantor type in the theory of orthogonal polynomials (OP) with respect to the equilibrium measure μ_K (see e.g. [2, 3, 4]). Whereas, due to B. Simon et al., there is a comprehensive theory for the finite gap Jacobi matrices, until now there is no such theory for purely singular continuous measures. In order to analyze the infinite gap Jacobi matrices, the class of Parreau-Widom sets was suggested (see e.g. [5]). This notion includes Cantor sets of positive Lebesgue measure, so the Szegő condition can be applied in this case. Recently numerous attempts were undertaken to investigate the spectral problem for self-similar measures with Cantor supports. By means of computational methods, some conjectures (see e.g. [11, 12]) about the asymptotic behavior of the Jacobi parameters and other related values were posed.

Here we consider one more example of a family of OP with respect to $\mu_{K(\gamma)}$ on the Cantor set $K(\gamma)$, introduced in [10]. The set $K(\gamma)$ depends on a parameter $\gamma = (\gamma_s)_{s=1}^\infty$ and is constructed by means of a recurrent procedure. If γ_s are not very small then the set $K(\gamma)$ is not polar. At least in known cases, the set $K(\gamma)$ is dimensional and, by [1], $\mu_{K(\gamma)}$ is mutually absolutely continuous with the corresponding Cantor-Hausdorff measure. This is not valid for geometrically symmetric zero Lebesgue measure Cantor sets, where, by [13] and followers, these measures are mutually singular.

In Section 2 we show that the monic OP Q_{2^s} coincide with the corresponding Chebyshev polynomials. In Sections 3 and 4 we suggest a procedure to find Q_n for $n \neq 2^s$. This allows to analyze the asymptotics of the Jacobi parameters $(a_n)_{n=1}^\infty$. Since $\text{Cap}(K(\gamma))$ is known, we estimate (Section 5) the *Widom factors* $W_n := \frac{a_1 \cdots a_n}{\text{Cap}(K(\gamma))^n}$ and check the Widom condition, that characterizes the Szegő class of Jacobi matrices in the finite gap case. In the last section we discuss a possible version of the Szegő condition for singular continuous measures. At least for $\gamma_s \leq$

$1/6, s \in \mathbb{N}$, the Lebesgue measure of the set $K(\gamma)$ is zero, so it is not a Parreau-Widom set.

For the basic concepts of the theory of logarithmic potential see e.g. [15], \log denotes the natural logarithm, $Cap(\cdot)$ stands for the logarithmic capacity, $0^0 := 1$.

2. ORTHOGONAL POLYNOMIALS

Given a sequence $\gamma = (\gamma_s)_{s=1}^\infty$ with $0 < \gamma_s < 1/4$ define $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$. Let

$$(2.1) \quad P_1 := x - 1 \text{ and } P_{2^{s+1}}(x) := P_{2^s}(x) \cdot (P_{2^s}(x) + r_s)$$

for $s \in \mathbb{N}_0$ in a recursive fashion. Thus, $P_2(x) = x \cdot (x - 1)$ for each γ , whereas, for $s \geq 2$, the polynomial P_{2^s} essentially depends on the parameter γ . For $s \in \mathbb{N}_0$ consider

$$E_s = \{x \in \mathbb{R} : P_{2^{s+1}}(x) \leq 0\} = \left(\frac{2}{r_s} P_{2^s} + 1\right)^{-1}([-1, 1]) = \cup_{j=1}^{2^s} I_{j,s},$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $I_{j,s}$ are closed *basic* intervals of the s -th level which are necessarily disjoint. Then $E_{s+1} \subset E_s$ and the set $K(\gamma) := \cap_{s=0}^\infty E_s$ is a Cantor set by Lemma 2 in [10].

Let $l_{i,s}$ stand for the length of $I_{j,s}$ when we enumerate them from the left to the right. By Lemma 6 in [10],

$$\gamma_1 \dots \gamma_s < l_{i,s} < \exp\left(16 \sum_{k=1}^s \gamma_k\right) \gamma_1 \dots \gamma_s, \quad 1 \leq i \leq 2^s,$$

provided $\gamma_k \leq 1/32$ for all k . Therefore, the Lebesgue measure $|K(\gamma)|$ of the set with this condition is zero and, by [5], $K(\gamma)$ is not a Parreau-Widom set. For the definition of Parreau-Widom sets see e.g. [5]. In Section 4 we will show that $|K(\gamma)| = 0$ as well if $\gamma_k \leq 1/6$ for all k .

On the other hand, by choosing $(\gamma_k)_{k=1}^\infty$ sufficiently close to $1/4$, we can obtain Cantor sets with positive Lebesgue measure. What is more, in the limit case, when all $\gamma_k = 1/4$, we get $E_s = [0, 1]$ for all s and $K(\gamma) = [0, 1]$ (see Example 1 in [10]).

In addition, by Corollary 1 in [10], $Cap(K(\gamma)) = \exp\left(\sum_{k=1}^\infty 2^{-k} \log \gamma_k\right)$. In the paper we assume $Cap(K(\gamma)) > 0$. Let $\mu_{K(\gamma)}$ denote the equilibrium measure on the set, $\|\cdot\|$ be the norm in the corresponding Hilbert space. From Corollary 3.2 in [1] we have $\mu_{K(\gamma)}(I_{j,s}) = 2^{-s}$ for all s and $1 \leq j \leq 2^s$, provided $\gamma_k \leq 1/32$ for all k .

From now on, by Q_n we denote the monic orthogonal polynomial of degree $n \in \mathbb{N}$ with respect to $\mu_{K(\gamma)}$. The main result of this section is that, for $n = 2^s$ with $s \in \mathbb{N}_0$, the polynomial Q_n coincides with the corresponding Chebyshev polynomial for $K(\gamma)$. The next two theorems will play a crucial role.

Theorem 2.1 ([10], Prop.1). *For each $s \in \mathbb{N}_0$ the polynomial $P_{2^s} + r_s/2$ is the Chebyshev polynomial for $K(\gamma)$.*

Remark 2.2. Only the values $s \in \mathbb{N}$ were considered in [10]. But, clearly, for $s = 0$ the polynomial $P_1 + 1/2 = x - 1/2$ is Chebyshev.

Remark 2.3. Since real polynomials are considered here and the alternating set for $P_{2^s} + r_s/2$ consists of $2^s + 1$ points, the Chebyshev property of this polynomial follows by the alternance argument.

Theorem 2.4 ([16], T.3.6). *Let $K \subset \mathbb{R}$ be a non-polar compact set. Then the normalized counting measures on the zeros of the Chebyshev polynomials converge to the equilibrium measure of K in the weak-star topology.*

For $s \in \mathbb{N}$, the polynomial $P_{2^s} + r_s / 2$ has simple real zeros $(x_k)_{k=1}^{2^s}$ which are symmetric about $x = 1/2$. Let us denote by ν_s the normalized counting measure at these points, that is $\nu_s = 2^{-s} \sum_{k=1}^{2^s} \delta_{x_k}$.

Lemma 2.5. *Let $s > m$ with $s, m \in \mathbb{N}_0$. Then $\int (P_{2^m} + \frac{r_m}{2}) d\nu_s = 0$.*

Proof. For $m = 0$ we have the result by symmetry. Suppose $m \geq 1$. By (2.1), at the points $(x_k)_{k=1}^{2^s}$ we have

$$P_{2^s} + \frac{r_s}{2} = (P_{2^{s-1}})^2 + r_{s-1}P_{2^{s-1}} + \frac{r_s}{2} = 0.$$

The discriminant of the equation is positive. Therefore, the roots satisfy

$$(P_{2^{s-1}} + \alpha_{s-1}^1)(P_{2^{s-1}} + \alpha_{s-1}^2) = 0,$$

where $\alpha_{s-1}^1 + \alpha_{s-1}^2 = r_{s-1}$ and $0 < \alpha_{s-1}^1, \alpha_{s-1}^2 < r_{s-1}$. Thus, a half of the points satisfy $P_{2^{s-1}} + \alpha_{s-1}^1 = 0$ while the other half satisfy $P_{2^{s-1}} + \alpha_{s-1}^2 = 0$.

Rewriting the equation $P_{2^{s-1}} + \alpha_{s-1}^1 = 0$, we see that

$$P_{2^{s-2}}^2 + r_{s-2}P_{2^{s-2}} + \alpha_{s-1}^1 = 0.$$

Since $r_{s-2}^2 > 4r_{s-1} > 4\alpha_{s-1}^1$, this yields

$$(P_{2^{s-2}} + \alpha_{s-2}^1)(P_{2^{s-2}} + \alpha_{s-2}^2) = 0$$

with $\alpha_{s-2}^1 + \alpha_{s-2}^2 = r_{s-2}$ and $0 < \alpha_{s-2}^1, \alpha_{s-2}^2 < r_{s-2}$. By the same argument, the second half of the roots satisfy

$$(P_{2^{s-2}} + \alpha_{s-2}^3)(P_{2^{s-2}} + \alpha_{s-2}^4) = 0$$

with $\alpha_{s-2}^3 + \alpha_{s-2}^4 = r_{s-2}$ and $0 < \alpha_{s-2}^3, \alpha_{s-2}^4 < r_{s-2}$.

Since at each step $r_{i-1}^2 > 4r_i$ we can continue this procedure until obtaining $P_{2^{m+1}}$. So we can decompose the Chebyshev nodes $(x_k)_{k=1}^{2^s}$ into 2^{s-m-1} groups. All 2^{m+1} nodes from the i -th group satisfy

$$P_{2^{m+1}} + \alpha_{m+1}^i = 0, \quad 0 < \alpha_{m+1}^i < r_{m+1}.$$

By using these 2^{s-m-1} equations we finally obtain

$$(P_{2^m} + \alpha_m^{2^{i-1}})(P_{2^m} + \alpha_m^{2^i}) = 0$$

where $\alpha_m^{2^{i-1}} + \alpha_m^{2^i} = r_m$. Thus, given fixed i -th group, for 2^m points from the group we have $P_{2^m} = -\alpha_m^{2^{i-1}}$, whereas for another half, $P_{2^m} = -\alpha_m^{2^i}$. Consequently, we have

$$\int \left(P_{2^m} + \frac{r_m}{2} \right) d\nu_s = \int P_{2^m} d\nu_s + \frac{r_m}{2} = \frac{\sum_{i=1}^{2^{s-m-1}} 2^m (-\alpha_m^{2^{i-1}} - \alpha_m^{2^i})}{2^s} + \frac{r_m}{2} = 0.$$

□

Lemma 2.6. *Let $0 \leq i_1 < i_2 < \dots < i_n < s$. Then*

(a)

$$\begin{aligned} \int P_{2^{i_1}} P_{2^{i_2}} \dots P_{2^{i_n}} d\nu_s &= \int P_{2^{i_1}} d\nu_s \int P_{2^{i_2}} d\nu_s \dots \int P_{2^{i_n}} d\nu_s \\ &= (-1)^n \prod_{k=1}^n \frac{r_{i_k}}{2}. \end{aligned}$$

$$(b) \int \left(P_{2^{i_1}} + \frac{r_{i_1}}{2}\right) \left(P_{2^{i_2}} + \frac{r_{i_2}}{2}\right) \dots \left(P_{2^{i_n}} + \frac{r_{i_n}}{2}\right) d\nu_s = 0.$$

Proof. (a) Suppose that $i_1 \geq 1$. As above, we can decompose the nodes $(x_k)_{k=1}^{2^s}$ into 2^{s-i_1-1} equal groups such that the nodes from the j -th group satisfy an equation

$$(P_{2^{i_1}} + \alpha_{i_1}^{2j-1})(P_{2^{i_1}} + \alpha_{i_1}^{2j}) = 0$$

with $\alpha_{i_1}^{2j-1} + \alpha_{i_1}^{2j} = r_{i_1}$. If, on some set, $(P_{2^k} + \alpha)(P_{2^k} + \beta) = 0$ with $\alpha + \beta = r_k$, then $P_{2^{k+1}} = P_{2^k}^2 + P_{2^k} r_k = -\alpha\beta$ and each of the next polynomials $P_{2^{k+i}}$ is constant on this set. Therefore the function $P_{2^{i_2}} \dots P_{2^{i_n}}$ takes the same value for all x_k from the j -th group. This allows to apply the argument of Lemma 2.5:

$$\int P_{2^{i_1}} P_{2^{i_2}} \dots P_{2^{i_n}} d\nu_s = -\frac{r_{i_1}}{2} \int P_{2^{i_2}} P_{2^{i_3}} \dots P_{2^{i_n}} d\nu_s.$$

This equality is valid also for $i_1 = 0$ since

$$\int \left(P_1 + \frac{1}{2}\right) P_{2^{i_2}} \dots P_{2^{i_n}} d\nu_s = 0,$$

by symmetry. Proceeding this way, the result follows, since $-r_m/2 = \int P_{2^m} d\nu_s$, by Lemma 2.5.

(b) Opening the parentheses yields

$$\int P_{2^{i_1}} P_{2^{i_2}} \dots P_{2^{i_n}} d\nu_s + \sum_{k=1}^n \frac{r_{i_k}}{2} \int \prod_{j \neq k} P_{2^{i_j}} d\nu_s + \dots + \prod_{k=1}^n \frac{r_{i_k}}{2}.$$

By Lemma 2.5 and part (a), this is

$$\prod_{k=1}^n \frac{r_{i_k}}{2} \cdot \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} = 0.$$

□

Remark 2.7. We can use $\mu_{K(\gamma)}$ instead of ν_s in Lemma 2.5 and Lemma 2.6, since, by Theorem 2.4, $\nu_s \rightarrow \mu_{K(\gamma)}$ in the weak-star topology.

Theorem 2.8. *The monic orthogonal polynomial Q_{2^s} with respect to the equilibrium measure $\mu_{K(\gamma)}$ coincides with the corresponding Chebyshev polynomial $P_{2^s} + r_s/2$ for all $s \in \mathbb{N}_0$.*

Proof. For $s = 0$ we have the result by symmetry. Let $s \geq 1$. Each polynomial P of degree less than 2^s is a linear combination of polynomials of the type

$$\left(P_{2^{s-1}} + \frac{r_{s-1}}{2}\right)^{n_{s-1}} \dots \left(P_2 + \frac{r_1}{2}\right)^{n_1} \left(x - \frac{1}{2}\right)^{n_0}$$

with $n_i \in \{0, 1\}$. By Lemma 2.6, $P_{2^s} + r_s / 2$ is orthogonal to all polynomials of degree less than 2^s , so it is Q_{2^s} . \square

By (2.1), we immediately have

Corollary 2.9. $Q_{2^{s+1}} = Q_{2^s}^2 - (1 - 2\gamma_{s+1})r_s^2/4$ for $s \in \mathbb{N}_0$.

3. SOME PRODUCTS OF ORTHOGONAL POLYNOMIALS

So far we only obtain 2^s degree orthogonal polynomials. We try to find Q_n for other degrees. By Corollary 2.9, since $\int Q_{2^{s+1}} d\mu_{K(\gamma)} = 0$, we have

$$(3.1) \quad \|Q_{2^s}\|^2 = \int Q_{2^s}^2 d\mu_{K(\gamma)} = (1 - 2\gamma_{s+1})r_s^2/4$$

and

$$(3.2) \quad Q_{2^{s+1}} = Q_{2^s}^2 - \|Q_{2^s}\|^2, \quad \forall s \in \mathbb{N}_0.$$

Our next goal is to evaluate $\int A d\mu_{K(\gamma)}$ for A -polynomial of the form

$$(3.3) \quad A = (Q_{2^{s_n}})^{i_n} (Q_{2^{s_{n-1}}})^{i_{n-1}} \cdots (Q_{2^{s_1}})^{i_1},$$

where $s_n > s_{n-1} > \cdots > s_1 > 0$ and $i_1, i_2, \dots, i_n \in \{1, 2\}$.

The next lemma is basically a consequence of (3.2).

Lemma 3.1. *Let A be a polynomial satisfying (3.3). Then the following propositions hold:*

- (a) *If $i_n = 2$ then $\int A d\mu_{K(\gamma)} = \|Q_{2^{s_n}}\|^2 \int Q_{2^{s_{n-1}}}^{i_{n-1}} \cdots Q_{2^{s_1}}^{i_1} d\mu_{K(\gamma)}$.*
- (b) *If $i_n = i_{n-1} = \cdots = i_{n-k} = 1$ and $i_{n-k-1} = 2$ where $k \in \{0, 1, \dots, n-2\}$, then*

$$\int A d\mu_{K(\gamma)} = \|Q_{2^{s_n}}\|^2 \int Q_{2^{s_{n-k-2}}}^{i_{n-k-2}} \cdots Q_{2^{s_1}}^{i_1} d\mu_{K(\gamma)},$$

provided $s_n = s_{n-1} + 1 = s_{n-2} + 2 = \cdots = s_{n-k-1} + k + 1$.

- (c) *If $i_k = 1$ and $s_k \geq s_{k-1} + 2$ for some $k \in \{2, 3, \dots, n\}$, then $\int A d\mu_{K(\gamma)} = 0$.*
- (d) *If $i_1 = i_2 = \cdots = i_k = 1$ for some $k \in \{1, 2, \dots, n\}$, then $\int A d\mu_{K(\gamma)} = 0$.*

Proof. (a) Using (3.2), we have $Q_{2^{s_n}}^2 = Q_{2^{s_n+1}} + \|Q_{2^{s_n}}\|^2$. The result easily follows since the degree of $Q_{2^{s_{n-1}}}^{i_{n-1}} \cdots Q_{2^{s_1}}^{i_1}$ is less than 2^{s_n+1} .

- (b) Here $A = Q_{2^{s_n}} Q_{2^{s_{n-1}}} \cdots Q_{2^{s_{n-k}}} Q_{2^{s_{n-k-1}}}^2 \cdot P$ with $P = Q_{2^{s_{n-k-2}}}^{i_{n-k-2}} \cdots Q_{2^{s_1}}^{i_1}$. Observe that the degrees of the first $k+2$ polynomials are consecutive and $\deg P < 2^{s_{n-k-2}+2} \leq 2^{s_{n-k}}$. We apply (3.2) repeatedly. First, $Q_{2^{s_{n-k-1}}}^2 = Q_{2^{s_{n-k}}} + \|Q_{2^{s_{n-k-1}}}\|^2$. Then $Q_{2^{s_{n-k}}} Q_{2^{s_{n-k-1}}}^2 = Q_{2^{s_{n-k+1}}} + \|Q_{2^{s_{n-k}}}\|^2 + \|Q_{2^{s_{n-k-1}}}\|^2 \cdot Q_{2^{s_{n-k}}}$. After $k+1$ steps we write A in the form $(Q_{2^{s_{n+1}}} + \|Q_{2^{s_n}}\|^2 + \mathcal{L})P$, where \mathcal{L} is a linear combination of the polynomials $Q_{2^{s_n}}, Q_{2^{s_n}} Q_{2^{s_{n-1}}}, \dots, Q_{2^{s_n}} Q_{2^{s_{n-1}}} \cdots Q_{2^{s_{n-k}}}$. By orthogonality, all terms vanish after integration, except $\|Q_{2^{s_n}}\|^2 P$, which is the desired conclusion.

(c) Let us take the maximal k with such property. Repeated application of (a) and (b) enables us to reduce $\int A d\mu_{K(\gamma)}$ to $C \int A_1 d\mu_{K(\gamma)}$ with $C > 0$ and $A_1 = Q_{2^{s_m}} \cdots Q_{2^{s_k}} \cdot R$, where $R = Q_{2^{s_{k-1}}}^{i_{k-1}} \cdots Q_{2^{s_1}}^{i_1}$ with $\deg R < 2^{s_{k-1}+2} \leq 2^{s_k}$. Comparing the degrees gives the result.

(d) Similarly, $\int A d\mu_{K(\gamma)} = C \cdot \int Q_{2^{s_k}} \cdots Q_{2^{s_1}} d\mu_{K(\gamma)} = 0$.

□

Theorem 3.2. *For A -polynomial given in (3.3) with $s_0 := -1, i_{n+1} := 2$, let $c_k = (i_k - 1)^{s_k - s_{k-1} - 1}$ and $c = \prod_{k=1}^n c_k$. Then $\int A d\mu_{K(\gamma)} = c \cdot \prod_{k=1}^n \|Q_{2^k}\|^{2(i_{k+1}-1)}$.*

Proof. First we remark that $c \in \{0, 1\}$ and $c = 0$ if and only if $i_1 = 1$ or $i_k = 1$ for some $k \in \{2, 3, \dots, n\}$ with $s_k > s_{k-1} + 1$, that is just in the cases (c) and (d) above.

Secondly, the procedures (a) – (d) of Lemma 3.1 allow to find $\int A d\mu_{K(\gamma)}$ for all admissible values of $(i_k)_{k=1}^n$ and $(s_k)_{k=1}^n$. Consider the string $\mathcal{I} = \{i_n, i_{n-1}, \dots, i_1\}$. If $i_n = 2$ then we use the procedure (a). Suppose $i_n = 1$. If all other $i_k = 1$ then $\int A d\mu_{K(\gamma)} = 0$, by (d). Otherwise we have a string $\{1, \dots, 1, 2, \dots\}$. Here we check the values s_k . If they are consecutive (including the first appearance of 2), then we use (b) and get the term $\|Q_{2^{s_n}}\|^{2(i_{n+1}-1)}$ in front of the new integral. Otherwise, by (c), the integral is zero.

Thus, in the case of non zero integral, we decompose \mathcal{I} into substrings of the types $\{2\}, \{1, 2\}, \dots, \{1, \dots, 1, 2\}$ without gaps between consecutive s_k in the same substring. If there are two or more consecutive substrings $\{2\}$ then gaps between corresponding s_k are allowed. Let k with $k < n$ be the index of the first term in $\mathcal{I}_k = \{1, \dots, 1, 2\}$. Then $i_{k+1} = 2$ and, by (b), \mathcal{I}_k brings in the term $\|Q_{2^k}\|^2$ into the common product. □

Corollary 3.3. *For A -polynomial given in (3.3), let $A = A_1 \cdot Q_{2^{s_1}}^{i_1}$, so A_1 contains all terms of A except the last. Suppose $i_1 = i_2 = 2$. Then $\int A d\mu_{K(\gamma)} = \|Q_{2^{s_1}}\|^2 \int A_1 d\mu_{K(\gamma)}$.*

We will represent Q_n in terms of B -polynomials that are defined, for $2^m \leq n < 2^{m+1}$ with $m \in \mathbb{N}_0$, as

$$B_n = (Q_{2^m})^{i_m} (Q_{2^{m-1}})^{i_{m-1}} \dots (Q_1)^{i_1},$$

where $i_k \in \{0, 1\}$ is the k -th coefficient in the binary representation $n = i_m 2^m + \dots + i_0$.

Thus, B_n is a monic polynomial of degree n . The polynomials $B_{(2k+1) \cdot 2^s}$ and $B_{(2j+1) \cdot 2^m}$ are orthogonal for all $j, k, m, s \in \mathbb{N}_0$ with $s \neq m$. Indeed, if $\min\{m, s\} = 0$ then $\int B_{(2k+1) \cdot 2^s} B_{(2j+1) \cdot 2^m} d\mu_{K(\gamma)} = 0$, since one polynomial is symmetric about $x = 1/2$, whereas another is antisymmetric. Otherwise we use Lemma 3.1 (d). By (a), we have

$$\|B_n\|^2 = \prod_{k=0}^m \|Q_{2^k}\|^{2i_k} = \prod_{k=0, i_k \neq 0}^m \|Q_{2^k}\|^2.$$

Theorem 3.4. *For each $n \in \mathbb{N}$, let $n = 2^s(2k+1)$, the polynomial Q_n has a unique representation as a linear combination of $B_{2^s}, B_{3 \cdot 2^s}, \dots, B_{(2k-1) \cdot 2^s}, B_{(2k+1) \cdot 2^s}$.*

Proof. Consider $P = a_0 B_{2^s} + a_1 B_{3 \cdot 2^s} + \dots + a_{k-1} B_{(2k-1) \cdot 2^s} + B_{(2k+1) \cdot 2^s}$, where $(a_j)_{j=0}^{k-1}$ are chosen such that P is orthogonal to all $B_{(2j+1) \cdot 2^s}$ with $j = 0, 1, \dots, k-1$.

This gives a system of k linear equations with k unknowns $(a_j)_{j=0}^{k-1}$. The determinant of this system is the Gram determinant of linearly independent functions $(B_{(2j+1)2^s})_{j=0}^{k-1}$. Therefore it is positive and the system has a unique solution. In addition, as was remarked above, P is orthogonal to all $B_{(2j+1)2^m}$ with $m \neq s$. Thus, P is a monic polynomial of degree n that is orthogonal to all polynomials of degree $< n$, so $P = Q_n$. \square

Corollary 3.5. *The polynomial $Q_{2^s(2k+1)}$ is a linear combination of products of the type $Q_{2^s m} Q_{2^s m-1} \cdots Q_{2^s}$, so the smallest degree of $Q_{2^s j}$ in every product is 2^s .*

To illustrate the theorem, we consider, for given $s \in \mathbb{N}_0$, the easiest cases with $k \leq 2$. Clearly, $Q_{2^s} = B_{2^s}$. Since $B_{3 \cdot 2^s} = Q_{2^s} Q_{2^{s+1}}$, we take $Q_{3 \cdot 2^s} = a_0 Q_{2^s} + Q_{2^{s+1}} Q_{2^s}$, where a_0 is such that $\int Q_{3 \cdot 2^s} Q_{2^s} d\mu_{K(\gamma)} = 0$. By Lemma 3.1,

$$Q_{3 \cdot 2^s} = Q_{2^{s+1}} Q_{2^s} - \frac{\|Q_{2^{s+1}}\|^2}{\|Q_{2^s}\|^2} Q_{2^s}.$$

Similarly, $B_{5 \cdot 2^s} = Q_{2^s} Q_{2^{s+2}}$ and $Q_{5 \cdot 2^s} = a_0 Q_{2^s} + a_1 Q_{2^{s+1}} Q_{2^s} + Q_{2^s} Q_{2^{s+2}}$ with

$$a_0 = \frac{\|Q_{2^{s+2}}\|^2}{\|Q_{2^s}\|^4 - \|Q_{2^{s+1}}\|^2}, \quad a_1 = -a_0 \frac{\|Q_{2^s}\|^2}{\|Q_{2^{s+1}}\|^2}.$$

Using (3.1), all coefficient can be expressed only in terms of $(\gamma_k)_{k=1}^\infty$. As k gets bigger, the complexity of calculations increases.

Remark 3.6. In general, the polynomial Q_n is not Chebyshev. For example, $Q_3 = Q_1(Q_2 + a_0)$ with $a_0 = -\frac{(1-2\gamma_2)\gamma_1^2}{1-2\gamma_1}$. At least for small γ_1 , the polynomial $Q_3(x) = (x - 1/2)(x^2 - x + \gamma_1/2 + a_0)$ increases on the first basic interval $I_{1,1} = [0, l_{1,1}]$. Here, $l_{1,1}$ is the first solution of $P_2 = -r_1$, so $l_{1,1} = (1 - \sqrt{1 - 4\gamma_1})/2$. If Q_3 is the Chebyshev polynomial then, by the alternance argument, $Q_3(l_{1,1}) = Q_3(1)$, but it is not the case.

4. JACOBI PARAMETERS

Since $\mu_{K(\gamma)}$ is supported on the real line, the polynomials $(Q_n)_{n=0}^\infty$ satisfy a three-term recurrence relation

$$Q_{n+1}(x) = (x - b_{n+1})Q_n(x) - a_n^2 Q_{n-1}(x), \quad n \in \mathbb{N}_0.$$

The recurrence starts from $Q_{-1} := 0$ and $Q_0 = 1$. The Jacobi parameters $\{a_n, b_n\}_{n=1}^\infty$ define the matrix

$$(4.1) \quad \begin{pmatrix} b_1 & a_1 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & a_2 & b_2 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\mu_{K(\gamma)}$ is the spectral measure for the unit vector δ_1 and the self-adjoint operator J on $l_2(\mathbb{N})$, which is defined by this matrix. We are interested in the analysis of asymptotic behavior of $(a_n)_{n=1}^\infty$. Since $\mu_{K(\gamma)}$ is symmetric about $x = 1/2$, all b_n are equal to $1/2$. It is known (see e.g. [20]) that $a_n > 0$, $\|Q_n\| = a_1 \cdots a_n$, which, in turn, is the reciprocal to the leading coefficient of the orthonormal polynomial of degree n .

In the next lemmas we use the equality $\int Q_n Q_m Q_{n+m} d\mu_{K(\gamma)} = \|Q_{n+m}\|^2$, which follows by orthogonality of Q_{n+m} to all polynomials of smaller degree.

Lemma 4.1. *For all $s \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we have*

$$Q_{2^s(2k+1)} = Q_{2^s} \cdot Q_{2^{s+1}k} - \frac{\|Q_{2^{s+1}k}\|^2}{\|Q_{2^s(2k-1)}\|^2} Q_{2^s(2k-1)}.$$

Proof. Consider the polynomial $P = Q_{2^s} \cdot Q_{2^{s+1}k} - \frac{\|Q_{2^{s+1}k}\|^2}{\|Q_{2^s(2k-1)}\|^2} Q_{2^s(2k-1)}$. It is a monic polynomial of degree $2^s(2k+1)$. Let us show that P is orthogonal to Q_n for all n with $0 \leq n < 2^s(2k+1)$. This will mean that $P = Q_{2^s(2k+1)}$.

Suppose $0 \leq n < 2^s(2k-1)$. Then orthogonality follows by comparison of the degrees.

If $n = 2^s(2k-1)$ then $\int P Q_n d\mu_{K(\gamma)} = 0$ due to the choice of coefficient of the addend in P and the remark above.

Let $2^s(2k-1) < n < 2^s(2k+1)$. We show that $\int Q_{2^s} Q_{2^{s+1}k} Q_n d\mu_{K(\gamma)} = 0$. We write k in the form $k = 2^q(2l+1)$ with some $q, l \in \mathbb{N}_0$. In turn, $n = 2^m(2p+1)$ with $m \neq s$. By Corollary 3.5, $Q_{2^{s+1}k}$ is a linear combination of products of $Q_{2^{s_j}}$ with $\min s_j = s+1+q$ in every product. Similarly for Q_n , but here the smallest degree is 2^m . Therefore, $Q_{2^s} Q_{2^{s+1}k} Q_n$ is a linear combination of A -polynomials and for each A -polynomial the exponent of the smallest term is 1. By Lemma 3.1(d), the corresponding integral is zero. \square

Lemma 4.2. *For all $s \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we have*

$$a_{2^s(2k+1)}^2 a_{2^s(2k+1)-1}^2 \cdots a_{2^{s+1}k+1}^2 + a_{2^{s+1}k}^2 a_{2^{s+1}k-1}^2 \cdots a_{2^{s+1}k-2^s+1}^2 = \|Q_{2^s}\|^2.$$

Proof. By Lemma 4.1 and the remark above,

$$(4.2) \quad \|Q_{2^s(2k+1)}\|^2 = \int Q_{2^s}^2 Q_{2^{s+1}k}^2 d\mu_{K(\gamma)} - \frac{\|Q_{2^{s+1}k}\|^4}{\|Q_{2^s(2k-1)}\|^2}.$$

Let us show that

$$\int Q_{2^s}^2 Q_{2^{s+1}k}^2 d\mu_{K(\gamma)} = \|Q_{2^s}\|^2 \|Q_{2^{s+1}k}\|^2.$$

If $k = 2^m$, we have this immediately, by Lemma 3.1(a).

Otherwise, $2^{s+1}k = 2^m(2l+1)$ with $l \in \mathbb{N}$ and $m \geq s+1$. Then, by Corollary 3.5, $Q_{2^{s+1}k}$ is a linear combination of products $Q_{2^{s_q}} \cdots Q_{2^{s_j}} \cdots Q_{2^m}$ with $s_j > m$ except the last term. From here, $Q_{2^{s+1}k}^2 = Q_{2^m}^2 \cdot \sum \alpha_j A_j$, where $\sum \alpha_j A_j$ is a linear combination of A -type polynomials with $s_1 > m$ for each A_j . Therefore,

$$\|Q_{2^{s+1}k}\|^2 = \sum \alpha_j \int A_j Q_{2^m}^2 d\mu_{K(\gamma)}.$$

On the other hand,

$$\int Q_{2^s}^2 Q_{2^{s+1}k}^2 d\mu_{K(\gamma)} = \sum \alpha_j \int A_j Q_{2^m}^2 Q_{2^s}^2 d\mu_{K(\gamma)}.$$

By Corollary 3.3, this is $\|Q_{2^s}\|^2 \|Q_{2^{s+1}k}\|^2$.

Therefore, (4.2) can be written as

$$\frac{\|Q_{2^s(2k+1)}\|^2}{\|Q_{2^{s+1}k}\|^2} = \|Q_{2^s}\|^2 - \frac{\|Q_{2^{s+1}k}\|^2}{\|Q_{2^s(2k-1)}\|^2},$$

which is the desired result, as $a_n = \|Q_n\| / \|Q_{n-1}\|$. \square

Theorem 4.3. *The recurrence coefficients $(a_n)_{n=1}^\infty$ can be calculated recursively by using Lemma 4.2 and (3.1).*

Proof. We already have $a_1 = \|Q_1\|$ and $a_2 = \|Q_2\| / \|Q_1\|$. Suppose, by induction, that all a_i are given up to $i = n$. If $n + 1 = 2^s > 2$ then

$$(4.3) \quad a_{n+1} = \frac{\|Q_{2^s}\|}{\|Q_{2^{s-1}}\| \cdot a_{2^{s-1}+1} \cdot a_{2^{s-1}+2} \cdots a_{2^s-1}},$$

where the norms of polynomials can be found by (3.1).

Otherwise, $n + 1 = 2^s(2k + 1)$ for some $s \in \mathbb{N}_0$ and $k \in \mathbb{N}$. By Lemma 4.2, we have

$$(4.4) \quad a_{n+1}^2 = a_{2^s(2k+1)}^2 = \frac{\|Q_{2^s}\|^2 - a_{2^{s+1}k}^2 \cdots a_{2^{s+1}k-2^s+1}^2}{a_{2^s(2k+1)-1}^2 \cdots a_{2^{s+1}k+1}^2},$$

provided $s \neq 0$. If $s = 0$ then the denominator in the fraction above is absent. This gives a_{n+1} , since the recurrence coefficients are positive. \square

In order to illustrate the theorem, let us consider the cases of small s .

If $s = 0$ then $n + 1 = 2k + 1$ and $a_{2k+1}^2 = a_1^2 - a_{2k}^2$. Next, for $s = 1$ and $s = 2$,

$$a_{4k+2}^2 = \frac{\|Q_2\|^2 - a_{4k}^2 a_{4k-1}^2}{a_{4k+1}^2}, \quad a_{8k+4}^2 = \frac{\|Q_4\|^2 - a_{8k}^2 a_{8k-1}^2 a_{8k-2}^2 a_{8k-3}^2}{a_{8k+3}^2 a_{8k+2}^2 a_{8k+1}^2}, \text{ etc.}$$

Thus, $a_1 = \frac{\sqrt{1-2\gamma_1}}{2}$, $a_2 = \frac{\sqrt{1-2\gamma_2}}{\sqrt{1-2\gamma_1}} \gamma_1$, $a_3^2 = a_1^2 - a_2^2$, $a_4 = \frac{\gamma_1 \gamma_2 \sqrt{1-2\gamma_3}}{a_3 \sqrt{1-2\gamma_2}}$, $a_5^2 = a_1^2 - a_4^2$, etc.

Remark 4.4. If $\gamma_n < 1/4$ for $1 \leq n \leq s$ and $\gamma_n = 1/4$ for $n > s$ then $K(\gamma) = E_s = (2P_{2^s}/r_s + 1)^{-1}[-1, 1]$. Here $(P_{2^n} + r_n/2)_{n=0}^\infty$ are the Chebyshev polynomials for E_s , as is easy to check. Therefore Theorems 2.8 and 4.3 are applicable for this case as well. For further information about Jacobi parameters corresponding equilibrium measures of polynomial inverse images, we refer the reader to the article [9].

Remark 4.5. Suppose $\gamma_n = 1/4$ for $n \leq N$ with $2^s \leq N < 2^{s+1}$. Then $a_1 = 1/\sqrt{8}$ and $a_2 = a_3 = \cdots = a_{2^{s+1}-1} = 1/4$. In particular, if $\gamma_n = 1/4$ for all n then $a_n = 1/4$ for all $n \geq 2$, which corresponds to the case of the Chebyshev polynomials of first kind on $[0, 1]$.

Lemma 4.6. Suppose $\gamma_s \leq 1/6$ for all s . For fixed $s \in \mathbb{N}_0$, let $c = \frac{4\gamma_{s+1}^2}{(1-2\gamma_{s+1})^2}$ and $C = \frac{2}{1+\sqrt{1-4c}}$. Then the following inequalities hold with $k \in \mathbb{N}_0$:

(a) If $n = 2^s(2k + 1)$ then

$$\frac{1}{2} \|Q_{2^s}\|^2 \leq C^{-1} \|Q_{2^s}\|^2 \leq a_n^2 \cdots a_{n-2^s+1}^2 \leq \|Q_{2^s}\|^2.$$

(b) If $n = 2^s(2k + 2)$ then

$$a_n^2 \cdots a_{n-2^s+1}^2 \leq C \frac{\|Q_{2^{s+1}}\|^2}{\|Q_{2^s}\|^2} \leq 2 \frac{\|Q_{2^{s+1}}\|^2}{\|Q_{2^s}\|^2}.$$

Proof. Note that, if γ_{s+1} increases from 0 to $1/6$, then c increases from 0 to $1/4$ and C increases from 1 to 2. By (3.1) and the definition of r_s , we get

$$(4.5) \quad \frac{\|Q_{2^{s+1}}\|^2}{\|Q_{2^s}\|^2} = \gamma_{s+1}^2 r_s^2 \frac{1-2\gamma_{s+2}}{1-2\gamma_{s+1}} = (1-2\gamma_{s+2}) c \|Q_{2^s}\|^2 < \|Q_{2^s}\|^2/4.$$

We proceed by induction. For a fixed $s \in \mathbb{N}_0$, let $k = 0$. Then we have at once

$$a_{2^s}^2 \cdots a_1^2 = \|Q_{2^s}\|^2 \quad \text{and} \quad a_{2^{s+1}}^2 \cdots a_{2^s+1}^2 = \frac{\|Q_{2^{s+1}}\|^2}{\|Q_{2^s}\|^2}.$$

Suppose (a), (b) are satisfied for $k \leq m$. We apply Lemma 4.2 with $k = m + 1$:

$$a_{2^s(2m+3)}^2 \cdots a_{2^s(2m+2)+1}^2 + a_{2^s(2m+2)}^2 \cdots a_{2^s(2m+2)-2^s+1}^2 = \|Q_{2^s}\|^2,$$

where for the addend we can use (b) for $k = m$. Therefore,

$$\|Q_{2^s}\|^2 - C \frac{\|Q_{2^{s+1}}\|^2}{\|Q_{2^s}\|^2} \leq a_{2^s(2m+3)}^2 \cdots a_{2^s(2m+2)+1}^2 \leq \|Q_{2^s}\|^2,$$

which is (a) for $k = m + 1$, by (4.5).

Next, we claim that

$$(4.6) \quad a_{2^{s+1}(2m+4)}^2 \cdots a_{2^s(2m+2)+1}^2 \leq \|Q_{2^{s+1}}\|^2$$

for $m \in \mathbb{N}_0$. If $m = 2l + 1$ then we use Lemma 4.2 with $s + 1$ instead of s :

$$a_{2^{s+1}(2k+1)}^2 \cdots a_{2^{s+2}k+1}^2 + \text{positive term} = \|Q_{2^{s+1}}\|^2,$$

which implies (4.6), if we take $k = l + 1$, as $2(2k + 1) = 2m + 4$, $4k = 2m + 2$.

Suppose m is even. Lemma 4.2 now gives

$$\text{positive term} + a_{2^{s+2}k}^2 \cdots a_{2^{s+2}k-2^{s+1}+1}^2 = \|Q_{2^{s+1}}\|^2,$$

where we take $k = m/2 + 1$. Thus, (4.6) holds true in both cases.

Putting together (a) for $k = m + 1$ and (4.6) we get (b) for $k = m + 1$. \square

Theorem 4.7. *Let $\gamma_s \leq 1/6$ for all s . Then $\lim_{s \rightarrow \infty} a_{j \cdot 2^s + n} = a_n$ for $j \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Here, $a_0 := 0$. In particular, $\liminf a_n = 0$.*

Proof. We first show that $\lim_{s \rightarrow \infty} a_{j \cdot 2^s} = 0$ for all $j \in \mathbb{N}$. Let $j = 2^l(2k + 1)$ where $k, l \in \mathbb{N}_0$. For $s > 0$, the Jacobi parameters admit the following inequality by Lemma 4.6(a):

$$(4.7) \quad a_{2^{s+l}(2k+1)}^2 \cdots a_{2^{s+l+1}k+1}^2 \leq \|Q_{2^{s+l}}\|^2.$$

If $i < s + l$ where $i \in \mathbb{N}_0$, we have $2^{s+l}(2k + 1) - 2^i = 2^i(2^{s+l-i}(2k + 1) - 1)$. Since $2^{s+l-i}(2k + 1) - 1$ is a positive odd number, by Lemma 4.6(a), we have the inequalities

$$\frac{1}{2} \|Q_{2^i}\|^2 \leq a_{2^{s+l}(2k+1)-2^i}^2 \cdots a_{2^{s+l}(2k+1)-2^{i+1}+1}^2 \quad \text{for } i = 0, \dots, s + l - 1.$$

We multiply these $s + l$ inequalities side by side:

$$2^{-s-l} \|Q_1\|^2 \cdots \|Q_{2^{s+l-1}}\|^2 \leq a_{2^{s+l}(2k+1)-1}^2 \cdots a_{2^{s+l+1}k+1}^2$$

and use (4.7):

$$a_{j \cdot 2^s}^2 = a_{2^{s+l}(2k+1)}^2 \leq \frac{2^{s+l} \|Q_{2^{s+l}}\|^2}{\|Q_{2^{s+l-1}}\|^2 \|Q_{2^{s+l-2}}\|^2 \cdots \|Q_1\|^2}.$$

By (4.5), the fraction above is bounded by 2^{-s-l+2} . Thus, $\lim_{s \rightarrow \infty} a_{j \cdot 2^s} = 0$.

If $n = 1$ then $a_{j \cdot 2^s + 1}^2 = a_1^2 - a_{j \cdot 2^s}^2 \rightarrow a_1^2$, which is our claim.

Suppose, by induction, that $\lim_{s \rightarrow \infty} a_{j \cdot 2^s + n} = a_n$ for $n = 0, 1, \dots, m$ and all $j \in \mathbb{N}$. Let $m+1 = 2^p(2q+1)$ where $p, q \in \mathbb{N}_0$. If $q = 0$ then $j \cdot 2^s + m+1 = j \cdot 2^{s-p} + 1$, so we get the case with $n = 1$. Thus, we can suppose $q \in \mathbb{N}$. Then $j \cdot 2^s + m+1 = 2^p(2^{s+l-p}(2k+1) + 2q+1)$ and, for large enough s , we can apply Lemma 4.2:

$$a_{j \cdot 2^s + m+1}^2 a_{j \cdot 2^s + m}^2 \cdots a_{j \cdot 2^s + m - 2^p + 1}^2 + a_{j \cdot 2^s + m - 2^p}^2 \cdots a_{j \cdot 2^s + m+1 - 2^{p+1}}^2 = \|Q_{2^p}\|^2.$$

Here all indices, except the first, are of the form $j \cdot 2^s + n$ with $n < m+1$. Therefore, by induction hypothesis, $a_{j \cdot 2^s + n}^2 \rightarrow a_n^2$ as $s \rightarrow \infty$ and

$$\left(\lim_{s \rightarrow \infty} a_{j \cdot 2^s + m+1}^2\right) a_m^2 \cdots a_{m-2^p+1}^2 + a_{m-2^p}^2 \cdots a_{m+1-2^{p+1}}^2 = \|Q_{2^p}\|^2.$$

On the other hand, if we apply Lemma 4.2 to the number $m+1$, then we get the same equality with a_{m+1}^2 instead of $\lim_{s \rightarrow \infty} a_{j \cdot 2^s + m+1}^2$. Since all a_k are positive, we have the desired result. \square

Remark 4.8. Since $\liminf a_n = 0$, by [8], the set $K(\gamma)$ with $\gamma_s \leq 1/6$ has zero Lebesgue measure.

5. WIDOM FACTORS

A finite Borel measure μ supported on a non-polar compact set $K \subset \mathbb{C}$ is said to be regular in the Stahl-Totik sense if $\lim_{n \rightarrow \infty} \|Q_n\|^{\frac{1}{n}} = \text{Cap}(K)$ where Q_n is the monic orthogonal polynomial of degree n corresponding to μ . It is known (see, e.g. [18, 21]) that the equilibrium measure is regular in the Stahl-Totik sense. While $\|Q_n\|^{\frac{1}{n}} / \text{Cap}(K)$ has limit 1, the ratio $W_n = \|Q_n\| / (\text{Cap}(K))^n$ may have various asymptotic behavior. We call W_n the *Widom factor* due to the paper [22]. These values play an important role in spectral theory of orthogonal polynomials on several intervals.

Let $E = [\alpha, \beta] \setminus \bigcup_{i=1}^n (\alpha_i, \beta_i)$ where $\alpha, \beta \in \mathbb{R}$ and the intervals (α_i, β_i) are disjoint subsets of $[\alpha, \beta]$. Let $d\mu = f(t)dt$ on E and $(a_n)_{n=1}^\infty$ be the Jacobi parameters corresponding to μ . Then [6]

$$(5.1) \quad \int \log f(t) d\mu_E(t) > -\infty \iff \limsup_{n \rightarrow \infty} \frac{a_1 \cdots a_n}{\text{Cap}(E)^n} > 0.$$

For further generalizations and different aspects of this result, see [5, 6, 7, 14, 19].

We already know that $\text{Cap}(K(\gamma)) = \exp(\sum_{k=1}^\infty 2^{-k} \log \gamma_k)$. In terms of $(\gamma_k)_{k=1}^\infty$ we can rewrite $\|Q_{2^s}\|$ as

$$(5.2) \quad \frac{\sqrt{1-2\gamma_{s+1}}}{2} \exp\left(2^s \sum_{k=1}^s 2^{-k} \log \gamma_k\right).$$

Therefore,

$$(5.3) \quad W_{2^s} = \frac{\sqrt{1-2\gamma_{s+1}}}{2 \exp(\sum_{k=s+1}^\infty 2^{s-k} \log \gamma_k)} \geq \sqrt{2},$$

since $\gamma_s \leq 1/4$. The limit values $\gamma_s = 1/4$ for all s give the Widom Factors for the equilibrium measure on $[0, 1]$.

Clearly, (5.3) implies that $\limsup W_n > 0$. If $\gamma_s \leq 1/6$ for all s then

$$(5.4) \quad W_{2^s} \geq \sqrt{6}.$$

Let us show that, in this case, $\liminf W_n > 0$.

Theorem 5.1. *Let $(W_n)_{n=1}^\infty$ be Widom factors for $\mu_{K(\gamma)}$ where $\gamma_s \leq 1/6$ for all s . Then*

$$(a) \quad \liminf_{s \rightarrow \infty} W_{2^s} = \liminf_{n \rightarrow \infty} W_n.$$

$$(b) \quad \limsup_{n \rightarrow \infty} W_n = \infty.$$

Proof. (a) We show that $W_n > W_{2^s}$ for $2^s < n < 2^{s+1}$. Let $n = 2^s + 2^{s_1} + \dots + 2^{s_m}$ with $s > s_1 > s_2 > \dots > s_m \geq 0$. Then we decompose $a_1 \cdots a_n$ into groups $(a_1 \cdots a_{2^s}) \cdot (a_{2^s+1} \cdots a_{2^s+2^{s_1}}) \cdots (a_{2^s+\dots+2^{s_{m-1}}+1} \cdots a_n)$. For the first group we have $a_1 \cdots a_{2^s} = \|Q_{2^s}\|$. For the second group we use Lemma 4.6(a) with $n = 2^s + 2^{s_1} : a_{2^s+1} \cdots a_{2^s+2^{s_1}} \geq \|Q_{2^{s_1}}\|/\sqrt{2}$. Similar estimation is valid for all other groups. Therefore,

$$\begin{aligned} W_n &= \frac{a_1 \cdots a_{2^s}}{\text{Cap}(K(\gamma)^{2^s})} \frac{a_{2^s+1} \cdots a_{2^s+2^{s_1}}}{\text{Cap}(K(\gamma)^{2^{s_1}})} \cdots \frac{a_{2^s+\dots+2^{s_{m-1}}+1} \cdots a_n}{\text{Cap}(K(\gamma)^{2^{s_m}})} \\ &\geq W_{2^s} W_{2^{s_1}} \cdots W_{2^{s_m}} (\sqrt{2})^{-m}, \end{aligned}$$

which exceeds $W_{2^s}(\sqrt{3})^m$, by (5.4). From here, $\min_{2^s \leq n < 2^{s+1}} W_n = W_{2^s}$ and the result follows.

(b) Applying the procedure above to W_{2^s-1} and taking the limit gives the desired result. \square

In order to illustrate the behavior of Widom factors, let us consider some examples. Suppose $\gamma_s \leq 1/6$ for all s .

Example 5.2. If $\gamma_s \rightarrow 0$ then $W_n \rightarrow \infty$. Indeed, $W_{2^s} \geq \frac{1}{\sqrt{6}} \exp(\frac{1}{2} \log \frac{1}{\gamma_{s+1}})$.

Example 5.3. There exists $\gamma_s \rightarrow 0$ with $W_n \rightarrow \infty$. Indeed, we can take $\gamma_{2^k} = 1/6$, $\gamma_{2^k-1} = 1/k$.

Example 5.4. If $\gamma_s \geq c > 0$ for all s then $\liminf_{n \rightarrow \infty} W_n \leq 1/2c$.

Example 5.5. There exists γ with $\inf \gamma_s = 0$ and $\liminf_{n \rightarrow \infty} W_n < \infty$. Here we can take $\gamma_s = 1/6$ for $s \neq s_k$ and $\gamma_{s_k} = 1/k$ for a sparse sequence $(s_k)_{k=1}^\infty$. Then $(W_{2^{s_k}})_{k=1}^\infty$ is bounded.

6. TOWARDS THE SZEGŐ CLASS

The convergence of the integral on the left-hand side of (5.1) defines the Szegő class of spectral measures for the finite gap Jacobi matrices. The Widom condition on the right-hand side is the main candidate to characterize the Szegő class for the general case, see [5, 14, 19].

The equilibrium measure is the most natural measure in the theory of orthogonal polynomials. In particular, for known examples, the values $\limsup W_n$ associated with equilibrium measures are bounded below by positive numbers. So we make the following conjecture:

Conjecture 6.1. *If a compact set $K \subset \mathbb{R}$ is regular with respect to the Dirichlet problem then the Widom condition $W_n \rightarrow 0$ is valid for the equilibrium measure μ_K .*

Concerning the Szegő condition, one can conjecture that, in the case of measures with non-polar perfect support K , the left-hand side of (5.1) can be written as

$$(6.1) \quad I(\mu) := \int \log(d\mu/d\mu_K) d\mu_K(t) > -\infty.$$

Indeed, for the finite gap case, this coincides with the condition in (5.1), since the integral $\int \log(d\mu_K/dt) d\mu_K(t)$ converges. By Jensen's inequality (see also Section 4 in [6]), the value $I(\mu)$ is nonpositive and it attains its maximum 0 just in the case $\mu = \mu_K$ a.e. with respect to μ_K . On the other hand, there are strong objections to (6.1), based on the numerical evidence from [12], where, for the Cantor-Lebesgue measure μ_{CL} on the classical Cantor set K_0 , the Jacobi parameters (a_n) were calculated for $n \leq 200.000$. The Widom factors for such values behave as a bounded below (by a positive number) sequence. Therefore, if we wish to preserve the Widom characterization of the Szegő class, the integral $I(\mu_{CL})$ must converge, but, since μ_{CL} and μ_{K_0} are mutually singular, it is not the case.

REFERENCES

- [1] Alpan, G., Goncharov, A.: Two measures on Cantor sets. *J. Approx. Theory.* **186**, 28–32 (2014)
- [2] Barnsley M.F., Geronimo, J.S., Harrington, A.N.: Infinite-Dimensional Jacobi Matrices Associated with Julia Sets. *Proc. Amer. Math. Soc.* **88**(4), 625–630 (1983)
- [3] Barnsley, M.F., Geronimo, J.S., Harrington, A.N.: Almost periodic Jacobi matrices associated with Julia sets for polynomials. *Comm. Math. Phys.* **99**(3), 303–317 (1985)
- [4] Bellissard, J., Bessis, D., Moussa, P.: Chaotic States of Almost Periodic Schrödinger Operators. *Phys. Rev. Lett.* **49**, 701–704 (1982)
- [5] Christiansen, J.S.: Szegő's theorem on Parreau-Widom sets. *Adv. Math.* **229**, 1180–1204 (2012)
- [6] Christiansen, J.S., Simon, B., Zinchenko, M.: Finite Gap Jacobi Matrices, II. The Szegő Class. *Constr. Approx.* **33**(3), 365–403 (2011)
- [7] Christiansen, J.S., Simon, B., Zinchenko, M.: Finite gap Jacobi matrices: a review.. *American Mathematical Society. Proceedings of Symposia in Pure Mathematics*, p. 87–103 (2013)
- [8] Dombrowski, J.: Quasitriangular matrices. *Proc. Amer. Math. Soc.* **69**, p. 95–96 (1978)
- [9] Geronimo, J.S., Van Assche, W.: Orthogonal polynomials on several intervals via a polynomial mapping. *Trans. Amer. Math. Soc.* **308**, 559–581 (1988)
- [10] Goncharov, A.: Weakly equilibrium Cantor type sets. *Potential Anal.* **40**, 143–161 (2014)
- [11] Heilman, S.M., Owrutsky, P., Strichartz, R.: Orthogonal polynomials with respect to self-similar measures. *Exp. Math.* **20**, 238–259 (2011)
- [12] Krüger, H., Simon, B.: Cantor polynomials and some related classes of OPRL. *J. Approx. Theory* (2014), <http://dx.doi.org/10.1016/j.jat.2014.04.003>
- [13] Makarov, N.G., Volberg, A.: On the harmonic measure of discontinuous fractals. LOMI preprint, E-6-86, Steklov Mathematical Institute, Leningrad (1986)
- [14] Peherstorfer, F., Yuditskii, P.: Asymptotic behavior of polynomials orthonormal on a homogeneous set. *J. Anal. Math.* **89**, 113–154 (2003)
- [15] Ransford, T.: *Potential Theory in the Complex Plane*. Cambridge University Press, (1995)
- [16] Saff, E.B., Totik, V.: *Logarithmic potentials with external fields*. Springer-Verlag, New York (1997)
- [17] Simon, B.: Almost periodic Schrödinger operators: A Review. *Adv. Appl. Math.* **3**, 463–490 (1982)
- [18] Simon, B.: Equilibrium measures and capacities in spectral theory. *Inverse Probl. Imaging* **1**, 713–772 (2007)
- [19] Sodin, M., Yuditskii, P.: Almost periodic Jacobi matrices with homogeneous spectrum, infinite-dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions. *J. Geom. Anal.* **7** (1997), 387–435
- [20] Totik, V.: Orthogonal polynomials. *Surv. Approx. Theory* **1**, 70–125, (2005)

- [21] Widom, H.: Polynomials associated with measures in the complex plane. J. Math. Mech. **16**, 997–1013 (1967)
- [22] Widom, H: Extremal polynomials associated with a system of curves in the complex plane. Adv. Math. **3**, 127–232 (1969)

DEPARTMENT OF MATHEMATICS, BILKENT UNIVERSITY, 06800 ANKARA, TURKEY
E-mail address: `gokalp@fen.bilkent.edu.tr`

DEPARTMENT OF MATHEMATICS, BILKENT UNIVERSITY, 06800 ANKARA, TURKEY
E-mail address: `goncha@fen.bilkent.edu.tr`