Gravity Waves in Three Dimensions

Metin Gürses,[1,*] Tahsin Çağrı Şişman,[2,†] and Bayram Tekin[3,‡]

[1]*Department of Mathematics, Faculty of Sciences*
Bilkent University, 06800 Ankara, Turkey
[2]*Department of Astronautical Engineering,*
University of Turkish Aeronautical Association, 06790 Ankara, Turkey
[3]*Department of Physics,*
Middle East Technical University, 06800 Ankara, Turkey

(Dated: October 8, 2015)

We find the explicit forms of the anti-de Sitter plane, anti-de Sitter spherical, and pp waves that solve both the linearized and exact field equations of the most general higher derivative gravity theory in three dimensions. As a sub-class, we work out the six derivative theory and the critical version of it where the masses of the two spin-2 excitations vanish and the spin-0 excitations decouple.

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I. INTRODUCTION

Due to the nonlinearity of Einstein's equations, it is highly difficult to find exact solutions. This is even more so in modified gravity theories where more powers of curvature added to the Einstein-Hilbert action to make the theory better behaved in the UV region. Therefore, it is quite important to find exact solutions of higher derivative gravity theories. Especially, for the purposes of the anti-de Sitter/conformal field theory correspondence, it is highly

*Electronic address: gurses@fen.bilkent.edu.tr
†Electronic address: tahsin.c.sisman@gmail.com
‡Electronic address: btekin@metu.edu.tr

desirable to find some "neighboring" solutions to the AdS spacetime. This work started with the purpose of providing some AdS-related solutions to generic gravity theories in three dimensions. For a specific quadratic curvature gravity, called the new massive gravity (NMG) [1], these kind of solutions were studied in [2–5]. Recently [6, 7], using general arguments, we have shown that the AdS-wave and the pp-wave metrics solve the most general gravity theory with the action in the n-dimensional spacetime given as

$$I=\int d^n x\sqrt{-g}\,F\left(g^{\alpha\beta},R^{\mu}{}_{\nu\gamma\sigma},\nabla_{\rho}R^{\mu}{}_{\nu\gamma\sigma},\ldots,\left(\nabla_{\rho_1}\nabla_{\rho_2}\ldots\nabla_{\rho_M}\right)R^{\mu}{}_{\nu\gamma\sigma},\ldots\right), \tag{1}$$

where F is a differentiable function of its arguments. In this work, as an explicit example to our formalism, we shall provide the solutions of the most general sixth order theory in three dimensions. These wave solutions can be written in the Kerr-Schild form

$$g_{\mu\nu}=\bar{g}_{\mu\nu}+2V\lambda_\mu\lambda_\nu, \tag{2}$$

with $\bar{g}_{\mu\nu}$ as the "background metric" which is the flat Minkowski metric for the pp-waves and the AdS spacetime for the AdS-wave metrics. The properties of the λ^μ-vector are crucial: it is a null and a geodesic vector. Namely, it satisfies the following expressions for both $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$:

$$\lambda_\mu\lambda^\mu=g_{\mu\nu}\lambda^\mu\lambda^\nu=\bar{g}_{\mu\nu}\lambda^\mu\lambda^\nu=0, \tag{3}$$

$$\lambda^\mu\nabla_\mu\lambda_\rho=\lambda^\mu\bar{\nabla}_\mu\lambda_\rho=0, \tag{4}$$

$$\nabla_\mu\lambda_\nu=\bar{\nabla}_\mu\lambda_\nu=\lambda_{(\mu}\xi_{\nu)}=\frac{1}{2}\left(\lambda_\mu\xi_\nu+\xi_\mu\lambda_\nu\right), \qquad \xi^\mu\lambda_\mu=0, \tag{5}$$

where $\bar{\nabla}_\mu$ is the covariant derivative with respect to the background metric. The last property restricts the Kerr-Schild metric to the Kundt class where the λ^μ vector is nonexpanding, shear-free, and nontwisting. Due to this property, we denote this class of metrics as Kerr-Schild-Kundt (KSK) metrics. The new vector ξ^μ that appears in (5) is defined via that equation. The metric function V satisfies $\lambda^\mu\partial_\mu V=0$. Let us suppose that the most general theory is a $2N+2$ derivative theory; namely, the highest partial derivative of the metric in the field equations is $2N+2$. For example, Einstein's gravity has $N=0$, any of the form f (Riemann) with no derivatives of the Riemann tensor but only quadratic and more contractions, has $N=1$. Explicit AdS-wave solutions of these theories have been considered before [3–5, 9, 10, 19]. Explicit solutions of the most general $N=2$ theory, namely the six derivative theory, have not been considered before. Here, we shall remedy this in three dimensions for the most general theory.

We have shown that for the metrics of the form (2) having the properties (3-5), all curvature scalars are constant and the scalar curvature is $R=-\frac{6}{\ell^2}$, and the traceless part of the Ricci tensor, that is $S_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{3}g_{\mu\nu}R$, reduces to the following simple expression [7]

$$S_{\mu\nu}=-\left(\bar{\Box}+\frac{2}{\ell^2}\right)\lambda_\mu\lambda_\nu V\equiv\lambda_\mu\lambda_\nu\mathcal{O}V, \tag{6}$$

where ℓ is the AdS radius, $\bar{\Box}$ is the Laplace-Beltrami operator of the background metric, and the operator $\mathcal{O}$ can be found in three dimensions as

$$\mathcal{O}=-\left(\bar{\Box}+2\xi^\mu\partial_\mu+\frac{1}{2}\xi^\mu\xi_\mu-\frac{2}{\ell^2}\right). \tag{7}$$

The field equations of the most general $(2N+2)$-derivative gravity theory splits into two parts: one is the trace part that determines the AdS radius in terms of the parameters given in the action such as the bare cosmological constant and the coefficients of the curvature terms. The other equation is the traceless part which reads as [7]

$$\sum_{n=0}^{N} a_n \,\bar{\Box}^n \, S_{\mu\nu} = 0 \tag{8}$$

where a_n's $(n = 0, 1, 2, \cdots)$ are constants which are again functions of the parameters of the theory whose proof for the AdS-spherical wave will be given in [8]. Equation (8) can factored as

$$\prod_{n=1}^{N} \left(\bar{\Box} + c_n\right) S_{\mu\nu} = 0, \tag{9}$$

where c_n's are the roots of the polynomial

$$a_N y^N + a_{N-1} y^{N-1} + \cdots + a_1 y + a_0 = 0, \tag{10}$$

which are a priori complex in general. But, in order for the theory to be free of tachyons, all the roots must be real since they are related to the masses of the spin-2 excitations about the AdS background through the relation

$$c_n = \frac{2}{\ell^2} - m_n^2, \quad n = 1, 2, \cdots, N. \tag{11}$$

This can be understood as follows: once a perturbation about the AdS background $h_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}$ is defined as $h_{\mu\nu} \equiv 2V\lambda_\mu\lambda_\nu$, with all the properties of λ_μ and V intact as in the exact solution, then the exact solution and the perturbative solution for this particular transverse-traceless $h_{\mu\nu}$ representing spin-2 modes become equal. Note that the spin-0 modes cannot be obtained this way as the full spacetime is a constant curvature spacetime, namely linearized part of the scalar curvature is zero. For the case of the pp-wave metrics, one takes the limit $\ell \to \infty$. Note that the naïve counting of the degrees of freedom in terms of the metric *alone* in these higher derivative theories would take one astray: for example, in four dimensions, one would conclude that a symmetric two-tensor, $h_{\mu\nu}$, could have at most 10 propagating degrees of freedom. This could only be true in a second derivative theory without any symmetries. On the other hand, in higher derivative theories, $\partial^p h_{\mu\nu}$ type objects should be considered as independent fields as was done by Pais and Uhlenbeck [11]. Since $S_{\mu\nu}$ satisfies (6) then (9) reduces to

$$\prod_{n=1}^{N} \left(\mathcal{O} + m_n^2\right) \mathcal{O} V = 0, \tag{12}$$

where we also used the relations

$$\Box \left(\phi\lambda_\alpha\lambda_\beta\right) = \bar{\Box} \left(\phi\lambda_\alpha\lambda_\beta\right) = -\lambda_\alpha\lambda_\beta \left(\mathcal{O} + \frac{2}{\ell^2}\right) \phi, \tag{13}$$

which are valid for any function ϕ satisfying $\lambda^\mu \nabla_\mu \phi = 0$. This also leads to $\lambda^\mu \nabla_\mu \mathcal{O}\phi = 0$. Provided that that all m_n^2's are different, the most general solution to (12) can be written as

$$V = V_E + \sum_{n=1}^{N} V_n, \tag{14}$$

where V_E represents the solution to the cosmological Einstein's theory satisfying

$$\mathcal{O}V_E = 0. \tag{15}$$

In three dimensions, the solutions of this equation can be "gauged away": namely, the metric $\bar{g}_{\mu\nu} + 2V_E\lambda_\mu\lambda_\nu$ is that of AdS_3. This is related to the fact that cosmological Einstein's theory does not have any propagating degree of freedom in three dimensions. In other dimensions, on the other hand, (15) does have nontrivial solutions. In what follows, since we work explicitly in three dimensions, we shall gauge away this Einsteinian solution and not write it. In (14), each V_n satisfies

$$\left(\mathcal{O} + m_n^2\right) V_n = 0. \tag{16}$$

In the case that two or more coalescing m_n^2's , the structure of the solution changes dramatically; for example, the asymptotic behavior is no longer that of AdS. Let r be the number (multiplicity) of m_n^2's that are equal to say m_r^2, then the corresponding V_r satisfies an nonfactorizable higher derivative equation;

$$\left(\mathcal{O} + m_r^2\right)^r V_r = 0. \tag{17}$$

The most general solution now becomes

$$V = V_r + \sum_{n=1}^{N-r} V_n, \tag{18}$$

where V_r contains $\log^p$ terms with $p = 1, 2, \ldots, r-1$. Such theories are called critical (r-critical). Note that m_r^2 may also be zero. Then, the most general solution is in the form

$$V = V_{r0} + \sum_{n=1}^{N-r} V_n, \tag{19}$$

where V_{r0} is the solution of $\mathcal{O}^{r+1} V_{r0} = 0$ and involves $\log^p$ terms with $p = 1, 2, \ldots, r$. If all the mass parameters are all equal to zero, then the criticality reaches its maximum value of $N+1$. Furthermore, the relation between the maximum criticality and the derivative order of any gravity theory is worth mentioning: $\frac{\text{derivative order}}{\text{maximum criticality}} = 2$. For the case of maximum criticality, the field equations take the form

$$\mathcal{O}^{N+1} V = 0. \tag{20}$$

As noted above, for the pp-wave metrics, the above discussions are also valid but in the limit the AdS radius goes to infinity, $\ell \to \infty$.

The layout of the paper is as follows: In Sec. II, we define the most general sixth order theory in three dimensions and give its field equations for KSK metrics from which the masses of the spin-2 excitations around the (A)dS background can be obtained. In Sec. III, we give the solutions of the sixth-order theory and in the ensuing section we extend these solutions to all higher order derivative theories. In Sec. V, we also give the pp-wave solutions of sixth-order theories and beyond.

II. SIXTH ORDER THEORY IN THREE DIMENSIONS

To give a nontrivial explicit example in full detail, let us consider the action

$$I=\frac{1}{\kappa^{2}}\int d^{3}x\sqrt{-g}\left(F\left(R_{\nu}^{\mu}\right)+\mathcal{L}_{R\square R}\right), \tag{21}$$

where at this stage, $F\left(R_{\nu}^{\mu}\right)$ is an arbitrary differentiable function of the Ricci tensor but not its derivatives and the second piece in the action constitutes of the two possible second derivative terms (up to boundary terms):

$$\mathcal{L}_{R\square R}=b_{1}\nabla_{\mu}R\,\nabla^{\mu}R+b_{2}\nabla_{\rho}R_{\alpha\beta}\nabla^{\rho}R^{\alpha\beta}. \tag{22}$$

In [12] (see also [13]), it was shown that the $F\left(R_{\nu}^{\mu}\right)$ function can be represented more compactly as $F\left(R_{\nu}^{\mu}\right)=F\left(R,S_{\nu}^{\mu}S_{\mu}^{\nu},S_{\rho}^{\mu}S_{\mu}^{\nu}S_{\nu}^{\rho}\right)$ after the use of Schouten identities to represent higher curvature scalars in terms of these three curvature scalars, so that the most general six-derivative theory takes the form[1]

$$I=\frac{1}{\kappa^{2}}\int d^{3}x\sqrt{-g}\left(F\left(R,A,B\right)+\mathcal{L}_{R\square R}\right), \tag{23}$$

where we have defined

$$A\equiv S_{\nu}^{\mu}S_{\mu}^{\nu},\qquad B\equiv S_{\rho}^{\mu}S_{\mu}^{\nu}S_{\nu}^{\rho}. \tag{24}$$

Let us write the field equations coming from the variation of (23) in two parts:

$$E_{\mu\nu}+H_{\mu\nu}=0, \tag{25}$$

where $E_{\mu\nu}$ comes from the $F\left(R,A,B\right)$ part as [13]

$$\begin{aligned}E_{\mu\nu}=&-\frac{1}{2}g_{\mu\nu}F+2F_{A}S_{\mu}^{\rho}S_{\rho\nu}+3F_{B}S_{\mu}^{\rho}S_{\rho\sigma}S_{\nu}^{\sigma}+\left(\square+\frac{2}{3}R\right)\left(F_{A}S_{\mu\nu}+\frac{3}{2}F_{B}S_{\mu}^{\rho}S_{\rho\nu}\right)\\&+\left(g_{\mu\nu}\square-\nabla_{\mu}\nabla_{\nu}+S_{\mu\nu}+\frac{1}{3}g_{\mu\nu}R\right)\left(F_{R}-F_{B}S_{\sigma}^{\rho}S_{\rho}^{\sigma}\right)\\&-2\nabla_{\alpha}\nabla_{(\mu}\left(S_{\nu)}^{\alpha}F_{A}+\frac{3}{2}S_{\nu)}^{\rho}S_{\rho}^{\alpha}F_{B}\right)+g_{\mu\nu}\nabla_{\alpha}\nabla_{\beta}\left(F_{A}S^{\alpha\beta}+\frac{3}{2}F_{B}S^{\alpha\rho}S_{\rho}^{\beta}\right).\end{aligned} \tag{26}$$

Here, the derivatives of the F function are represented as $F_{R}\equiv\frac{\partial F}{\partial R}$, $F_{A}\equiv\frac{\partial F}{\partial A}$, and $F_{B}\equiv\frac{\partial F}{\partial B}$. The second part of the field equations, that is $H_{\mu\nu}$, comes from the variation of $\mathcal{L}_{R\square R}$ and

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[1] Note that one does not have to use this procedure. A more direct way would be to work with R_{ν}^{μ} is only independent variable, but for our purposes the laid out method is better since $S_{\mu\nu}\sim\lambda_{\mu}\lambda_{\nu}$.

is given as

$$
\begin{aligned}
H_{\mu\nu} \ = \ & b_1 \left(\nabla_\mu R \nabla_\nu R - 2R_{\mu\nu}\Box R - 2\left(g_{\mu\nu}\Box^2 - \nabla_\mu \nabla_\nu \Box\right) R - \frac{1}{2} g_{\mu\nu} \nabla_\alpha R \nabla^\alpha R\right) \\
& + b_2 \Bigg(\nabla_\mu R_{\alpha\beta} \nabla_\nu R^{\alpha\beta} - \Box^2 R_{\mu\nu} - g_{\mu\nu} \nabla_\rho \nabla_\sigma \Box R^{\rho\sigma} + 2\nabla^\rho \nabla_{(\mu}\Box R_{\nu)\rho} \\
& + 2\nabla^\rho R_{\rho\sigma} \nabla_{(\mu} R^\sigma_{\nu)} + 2R_{\rho\sigma}\nabla^\rho \nabla_{(\mu} R^\sigma_{\nu)} - 2R_{\sigma(\mu}\Box R^\sigma_{\nu)} \\
& - 2\nabla_\rho R_{\sigma(\mu}\nabla_{\nu)} R^{\rho\sigma} - 2R_{\sigma(\mu}\nabla^\rho \nabla_{\nu)} R^\sigma_\rho - \frac{1}{2} g_{\mu\nu} \nabla_\rho R_{\alpha\beta} \nabla^\rho R^{\alpha\beta}\Bigg).
\end{aligned} \tag{27}
$$

For the metric (2), with the properties listed in (3–5), we noted that $S_{\mu\nu}$ is in the form (6) and furthermore, the following identities can be computed from the listed properties of the metric:

$$\nabla^\alpha \Box R_{\alpha\mu} = 0, \quad \nabla^\alpha R_{\alpha\mu} = 0 \tag{28}$$
$$\nabla_\mu R_{\alpha\beta} \nabla_\nu R^{\alpha\beta} = 0, \tag{29}$$
$$\nabla_\rho R_{\sigma\mu} \nabla_\nu R^{\rho\sigma} = 0, \tag{30}$$
$$\nabla^\rho \nabla_\mu R_{\rho\nu} = -\frac{3}{\ell^2} S_{\mu\nu}, \tag{31}$$
$$\nabla^\rho \nabla_\mu \Box R_{\nu\rho} = -\frac{3}{\ell^2}\Box S_{\mu\nu}, \tag{32}$$
$$R^\rho{}_\mu \nabla^\sigma \nabla_\nu R_{\rho\sigma} = \frac{6}{\ell^4} S_{\mu\nu}, \tag{33}$$
$$R^{\rho\sigma} \nabla_\sigma \nabla_\mu R_{\nu\rho} = \frac{6}{\ell^4} S_{\mu\nu}, \tag{34}$$
$$R^\rho{}_\mu \Box R_{\nu\rho} = -\frac{2}{\ell^2}\Box S_{\mu\nu}. \tag{35}$$

In deriving these identities, we have used the representation of the three-dimensional Riemann tensor in terms of the Ricci tensor and the scalar curvature, and also the identity $\lambda^\rho \nabla_\nu S_{\rho\sigma} = 0$ which is valid for the KSK class of metrics to which our gravity waves in AdS belong.

With these identities $H_{\mu\nu}$ reduces to the following from

$$H_{\mu\nu} = -b_2 \left(\Box + \frac{2}{\ell^2}\right)\Box S_{\mu\nu}. \tag{36}$$

The metric discussed above represents constant curvature, Type-N spacetimes as $S_{\mu\nu}$ has the form $S_{\mu\nu} = \rho\lambda_\mu\lambda_\nu$. Then, the field equations for these spacetimes (26) becomes

$$\left(\frac{1}{3}RF_R - \frac{1}{2}F\right) g_{\mu\nu} + \left[-b_2\Box^2 + \left(F_A - \frac{2b_2}{\ell^2}\right)\Box - \frac{1}{3}RF_A + F_R\right] S_{\mu\nu} = 0, \tag{37}$$

where for the $E_{\mu\nu}$ part, results of [13] was used. The trace of (37) yields

$$\frac{1}{3}RF_R - \frac{1}{2}F = 0, \tag{38}$$

which determines the cosmological constant or the AdS radius ℓ. The traceless part of (37) becomes the nonlinear equation

$$\left[-b_2\,\Box^2 + \left(F_A - \frac{2b_2}{\ell^2}\right)\Box - \frac{1}{3}RF_A + F_R\right]S_{\mu\nu} = 0. \tag{39}$$

which can be rewritten as a product of two operators in general;

$$\left(\Box + \frac{2}{\ell^2} - m_-^2\right)\left(\Box + \frac{2}{\ell^2} - m_+^2\right)S_{\mu\nu} = 0, \tag{40}$$

where the mass-squared parameters follow from (39) as

$$m_\pm^2 = \frac{1}{\ell^2} + \frac{F_A}{2b_2} \mp \sqrt{\left(\frac{1}{\ell^2} + \frac{F_A}{2b_2}\right)^2 + \frac{1}{b_2}F_R}. \tag{41}$$

This formula represents the masses of the two spin-2 excitations for the most general sixth order gravity theory. Once the explicit form of F is given, one can calculate the masses of these modes. For example, for the choice of the most general quadratic curvature gravity in three dimensions, $F\left(R^\mu_\nu\right)$ has the form

$$F\left(R^\mu_\nu\right) = \sigma R - 2\lambda_0 + \alpha R^2 + \beta R_{\alpha\beta}\,R^{\alpha\beta} = \sigma R - 2\lambda_0 + \left(\alpha + \frac{\beta}{3}\right)R^2 + \beta S_{\alpha\beta}\,S^{\alpha\beta}, \tag{42}$$

yielding

$$F_R = \sigma - \frac{12}{\ell^2}\left(\alpha + \frac{\beta}{3}\right), \qquad F_A = \beta, \tag{43}$$

and the square of the mass reads

$$m_\pm^2 = \frac{\beta}{2b_2} + \frac{1}{\ell^2} \mp \frac{1}{2b_2}\sqrt{\beta^2 + \frac{4b_2^2}{\ell^4} + 4b_2\sigma - \frac{12b_2}{\ell^2}\left(\beta + 4\alpha\right)}. \tag{44}$$

For a six-derivative theory, the mass-squared terms, $m_\pm^2$, can be arranged to be zero given that $\frac{1}{\ell^2} + \frac{F_A}{2b_2} = 0$ and $F_R = 0$. In this limit, the field equations of the so called tricritical theories reduce to the form

$$\mathcal{O}^3 V = 0, \tag{45}$$

and, hence, have the same logarithmic solutions that we discuss in the next section.

III. AdS-WAVE SOLUTIONS

Let us now discuss the exact solutions of (40) which fall into several distinct classes depending on the values of $m_\pm^2$. In the generic case, $m_+^2 \neq m_-^2$. As a second case, $m_+^2 = m_-^2 \neq 0$. In the third case, one of them could be zero. In the last case, $m_+^2 = m_-^2 = 0$.

<u>*Case 1– $m_+^2 \neq m_-^2$:*</u> For this case, (40) reduces to

$$\left(\mathcal{O}+m_-^2\right)\left(\mathcal{O}+m_+^2\right)\mathcal{O}V=0, \tag{46}$$

whose solutions can be obtained from the solutions of the lower derivative equations

$$\mathcal{O}V_E=0, \tag{47}$$

$$\left(\mathcal{O}+m_+^2\right)V_+=0, \tag{48}$$

$$\left(\mathcal{O}+m_-^2\right)V_-=0, \tag{49}$$

as $V=V_E+V_++V_-$. Here, V_E refers to the solution of $S_{\mu\nu}=0$. Let us note that $\left(\mathcal{O}+m^2\right)\mathcal{O}V=0$ is the traceless part of the field equation for these metrics of the quadratic curvature gravity, and hence, in some sense for these metrics the field equations of the sixth order theory reduce to two copies of the quadratic theory.

With the specific choices of λ^μ vector, one can get the AdS-plane and AdS-spherical wave solutions. The AdS-plane wave metric can be given as

$$\mathrm{d}s^2=\frac{\ell^2}{z^2}\left(2\mathrm{d}u\mathrm{d}v+\mathrm{d}z^2\right)+2V\left(u,z\right)\mathrm{d}u^2, \tag{50}$$

where the null coordinates are defined as $u=\frac{1}{\sqrt{2}}\left(x+t\right)$ and $v=\frac{1}{\sqrt{2}}\left(x-t\right)$. Then, the relevant differential equation becomes [7]

$$\left(\frac{z^2}{\ell^2}\frac{\partial^2}{\partial z^2}+\frac{3z}{\ell^2}\frac{\partial}{\partial z}-m_\pm^2\right)V\left(u,z\right)=0, \tag{51}$$

whose solution is

$$V_\pm\left(u,z\right)=\frac{1}{z}\left(c_1z^{p_\pm}+c_2z^{-p_\pm}\right), \tag{52}$$

where $p_\pm\equiv\sqrt{1+m_\pm^2\ell^2}$ and $c_{1,2}$ are functions of u. This solution was given in the case of NMG in [19]. In $p\to1$ limit, one obtains the Einsteinian solution

$$V_E\left(u,z\right)=\frac{1}{z^2}\left(c_1z^2+c_2\right). \tag{53}$$

From this form, it is easy to see that with this V_E, (50) is the AdS space.

The metric in the coordinates used in [14] reads

$$\mathrm{d}s^2=\frac{\ell^2}{\cos^2\theta}\left(\frac{4\mathrm{d}u\mathrm{d}v}{\left(u+v\right)^2}+\mathrm{d}\theta^2\right)+2V\left(u,\theta\right)\mathrm{d}u^2, \tag{54}$$

which is called the AdS-spherical wave as the null coordinates are defined as $u=\frac{1}{\sqrt{2}}\left(r+t\right)$ and $v=\frac{1}{\sqrt{2}}\left(r-t\right)$, so the AdS part is conformal to the flat space in spherical coordinates. Then, the relevant differential equation reduces to

$$\left[\cos^2\theta\frac{\partial^2}{\partial\theta^2}-3\sin\theta\cos\theta\frac{\partial}{\partial\theta}-\left(2\cos^2\theta+m_\pm^2\ell^2\right)\right]V_\pm\left(u,\theta\right)=0, \tag{55}$$

whose solution is

$$V_{\pm}\left(u,\theta\right)=\frac{1}{\cos\theta}\left[c_1\left(\frac{\cos\theta}{1+\sin\theta}\right)^{p_{\pm}}+c_2\left(\frac{\cos\theta}{1+\sin\theta}\right)^{-p_{\pm}}\right], \tag{56}$$

where $c_{1,2}$ are arbitrary functions of the null coordinate u. Again, this solution was given in the case of NMG in different coordinates in [3–5]. The $p=1$ case yields the Einsteinian solution

$$V_E\left(u,\theta\right)=\frac{1}{\cos^2\theta}\left(c_{+}+c_{-}\sin\theta\right), \tag{57}$$

where $c_{\pm}=c_2\pm c_1$ which can again be gauged away. Therefore, the general AdS-spherical wave solution to the most general sixth order gravity is

$$V\left(u,\theta\right)=V_{+}\left(u,\theta\right)+V_{-}\left(u,\theta\right), \tag{58}$$

where V_+ and V_- are given in (56) and the mass parameters m_+^2 and m_-^2 are given in (41).

Case 2– $m_+^2=m_-^2\neq 0$: The AdS-plane wave solution of the fourth order massive operator part is

$$V_m=\frac{1}{z}\left[c_1 z^p+c_2 z^{-p}+\ln\left(\frac{z}{\ell}\right)\left(c_3 z^p+c_4 z^{-p}\right)\right], \tag{59}$$

while the AdS-spherical wave solution is

$$\begin{aligned}V_m=\frac{1}{\cos\theta}\Bigg(&c_1\left(\frac{\cos\theta}{1+\sin\theta}\right)^{p}+c_2\left(\frac{\cos\theta}{1+\sin\theta}\right)^{-p}\\&+\ln\left(\frac{\cos\theta}{1+\sin\theta}\right)\left[c_3\left(\frac{\cos\theta}{1+\sin\theta}\right)^{p}+c_4\left(\frac{\cos\theta}{1+\sin\theta}\right)^{-p}\right]\Bigg).\end{aligned} \tag{60}$$

The log-terms appear because of the genuinely fourth order nature of the equation.

Case 3–one of the masses is zero: The solution is

$$V=V_m+V_{\log}, \tag{61}$$

where, for the AdS-plane wave, one finds

$$V_{\log}\left(u,z\right)=\frac{1}{z^2}\ln\left(\frac{z}{\ell}\right)\left(c_3 z^2+c_4\right), \tag{62}$$

also appeared in NMG case [19], and for the AdS-spherical wave, one finds

$$V_{\log}\left(u,\theta\right)=\frac{1}{\cos^2\theta}\ln\left(\frac{\cos\theta}{1+\sin\theta}\right)\left(c_3+c_4\sin\theta\right). \tag{63}$$

Case 4– $m_+^2=m_-^2=0$: In this case, the theory is called tricritical [15]. The AdS-plane wave solution is

$$V_{\log}\left(u,z\right)=\frac{1}{z^2}\ln\left(\frac{z}{\ell}\right)\left[c_3 z^2+c_4+\ln\left(\frac{z}{\ell}\right)\left(c_5 z^2+c_6\right)\right], \tag{64}$$

which was partially covered in [16], while the AdS-spherical wave solution is

$$V_{\log}\left(u,\theta\right)=\frac{1}{\cos^2\theta}\ln\left(\frac{\cos\theta}{1+\sin\theta}\right)\left[c_3+c_4\sin\theta+\ln\left(\frac{\cos\theta}{1+\sin\theta}\right)\left(c_5+c_6\sin\theta\right)\right]. \tag{65}$$

IV. EXTENSION TO ANY HIGHER DERIVATIVE ORDER

As noted above, for the AdS-wave metrics, the traceless part of the field equations of any $(2N+2)$-derivative theory in three dimensions reduce to the following product

$$\left(\mathcal{O}+m_1^2\right)\left(\mathcal{O}+m_2^2\right)\cdots\left(\mathcal{O}+m_N^2\right)\mathcal{O}V=0, \tag{66}$$

where m_i are the masses of the spin-2 excitations which can be found in a rather tedious procedure in terms of the parameters of the theory once the Lagrangian of the theory is given. In Sec. II, we gave an explicit example for the sixth order gravity. Solutions of (66) depend on whether the masses are equal or not.

Case 1–All the masses are distinct: For this case, the most general solution is the sum of the solutions of each massive operator part as

$$V=\sum_{i=1}^{N}V_i, \tag{67}$$

where the solutions V_i are given in (52) for the AdS-plane wave and in (54) for the AdS-spherical wave. Here, we again dropped the Einsteinian part.

Case 2–Some masses are equal but not zero: For the case where r number of masses are equal to m, the general solution takes the form

$$V=\sum_{i=0}^{r-1}V_m\left(c_i,c_{i+1}\right)\left(\ln f\right)^i+\sum_{i=1}^{N-r}V_i, \tag{68}$$

where $f=\frac{z}{\ell}$ and $V_m\left(c_i,c_{i+1}\right)$ is given in (52) for the AdS-plane wave, and $f=\frac{\cos\theta}{1+\sin\theta}$ and $V_m\left(c_i,c_{i+1}\right)$ is given in (52) for the AdS-spherical wave.

Case 3–some or all of the masses are zero: If r number of masses are zero, then the general solution is

$$V=\sum_{i=1}^{r}V_E\left(c_i,c_{i+1}\right)\left(\ln f\right)^i+\sum_{i=1}^{N-r}V_i, \tag{69}$$

where $f=\frac{z}{\ell}$ and $V_E\left(c_i,c_{i+1}\right)$ is given in (53) for the AdS-plane wave, and $f=\frac{\cos\theta}{1+\sin\theta}$ and $V_E\left(c_i,c_{i+1}\right)$ is given in (57) for the AdS-spherical wave. When all of the masses are zero, that is the maximal criticality case, then the general solution becomes

$$V=\sum_{i=1}^{N}V_E\left(c_i,c_{i+1}\right)\left(\ln f\right)^i. \tag{70}$$

V. pp-WAVE SOLUTIONS

Finally, let us discuss the pp-wave solutions which read in the Kerr-Schild form as

$$g_{\mu\nu}=\eta_{\mu\nu}+2V\lambda_\mu\lambda_\nu, \tag{71}$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric. The function V satisfies the property $\lambda^\mu\partial_\mu V=0$. The vector λ_μ is null $\lambda_\mu\lambda^\mu=0$ and satisfies $\nabla_\mu\lambda_\nu=0$.[2]

[2] One may consider the possibility of extending the condition $\nabla_\mu\lambda_\nu=0$ to the more general condition $\nabla_\mu\lambda_\nu=\frac{1}{2}\left(\lambda_\mu\xi_\nu+\xi_\mu\lambda_\nu\right)$.

It is well-known that for pp-wave spacetimes, the Ricci tensor takes the form $R_{\mu\nu} = -\lambda_\mu \lambda_\nu \partial^2 V$ where ∂^2 is the flat Laplacian. As discussed in [7], the field equations of the $(2N+2)$-derivative gravity theory for the pp-wave metrics reduce to the form

$$\sum_{n=0}^{N} a_n \,\Box^n\, R_{\mu\nu} = -\lambda_\mu \lambda_\nu \sum_{n=0}^{N} a_n \,\Box^n \partial^2 V = 0, \tag{72}$$

where $\Box$ is the Laplacian of the full metric and a_n are constants depending on the parameters of the theory. Here, the first equality follows from $\nabla_\mu \lambda_\nu = 0$. For the pp-wave spacetimes, a scalar ϕ satisfying $\lambda^\mu \nabla_\mu \phi = 0$ also satisfies $\Box\phi = \partial^2 \phi$ and in turn $\lambda^\mu \nabla_\mu \Box\phi = 0$ [7]. Using $\nabla_\mu \lambda_\nu = 0$, together with these results, it can be shown that $\Box^n \partial^2 V = \left(\partial^2\right)^{n+1} V$, so the field equations become

$$\sum_{n=0}^{N} a_n \left(\partial^2\right)^n \partial^2 V = 0. \tag{73}$$

Furthermore, this equation can also be factorized as in the case of the AdS-wave metrics:

$$\prod_{n=1}^{N} \left(\partial^2 - m^2_{n,\text{flat}}\right) \partial^2 V = 0, \tag{74}$$

where $m^2_{n,\text{flat}}$'s are the mass-squared terms for the massive spin-2 excitations around the flat spacetime. Note that $m^2_{n,\text{flat}}$'s are related to the m_n^2's in the limit $\lim_{\ell\to\infty} m_n^2 = m^2_{n,\text{flat}}$. If one assumes that the all $m^2_{n,\text{flat}}$'s are distinct , then the most general solution of (74) is again in the form

$$V = V_E + \sum_{n=1}^{N} V_n, \tag{75}$$

where V_E is the Einsteinian solution solving $\partial^2 V_E = 0$ and each V_n is the massive solution solving $\left(\partial^2 - m^2_{n,\text{flat}}\right) V_n = 0$. For the case of some $m^2_{n,\text{flat}}$'s are equal, the pp-wave solutions also follow the same pattern discussed for the AdS-wave solutions at the end of Sec. I after just changing $\mathcal{O} \to -\partial^2$.

Now, let us find the pp-wave solutions of the sixth order gravity for the four cases discussed above.

Case 1– $m_+^2 \neq m_-^2$: For this case, the field equation has the form

$$\left(\partial^2 - m_-^2\right)\left(\partial^2 - m_+^2\right) \partial^2 V = 0, \tag{76}$$

which has the solution $V = V_E + V_+ + V_-$ where V_E, V_+, and V_- satisfy $\partial^2 V = 0$, $\left(\partial^2 - m_+^2\right) V = 0$, and $\left(\partial^2 - m_-^2\right) V = 0$, respectively.

To find the explicit solutions, let us write the pp-wave metric in the null coordinates

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v + \mathrm{d}z^2 + 2V\left(u,z\right)\mathrm{d}u^2. \tag{77}$$

Then, the relevant differential equation becomes $\left(\partial_z^2 - m_\pm^2\right) V_\pm\left(u,z\right) = 0$ with the solution

$$V_\pm\left(u,z\right) = c_1 e^{m_\pm z} + c_2 e^{-m_\pm z}. \tag{78}$$

As we discussed, the Einsteinian part can be gauged away, so there is no need to consider V_E.

<u>Case 2– $m_+^2 = m_-^2 \neq 0$:</u> The pp-wave solution for this case becomes

$$V_m = c_1 e^{m\,z} + c_2 e^{-m\,z} + z\left(c_3 e^{m\,z} + c_4 e^{-m\,z}\right). \tag{79}$$

<u>Case 3–one of the masses is zero:</u> The solution is

$$V = V_m + V_0, \tag{80}$$

where

$$V_0\left(u, z\right) = c_3 z^3 + c_4 z^2. \tag{81}$$

<u>Case 4– $m_+^2 = m_-^2 = 0$:</u> In this case, the pp-wave solution is

$$V\left(u, z\right) = c_1 z^5 + c_2 z^4 + c_3 z^3 + c_4 z^2. \tag{82}$$

For the general case of $(2N+2)$-derivative theory, we have the similar cases:
<u>Case 1–All the masses are distinct:</u> The general solution is

$$V = \sum_{i=1}^{N} V_i, \tag{83}$$

where the solutions V_i are given in (78).
<u>Case 2–Some masses are equal but not zero:</u> For the case where r number of masses are equal to m, the general solution is

$$V = \sum_{i=0}^{r-1} V_m\left(c_i, c_{i+1}\right) z^i + \sum_{i=1}^{N-r} V_i, \tag{84}$$

where $V_m\left(c_i, c_{i+1}\right)$ is given in (78).
<u>Case 3–some or all of the masses are zero:</u> If r number of masses are zero, then the general solution is

$$V = \sum_{i=2}^{2r+1} c_i z^i + \sum_{i=1}^{N-r} V_i, \tag{85}$$

where $V_E\left(c_i, c_{i+1}\right) = c_i + c_{i+1} z$. When all of the masses are zero, that is the maximal criticality case, then the general solution becomes

$$V = \sum_{i=2}^{2N+1} c_i z^i. \tag{86}$$

Note that all the c_i's appearing in the solutions of this section are arbitrary functions of u.

VI. CONCLUSIONS

In this work, we studied wave-type exact solutions of any higher derivative gravity theory in three dimensions. These solutions also solve the linearized, perturbative, equations for the spin-2 sector as noted below (11). The field equations of the most general gravity theory are highly complicated and nonlinear that, a priori, it is hard to expect any exact solution (besides the maximally symmetric ones) to be found in closed form. But, rather remarkably, we found three different wave type solutions AdS-plane, AdS-spherical, and the pp-wave in any higher derivative theory which, by the way, do not exist in pure Einstein's gravity in three dimensions. The exact solutions, as well as the perturbative solutions, are parametrized by the values of the masses of the spin-2 excitations. Among the solutions, there are some critical cases that arise when some of the masses vanish or are equal to each other. For these critical cases, the operators take a nonfactorizable form and logarithmic terms appear in the solutions changing the asymptotic structures of the spacetime. As a specific example, we worked out the details of the most general sixth-order gravity for which we determined the field equations and the masses of the two spin-2 excitations explicitly. This example also covers the recently introduced tricritical gravity in three dimensions. It is an open question whether there could be other wave solutions in these theories.

Here, we were mainly interested in finding the exact wave solutions (in flat and AdS spacetimes) as well as the spin-2 spectrum of the generic theory while keeping in mind that these solutions, being the closest cousins of the globally AdS spacetime with the same curvature invariants as the latter, have potential applications in the AdS_3/CFT_2 context. For the generic solutions, where there are no logarithmic terms, Brown-Henneaux (BH) type boundary conditions [17] are applicable; while for the logarithmic solutions, one needs to relax these boundary conditions as was already noted in other theories [18, 19]. We have not studied the properties (c-charges, etc) of the putative CFT_2 theory, but it is quite possible that certain theories among the generic set we have studied will turn out to have a unitary CFT away from the special points. On the other hand, we expect that generically, the specific theories with the log terms will lead to non-unitary CFTs.

While we have studied a large class of gravity theories in 2+1 dimensions in this work, we have left several theories which need to be mentioned: to the most general action, one can add the parity-violating Chern-Simons term to obtain a new class of theories which will be extensions of Topologically Massive Gravity [20]. In principle, it is easy to extend our solutions to this more general parity violating theory. It would be interesting to see if such extensions and their chiral limits lead to viable boundary CFT theories. Finally, as was recently suggested [21, 22], a theory can be consistently defined without an action based on the metric alone, but with field equations, this theory is called the minimal massive gravity (MMG) with a single massive helicity 2 graviton with the property that the theory is unitary both in the bulk and on the boundary. Extension to the two spin-2 case was given in [23]. Exact solutions of these theories and their chiral limits were given in [24, 25].

VII. ACKNOWLEDGMENT

M. G. and B. T. are supported by the TÜBİTAK grant 113F155. T. C. S. thanks The Centro de Estudios Científicos (CECs) where part of this work was carried out under the support of Fondecyt with grant 3140127.

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