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Note

Covering a rectangular chessboard with staircase walks

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ABSTRACT

Let $C(n, m)$ be a $n \times m$ chessboard. An *ascending* (respectively *descending*) *staircase* walk on $C(n, m)$ is a rook's path on $C(n, m)$ that in every step goes either right or up (respectively right or down). We determine the minimal number of ascending and descending staircase walks covering $C(n, m)$.

1. Introduction

Let $C(n, m)$ be a $n \times m$ chessboard consisting of n rows and m columns. A rook's path on $C(n, m)$ that in every step goes either right or up (respectively right or down) is called an *ascending* (respectively *descending*) staircase walk. We find $f(n, m)$, the minimum number of ascending and descending staircase walks that together visit each unit square of $C(n, m)$ at least once. In the square case of $n = m$ the problem was recently solved in [1]:

Theorem 1 ([1]). $f(n, n) = \lceil \frac{2}{3} n \rceil$.

The main result of this paper is the following generalization of Theorem 1:

Theorem 2. $f(n, m) = \lceil \frac{2}{3}\left(n + m - \sqrt{n^2 + m^2 - nm}\right) \rceil$.

Due to the definitions of staircase walks $f(n, m) = f(m, n)$. Let $g(n, k)$ be a maximal integer such that there are k staircase walks that together visit each unit square of $C(n, g(n, k))$. Since $f(n, m) = min_{g(n,k) \geq m} k$, in order to determine $f(n, m)$ we will find an explicit expression for $g(n, k)$. Since for any $k \geq nk$ staircases can trivially cover $C(n, m)$ for any m, we will determine $g(n, k)$ for all $k < n$:

Theorem 3. *Let $k < n$. Then $g(n, k) = k + \lfloor \frac{k^2}{4(n-k)} \rfloor$.*

In order to prove Theorem 3 we will establish a slightly more general result. To be specific, giving a and b we will find the maximal value of m for which a ascending and b descending staircase walks together cover $C(n, m)$. Since for any a,b satisfying $a + b \geq na$ ascending and b descending staircase walks can trivially cover $C(n, m)$ for any m, we will focus on the case $a + b < n$.

Theorem 4. *Let $a + b < n$ and $g(n, a, b)$ be the maximal integer such that there are a ascending and b descending staircase walks that together visit each unit square of $C(n, g(n, a, b))$. Then*

$$g(n, a, b) = a + b + \left\lfloor \frac{ab}{n - a - b} \right\rfloor. \tag{1}$$

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2. Proofs

First of all, let us show that Theorem 4 implies Theorem 3. At fixed $k = a + b$ the expression ab takes its maximum when a and b are closest to each other. Therefore, by (1)

$$g(n,k) = \begin{cases} k + \left\lfloor \dfrac{k^2}{4(n-k)} \right\rfloor & \text{if } k \text{ is even} \\ k + \left\lfloor \dfrac{k^2-1}{4(n-k)} \right\rfloor & \text{if } k \text{ is odd.} \end{cases}$$

Let us show that for odd values of k the equality $\lfloor \frac{k^2}{4(n-k)} \rfloor = \lfloor \frac{k^2-1}{4(n-k)} \rfloor$ holds. Indeed, let $\delta = \frac{k^2}{4(n-k)} - \lfloor \frac{k^2}{4(n-k)} \rfloor$. Since $\frac{k^2}{4(n-k)}$ is not an integer number we readily get $\delta = \frac{d}{4(n-k)}$, where d is an integer number satisfying $1 \le d < 4(n-k)$. Since $\frac{k^2}{4(n-k)} - \frac{k^2-1}{4(n-k)} = \frac{1}{4(n-k)}$ we have $\lfloor \frac{k^2}{4(n-k)} \rfloor \le \frac{k^2-1}{4(n-k)} < \lfloor \frac{k^2}{4(n-k)} \rfloor + \delta$. Therefore, $\lfloor \frac{k^2}{4(n-k)} \rfloor = \lfloor \frac{k^2-1}{4(n-k)} \rfloor$ and Theorem 4 implies Theorem 3.

Let us show that Theorem 3 implies Theorem 2. Since $f(n,m) = min_{g(n,k) \ge m} k$, we will find a minimal integer k, $1 \le k < n$ such that $k + \lfloor \frac{k^2}{4(n-k)} \rfloor \ge m$. Since k and m are integers, the last inequality is equivalent to $k + \frac{k^2}{4(n-k)} - m \ge 0$. Let $H(x) = x + \frac{x^2}{4(n-x)} - m$, where x is a real positive variable not exceeding n. The inequality $H(x) \ge 0$ is equivalent to $P(x) \le 0$ where $P(x) = 3x^2 - 4(n+m)x + 4nm$ is a polynomial of the second degree. Two distinct positive roots of $P(x)$ are $x_{1,2} = \frac{2}{3}\big(n + m \mp \sqrt{n^2+m^2-nm}\big)$ and readily $x_2 - x_1 = \frac{4}{3}\big(\sqrt{n^2+m^2-nm}\big) > 1$. Therefore, the desired k is a smallest integer which is not less than the smallest root x_1: Theorem 3 implies Theorem 2.

Now we start to prove Theorem 4. Below we give several notations and definitions adopted from [1]. Let (i,j) be the unit square in the ith column and the jth row ($(1,1)$ denotes the bottom left square). Without loss of generality we suppose that all ascending staircase walks start at square $(1,1)$ and end at square (m,n), otherwise we can extend them to be such. Similarly, all descending staircase walks start at square $(1,n)$ and end at square $(m,1)$. Let D_i be the ith *descending diagonal* of $C(n,m)$: D_i consists of all unit squares satisfying $x+y = i+1$. Let E_i be the ith *ascending diagonal* of $C(n,m)$: E_i consists of all unit squares satisfying $x-y = i-n$. Let $U_a = \cup_{i=a}^{m+n-a} D_i$ and $V_b = \cup_{i=b}^{m+n-b} E_i$. Suppose that a ascending staircases $S_1, \dots, S_a$ and b descending staircases $T_1, \dots, T_b$ together cover $C(n,m)$. According to [1] (Lemmas 1 and 2) for $a+b < min(n,m)$ without loss of generality we can suppose that

- ∘ ascending staircases $S_1, \dots, S_a$ are disjoint in U_a and descending staircases $T_1, \dots, T_b$ are disjoint in V_b
- ∘ each ascending staircase S_i totally belongs to V_b and each descending staircase T_i totally belongs to U_b.

Therefore, since staircases start and end at opposite corner squares any pair (S_i, T_j) have at least one common unit square in $U_a \cap V_b$.

The key point of the proof is based on the following crucial simple observation which directly follows from the definition of staircase walks:

Observation. *The set of unit squares of each descending diagonal visited by at least one ascending walk is at most a and the set of unit squares of each ascending diagonal visited by at least one descending walk is at most b.*

We start with investigation of the upper bound of $g(n,a,b)$:

Lemma 1. *Let a, b and n be nonnegative integers such that $n > a+b$. Then*

$$g(n,a,b) \le a + b + \left\lfloor \frac{ab}{n-a-b} \right\rfloor.$$

Proof. Clearly for each $1 \le i \le a-1$ diagonal D_i contains i unit squares; for each $m+n-a+1 \le i \le m+n-1$ diagonal D_i contains $m+n-i$ unit squares. Similarly, for each $1 \le i \le b-1$ diagonal E_i contains i unit squares; for each $m+n-b+1 \le i \le m+n-1$ diagonal E_i contains $m+n-i$ unit squares. Due to the observation for each i satisfying $a \le i \le m+n-a$, a ascending staircases will cover together at most a unit squares of each D_i and therefore the set of all unit squares of $C(n,m)$ visited by at least one ascending walk is at most $1+2+\cdots+(a-1)+a(m+n-1-2(a-1))+(a-1)\cdots+2+1 = a(m+n)-a^2$. Similarly, due to the observation for each i satisfying $b \le i \le m+n-b$ b descending staircases will cover together at most b unit squares of each E_i and therefore the set of all unit squares of $C(n,m)$ visited by at least one descending walk is at most $1+2+\cdots+(b-1)+b(m+n-1-2(b-1))+(b-1)\cdots+2+1 = b(m+n)-b^2$. There are ab pairs (S_i, T_j) of ascending and descending walks. Since without loss of generality $S_i \subset V_b$, $T_i \subset U_b$ and any two different ascending (descending) staircases are disjoint [1], we conclude that: S_i and T_j have at least one common unit square in $U_a \cap V_b$ and for distinct pairs (S_i, T_j) these common unit squares are distinct. Therefore, there are at least ab revisits of unit squares. Thus, a ascending and b descending walks cover at most $a(n+m) - a^2 + b(n+m) - b^2 - ab$ unit squares. Since a ascending and b descending staircase walks have covered $C(n,m)$ consisting of nm squares we get

$$nm \le (a+b)(n+m) - a^2 - b^2 - ab \tag{2}$$

which in turn implies

$$m \le a + b + \frac{ab}{n-a-b}.$$

Thus, since m is an integer the desired upper bound for $g(n, a, b)$ is obtained and the lemma is proved. The proof is coherent with the estimation of lower bound of $f(n, n)$ in [1].

Now we explore the lower bound of $g(n, a, b)$:

Lemma 2. *Let a, b and n be nonnegative integers such that $n > a + b$. Then*

$$g(n, a, b) \ge a + b + \left\lfloor \frac{ab}{n-a-b} \right\rfloor. \tag{3}$$

Proof. A part of a walk starting at square (x, y) and successively visiting squares $(x+1, y), (x+2, y), \dots, (x+z, y)$ will be denoted by $(x, y) \to (x+z, y)$. A part of a walk starting at square (x, y) and successively visiting squares $(x, y+1), (x, y+2), \dots, (x, y+z)$ will be denoted by $(x, y) \to (x, y+z)$. We will label ascending and descending staircases by $S_1, \dots, S_a$ and $T_1, \dots, T_b$, respectively. For each $i = 1, 2, \dots, a$ we set the initial segment of each ascending staircase S_i as $(1, 1) \to (a-i+1, 1) \to (a-i+1, i)$ and the last segment of S_i as $(m-i+1, n-a+i) \to (m, n-a+i) \to (m, n)$. Thus, the initial segments of the a descending staircases will cover the first a ascending diagonals $D_1, \dots, D_a$ and the last segments of the a descending staircases will cover the last a descending diagonals $D_{m+n-a}, \dots, D_{m+n-1}$. For each $i = 1, 2, \dots, b$ we set the initial segment of each descending staircase T_i as $(1, n) \to (1, n-i+1) \to (b-i+1, n-i+1)$ and the last segment of T_i as $(m-i+1, b-i+1) \to (m, b-i+1) \to (m, 1)$. Thus, the initial segments of the b ascending staircases will cover the first b descending diagonals $E_1, \dots, E_b$ and the last segments of the b ascending staircases will cover the b descending diagonals $E_{m+n-b}, \dots, E_{m+n-1}$. Thus, below we can suppose that S_i, $i = 1, \dots, a$ starts at $(i, n-a+i)$, ends at $(n-a+i, i)$ and T_i, $i = 1, \dots, b$ starts at $(i, n-b+i)$ and ends at $(n-b+i, i)$.

Let $\Delta = n - a - b$. The inequality (3) in terms of a, b and $\Delta > 0$ will be denoted by $I(\Delta, a, b)$. Thus, our goal is to prove the following inequality

$$I(\Delta, a, b) : g(\Delta + a + b, a, b) \ge a + b + \left\lfloor \frac{ab}{\Delta} \right\rfloor. \tag{4}$$

We will prove the inequality $I(\Delta, a, b)$ for three variables Δ, a, b in finite sequence of steps, such that in each step we reduce the proof to the proof of the identical inequality where exactly one of the variables is decreased. In other words, $I(\Delta, a, b)$ will be reduced to $I(\Delta', a', b')$ where $\Delta' > 0$, $a' \ge 0$, $b' \ge 0$, two variables out of (Δ', a', b') coincide with the corresponding variable in (Δ, a, b) and exactly one variable is decreased. The process will be terminated if either one of the parameters a', b' becomes zero or $\lfloor \frac{a'b'}{\Delta} \rfloor = 0$. In the first case the terminal inequality becomes $I(\Delta', a', b')$ with $a' = 0$ or $b' = 0$. For example, $I(\Delta', a', 0)$ states that

$$g(n, a', 0) \ge a' + 0 + \left\lfloor \frac{a' \cdot 0}{n - a' - 0} \right\rfloor = a' \tag{5}$$

which is immediate since a' staircases can trivially cover first a' columns. In the second case the terminal inequality $I(\Delta', a', b')$ becomes

$$g(n, a', b') \ge a' + b' + \left\lfloor \frac{a' \cdot b'}{n - a' - b'} \right\rfloor = a' + b' \tag{6}$$

which is also immediate since a' ascending and b' descending staircases can trivially cover first $a' + b'$ columns.

The key point of the proof is the following constructions called ascending right and descending right moves.

If $a > 0$, $b > 0$, $a \ge b$ and $a \ge \Delta$ we define the ascending right move:

- ◦ for each $i = 1, \dots, b$: T_i starts as $(b-i+1, n-i+1) \to (b-i+1, a+b-i+1) \to (2b+\Delta-i+1, a+b-i+1)$.
- ◦ for each $i = 1, \dots, a-\Delta$: S_i starts as $(a-i+1, i) \to (a+b-i+1, i)$.
- ◦ for each $i = a-\Delta+1, \dots, a$: S_i is $(a-i+1, i) \to (a+b-i+1, i) \to (a+b-i+1, n-a+i) \to (m-i+1, n-a+i)$.

Note that the ascending right move completely sets walks of Δ ascending staircases S_i for $i = a-\Delta+1, \dots, a$. The move covers first $b+\Delta$ columns and last Δ rows of $C(n, m)$; the uncovered part of $C(n, m)$ is a rectangle $C(n-\Delta, m-b-\Delta)$. In order to prove (3) the uncovered $C(n-\Delta, m-b-\Delta)$ should be covered by remaining pieces of walks S_i, $i = 1, \dots, a-\Delta$ and T_i, $i = 1, \dots, b$. Since the terminal squares of these walks coincide with initial squares of corresponding to $C(n-\Delta, m-b-\Delta)$ $a-\Delta$ ascending and b descending staircases, the proof of $I(\Delta, a, b)$ will be a consequence of $I(\Delta, a-\Delta, b)$. Indeed, since $b+\Delta$ columns and Δ rows are already covered by the move we get

$$g(n, a, b) \ge b + \Delta + g(n-\Delta, a-\Delta, b) \ge b + \Delta + a - \Delta + b + \left\lfloor \frac{(a-\Delta)b}{\Delta} \right\rfloor = a + b + \left\lfloor \frac{ab}{\Delta} \right\rfloor$$

where the first inequality is due to the ascending right move and the second inequality is due to $I(\Delta, a-\Delta, b)$. Thus, as a result of ascending right move $I(\Delta, a, b)$ is reduced to $I(\Delta, a-\Delta, b)$.

Analogously if $a > 0$, $b > 0$, $b > a$ and $b \geq \Delta$ we define the descending right move:

- for each $i = 1, \dots, a$: S_i starts as $(a-i+1, i) \to (a-i+1, i+\Delta) \to (2a+\Delta-i+1, i+\Delta)$.
- for each $i = 1, \dots, b-\Delta$: T_i starts as $(b-i+1, n-i+1) \to (a+b-i+1, n-i+1)$.
- for each $i = b-\Delta+1, \dots, b$: T_i is $(b-i+1, n-i+1) \to (a+b-i+1, n-i+1) \to (a+b-i+1, b-i+1) \to (m-i+1, b-i+1)$.

Note that the descending right move completely sets walks of Δ descending staircases T_i for $i = b-\Delta+1, \dots, b$. The move covers first $a+\Delta$ columns and first Δ rows of $C(n, m)$; the uncovered part of $C(n, m)$ is a rectangle $C(n-\Delta, m-a-\Delta)$. In order to prove (3) the uncovered $C(n-\Delta, m-a-\Delta)$ should be covered by remaining pieces of walks S_i, $i = 1, \dots, a$ and T_i, $i = 1, \dots, b-\Delta$. Since the terminal squares of these walks coincide with initial squares of corresponding to $C(n-\Delta, m-a-\Delta)$ a ascending and $b-\Delta$ descending staircases, the proof of $I(\Delta, a, b)$ will be a consequence of $I(\Delta, a, b-\Delta)$. Indeed, since $a+\Delta$ columns and Δ rows are already covered by the move we get

$$g(n, a, b) \geq a + \Delta + g(n-\Delta, a, b-\Delta) \geq a + \Delta + a + b - \Delta + \left\lfloor \frac{a(b-\Delta)}{\Delta} \right\rfloor = a + b + \left\lfloor \frac{ab}{\Delta} \right\rfloor$$

where the first inequality is due to the descending right move and the second inequality is due to $I(\Delta, a, b-\Delta)$. Thus, as a result of ascending right move $I(\Delta, a, b)$ is reduced to $I(\Delta, a, b-\Delta)$.

Therefore, by finite number right moves the proof of $I(\Delta, a, b)$ will be reduced to the proof of the inequality

$$I(\Delta, r, q) : g(\Delta + r + q, r, q) \geq r + q + \left\lfloor \frac{rq}{\Delta} \right\rfloor \tag{7}$$

where $a = a_1\Delta + r$ and $b = b_1\Delta + q$ and $0 \leq r, q < \Delta$. Now if one of r, q is equal to zero or if $\lfloor \frac{rq}{\Delta} \rfloor = 0$ then the proof of (4) will follow from (5) or (6) respectively. On the contrary, suppose that $r, q > 0$ and $\lfloor \frac{rq}{\Delta} \rfloor > 0$ then $\Delta > r, q$ and the conditions for the ascending and descending right moves are not held. Let us show that the inequality (7) is a consequence of the following inequality

$$I\left(\left\lfloor \frac{qr}{\Delta} \right\rfloor, q, r\right) : g\left(\left\lfloor \frac{qr}{\Delta} \right\rfloor + q + r, q, r\right) \geq q + r + \Delta. \tag{8}$$

Indeed, in order to cover $C(r+q+\Delta, r+q+\lfloor \frac{rq}{\Delta} \rfloor)$ by r ascending and q descending staircases we can

- by (8) cover $C(r+q+\lfloor \frac{rq}{\Delta} \rfloor, r+q+\Delta)$ by q ascending and r descending staircases
- rotate the covered $C(r+q+\lfloor \frac{rq}{\Delta} \rfloor, r+q+\Delta)$ by $\pi/2$ counterclockwise
- place the rotated rectangle instead of $C(r+q+\Delta, r+q+\lfloor \frac{rq}{\Delta} \rfloor)$.

The operation defined above we will call a rotation. Since

$$\left\lfloor \frac{qr}{\lfloor \frac{rq}{\Delta} \rfloor} \right\rfloor \geq \left\lfloor \frac{qr}{rq/\Delta} \right\rfloor = \Delta$$

the inequality (8) in turn is a consequence of

$$I\left(\left\lfloor \frac{qr}{\Delta} \right\rfloor, q, r\right) : g\left(\left\lfloor \frac{qr}{\Delta} \right\rfloor + q + r, q, r\right) \geq q + r + \left\lfloor \frac{qr}{\lfloor \frac{rq}{\Delta} \rfloor} \right\rfloor. \tag{9}$$

Thus, the required inequality (8) is reduced to the inequality (9) which is the inequality $I(\Delta', q, r)$ with $\Delta' = \lfloor \frac{qr}{\Delta} \rfloor$. Now note that since $\Delta > r, q$ we have $\Delta' = \lfloor \frac{qr}{\Delta} \rfloor < r, q < \Delta$. Thus, by the rotation operation the inequality (7) has reduced to the inequality (9) in which $\Delta' < \Delta$ and since $\Delta' < q, r$ a new series of ascending and descending right moves is applicable to the triple Δ', q, r. Now in order to establish (9) we apply a new series of ascending and descending right moves and at each step decrease one of the parameters. Each time when parameter Δ' becomes greater than both parameters q, r we apply the rotation operation. We continue the process until one of the parameters $\tilde{a}, \tilde{b}$ or $\lfloor \frac{\tilde{a}\tilde{b}}{\tilde{\Delta}} \rfloor$ eventually becomes zero and finally complete the proof by applying (5) or (6). Lemma 2 is proved.

The proof of Lemma 2 allows us to construct the cover of $C(n, m)$ explicitly. In the special case $a = b$ and $n = m = 3a$ this cover coincides with the cover presented in [1]. Fig. 1 shows how the rectangle $C(21, 34)$ can be covered by 10 ascending and 7 descending staircases.

The rectangle $C(21, 34)$ is covered in eight steps:

1. Ascending right move at $a = 10$, $b = 7$, $\Delta = 4$.
2. Descending right move at $a = 6$, $b = 7$, $\Delta = 4$.
3. Ascending right move at $a = 6$, $b = 3$, $\Delta = 4$.

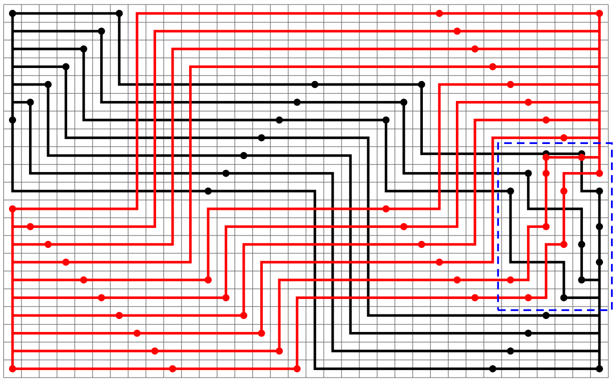

Fig. 1. The cover of $C(21, 34)$ by 10 ascending and 7 descending staircases.

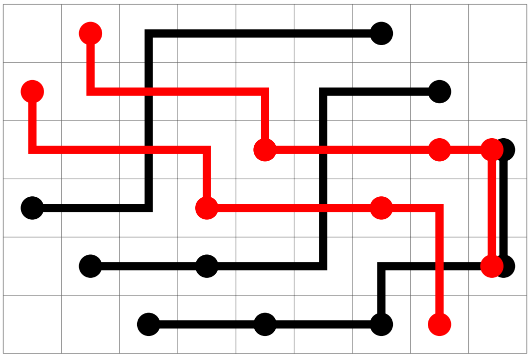

Fig. 2. The cover of $C(6, 9)$ by 3 ascending and 2 descending staircases.

After the third move $a = 2, b = 3, \Delta = 4$ no right moves are possible and in order to complete the cover $C(9, 34)$ we first will cover $C(6, 9)$ by 3 ascending and 2 descending staircases (see Fig. 2).

The rectangle $C(6, 9)$ is covered in four steps:

4. Ascending right move at $a = 3, b = 2, \Delta = 2$.
5. Ascending right move at $a = 3, b = 2, \Delta = 2$ (since $a = b = 2$ we can apply also descending right move).
6. Descending right move at $a = 1, b = 1, \Delta = 1$.
7. We trivially cover the rectangle $C(3, 1)$ by one ascending and one descending walks not using right moves, actually by right moves these staircases could cover $C(3, 3)$.
8. Finally we rotate the covered $C(6, 9)$ by $\pi/2$ counterclockwise and place on rectangle $C(9, 6) \subset C(21, 34)$ with the blue dashed boundary.

Note that since $\frac{a \cdot b}{\Delta} = \frac{10 \cdot 7}{4}$ is not an integer, ascending and descending staircases in Fig. 2 intersect at more than one unit square.

Theorem 4 readily follows from Lemmas 1 and 2.

Acknowledgments

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References

[1] E. Ackerman, R. Pinchasi, Covering a chessboard with staircase walks, Discrete Math. 313 (2013) 2547–2551.