STEADY-STATE ANALYSIS OF A MULTI-CLASS MAP/PH/C QUEUE WITH ACYCLIC PH RETRIALS

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Abstract

A multi-class c-server retrial queueing system in which customers arrive according to a class-dependent Markovian arrival process (MAP) is considered. Service and retrial times follow class-dependent phase-type (PH) distributions with the further assumption that PH distributions of retrial times are acyclic. A necessary and sufficient condition for ergodicity is obtained from criteria based on drifts. The infinite state space of the model is truncated with an appropriately chosen Lyapunov function. The truncated model is described as a multi-dimensional Markov chain and a Kronecker representation of its generator matrix is numerically analyzed.

Keywords: Markovian arrival process; phase-type service time distribution; acyclic phase-type retrial time distribution; Kronecker product; Markov chain

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1. Introduction

Retrial queues are queueing systems in which an arriving customer who finds all servers busy joins an infinite retrial queue (called the orbit) and retries to receive service. Such systems have been considered in various application areas such as call centers, computer networks, and telecommunication systems. Detailed overviews and bibliographical information about retrial queues may be obtained from the surveys by Gómez-Corral [29] and by Kim and Kim [33], the books by Artalejo and Gómez-Corral [4] and by Falin and Templeton [26], and also from the bibliographies by Artalejo [1, 2].

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Criteria based on Lyapunov functions have been widely used for the stability analysis of retrial queues. The following is a review to that end in which the list of references is not exhaustive but covers many related papers. For single- and multi-class M/M/c queues with exponential retrials respectively in [26] and [47], simple linear Lyapunov functions are used to show the sufficiency and necessity of an ergodicity condition. Artalejo and Phung-Duc [5] extend a single-class M/M/c queue with exponential retrial by allowing outgoing calls. Sakurai and Phung-Duc [44] consider a more general model than that in [5] with multiple types of outgoing calls. In both systems, simple linear Lyapunov functions are used in the sufficiency proofs of ergodicity conditions.

Analysis of a system in which arrival or service processes include phases is more complicated. For such systems, one possible approach is to obtain some of the parameters of the Lyapunov function using algebraic properties of the blocks of an auxiliary M/G/1 type matrix (see [38]). Phung-Duc and Kawanishi [40] consider an M/M/c retrial queue with exponential retrials and after-call work, and allow for customer abandonment in [41]. Diamond and Alfa [22] consider a retrial queue in which customers arrive according to a Markovian arrival processes (MAP) [37], service times follow phase-type (PH) [38] and retrial times follow exponential distributions. For these three systems, the auxiliary M/G/1 type matrices are obtained from their classical queueing system counterparts in which customers join infinite waiting lines instead of orbits. Dudin and Klimenok [24] define asymptotically quasi-Toeplitz Markov chains (AQTMCs) whose blocks converge to the blocks of an M/G/1 type matrix as the level numbers increase. The sufficiency of an ergodicity condition is then obtained by choosing a Lyapunov function whose parameters are determined using the results for the blocks in the M/G/1 type matrix. They also show that a single-server queue with batch MAP (BMAP) arrivals, semi-Markovian service process, and exponential retrials is in the class of AQTMCs. Breur et al. [10] show that a BMAP/PH/c queue with exponential retrials is in the class of AQTMCs and give a sufficient condition for its ergodicity. He et al. [30] consider a BMAP/PH/c retrial queue with a waiting line and PH retrials. They obtain a sufficient ergodicity condition by choosing a Lyapunov function whose parameters follow from a classical BMAP/PH/c queue with no retrials.

Artalejo and Gómez-Corral [4, p. 33] indicate that mathematical analysis of multi-class retrial queues in which all customers can join an orbit is more difficult compared

to its single-class counterpart since its joint queue length process is a random walk on the multi-dimensional integer lattice. Expected waiting time expressions are given for two- and multi-class M/G/1 queues with batch arrivals and exponential retrials in [34] and [25], respectively. Avrachenkov et al. [7] consider an M/G/c queue with waiting lines and constant retrial policy in which only one customer in the orbit can attempt to get service. Shin and Moon [47] show that the stationary distribution of a multi-class M/M/c queue with exponential retrials converges to that of a classical multi-class M/M/c queue with discriminatory random order service policy as retrial rates tend to infinity, and they present approximation formulae for some performance measures. Kim [31] considers a multi-server multi-class retrial queue in which customers arrive according to a class-dependent Poisson process, service and retrial times follow exponential distributions, and each server can serve a specific class of customers. They obtain a necessary and sufficient condition for positive recurrence by using the fluid limit approach. There are explicit results for a multi-class retrial queue with multiple servers and a few papers take a computational approach. The retrial queue in [16] has two customer classes in which customers either join an infinite waiting line or a finite orbit depending on their class if all servers are busy upon arrival. Therein, this system is modeled as an LDQBD and solved using the matrix-analytic method in [39]. Choi et al. [17] obtain several performance measures; they consider a different system than the one in [16] in that the orbit is infinite and the waiting line is finite. Gharbi et al. [28] model a finite source retrial system using generalized stochastic Petri nets and analyze it with the embedded Markov chain resolution algorithm in [15].

The analysis of a queueing system with PH retrials requires a state representation keeping the number of customers in all retrial phases. This leads to a random walk on the multi-dimensional integer lattice with more than one infinite dimension as for multiple customer classes. Therefore, there are only a handful of papers that consider PH retrials. In those, either simple arrival and service processes are assumed, or a methodology for numerical analysis is not proposed. M/M/c queues with PH retrials appear in [35, 45, 46]. Kumar et al. [35] analyze the waiting time distribution, and Shin and Moon [46] present approximation formulae for the distributions of numbers of customers in service and orbit. Shin [45] describes the model with two retrial phases as an LDQBD and gives an algorithmic solution. In [23], a stability condition for an

M/PH/1 queue with PH retrials is obtained. Besides, a method is proposed for approximating its stationary distribution and waiting time moments. In [3], a MAP/PH/c queue with PH retrials is numerically analyzed with a finite source assumption. Besides giving a sufficient condition for ergodicity, He et al. [30] show that the condition is also necessary for the ergodicity of a BMAP/PH/c queue with a waiting line and PH retrials. However, a methodology for numerical analysis is not proposed. Finally, Chakravarthy [14] studies a MAP/PH/c queue with PH retrials via simulation due to its complexity. To the best of our knowledge, a multi-class queueing system with MAP arrivals, PH services, and PH retrials has not been analyzed previously.

In this paper, a multi-class MAP/PH/c retrial queueing system with acyclic PH retrials is considered. The acyclic PH distribution is a subclass of the PH distribution, but is considered to be as powerful as a general PH distribution, since both are dense in the set of nonnegative distributions (see, for instance, [13]). Therefore the system under consideration is quite general. A necessary and sufficient condition for its ergodicity is obtained from criteria based on drifts by choosing appropriate Lyapunov functions. The system includes multiple customer classes and construction of a useful auxiliary M/G/1 type matrix is not obvious. Hence, it seems that the approach taken to choose the Lyapunov function for the BMAP/PH/c retrial queue in [30] is not applicable when there are multiple customer classes. The infinite state space of the model is truncated with the help of the Lyapunov function chosen in the sufficiency proof so that a finite state space including at least a given steady-state probability mass is determined [18]. Then the truncated model is described as a multi-dimensional Markov chain (MC) and a Kronecker representation of its underlying infinitesimal generator matrix is formed in a similar manner to those models in [8, 19]. Finally, the steady-state distribution of the truncated model is computed iteratively using successive over-relaxation (SOR) [49]. Here, the truncated model is not modeled as an infinite LDQBD and is not solved using the matrix-analytic method of Bright and Taylor [11] although it is possible to do so by choosing an appropriate level definition as in [8, 19, 21]. The reason for this choice is that the method does not scale well as the number of dimensions in the multi-dimensional MC increases. This is due to the increase in the order of the diagonal blocks as the level number increases in multi-dimensional MCs.

2. Mathematical model

The system under consideration is a multi-class MAP/PH/c queue with acyclic PH retrials, where c is the number of servers. We model this system as a multi-dimensional MC and give the generator matrix of the underlying Markov process.

Recall that a MAP can be viewed as a counting process or an irreducible MC with some marked transitions (describing arrivals) as in [13]. We will be using the definition of a MAP with the latter interpretation given below.

Definition 1. MAP with representation $(\boldsymbol{C},\boldsymbol{D})$ of order m is an irreducible MC with a state space of size m and irreducible generator matrix $(\boldsymbol{C}+\boldsymbol{D})$, where $\boldsymbol{C}$ is a nonsingular matrix with negative diagonal and nonnegative off-diagonal elements and $\boldsymbol{D} \geqslant 0$.

A state of the MAP in Definition 1 is said to be a phase. Without loss of generality, we assume that the phases of the MAP are numbered 1 through m. The MAP characterizes a stochastic process, where $\boldsymbol{C}$ includes transitions without an arrival and $\boldsymbol{D}$ includes transitions with one customer arrival. The definition of PH distribution which will be used to model service and retrial times is given next.

Definition 2. Let $m \in \mathbb{Z}_{\geqslant 0}$, $\boldsymbol{\beta} \in \mathbb{R}_{\geqslant 0}^{1 \times m}$, and $\boldsymbol{T} \in \mathbb{R}_{\geqslant 0}^{m \times m}$ be a nonsingular matrix with negative diagonal and nonnegative off-diagonal elements. A PH distribution with representation $(\boldsymbol{\beta},\boldsymbol{T})$ of order m is the distribution of time until absorption in a finite state space MC with generator matrix

$$\hat{\boldsymbol{T}} = \left[\begin{array}{cc} \boldsymbol{T} & \boldsymbol{T}^0 \\ \boldsymbol{0} & 0 \end{array} \right]_{(m+1) \times (m+1)}$$

and initial probability vector $(\boldsymbol{\beta}, 1 - \boldsymbol{\beta e}) \in \mathbb{R}_{\geqslant 0}^{1 \times (m+1)}$.

Here, $\boldsymbol{e}$ represents a column vector of 1's. Without loss of generality, we assume that the state space of $\hat{\boldsymbol{T}}$ is $\{1, \ldots, m+1\}$, where $m+1$ is the absorbing state and the other states are transient. The transient states in a PH distribution are called phases. We assume that the process does not start in the absorbing state; hence, $\boldsymbol{\beta e} = 1$ holds. Since $\hat{\boldsymbol{T}}$ is the generator matrix of a MC, $\boldsymbol{T}^0 = -\boldsymbol{Te}$ is a nonnegative column vector. Next, the definition of acyclic PH distribution is given.

Definition 3. PH distribution with representation $(\boldsymbol{\xi}, \boldsymbol{U})$ is said to be acyclic if its states can be ordered in such a way that $\boldsymbol{U}$ is an upper-triangular matrix.

We consider a retrial queueing system with $c \geqslant 1$ homogeneous servers and $K \geqslant 1$ customer classes. Customers of class $k \in \{1, \ldots, K\}$ arrive according to MAP with representation $(\boldsymbol{C}_k, \boldsymbol{D}_k)$ of order m_k^A. Since $(\boldsymbol{C}_k + \boldsymbol{D}_k)$ is irreducible by Definition 1, there exists a nonnegative row vector $\boldsymbol{\theta}_k \in \mathbb{R}_{\geqslant 0}^{1 \times m_k^A}$ such that $\boldsymbol{\theta}_k(\boldsymbol{C}_k + \boldsymbol{D}_k) = \boldsymbol{0}$ and $\boldsymbol{\theta}_k \boldsymbol{e} = 1$. Furthermore, the average arrival rate is given by $\lambda_k = \boldsymbol{\theta}_k \boldsymbol{D}_k \boldsymbol{e}$. If all servers are busy upon arrival, an arriving customer of class k joins orbit k and retries to capture a server after a random amount of time. A retrial customer is blocked if it attempts to receive service when there are no idle servers. The service time of a class k customer follows PH distribution with representation $(\boldsymbol{\beta}_k, \boldsymbol{T}_k)$ of order m_k^S and $\boldsymbol{T}_k^0 = -\boldsymbol{T}_k \boldsymbol{e}$. The retrial time of a class k customer follows acyclic PH distribution with representation $(\boldsymbol{\xi}_k, \boldsymbol{U}_k)$ of order m_k^R and $\boldsymbol{U}_k^0 = -\boldsymbol{U}_k \boldsymbol{e}$. Hence, we assume that $\boldsymbol{U}_k$ is upper-triangular. For a customer of class k, the average service rate is given by $\mu_k = \left[-\boldsymbol{\beta}_k(\boldsymbol{T}_k)^{-1}\boldsymbol{e}\right]^{-1}$ and the average retrial rate is given by $\delta_k = \left[-\boldsymbol{\xi}_k(\boldsymbol{U}_k)^{-1}\boldsymbol{e}\right]^{-1}$.

In [14], a single-class MAP/PH/c queue with PH retrials is modeled using a multi-dimensional MC. The multi-class counterpart can also be modeled similarly. To that end, we let $X_k(t)$, $X_{b_k^R + i_k^R}(t)$, and $X_{b_k^S + i_k^S}(t)$ respectively denote the phase of the arrival process of class k customers, the number of class k retrial customers in phase i_k^R, and the number of busy servers serving class k customers in phase i_k^S for $i_k^R = 1, \ldots, m_k^R$ and $i_k^S = 1, \ldots, m_k^S$, where

$$m^R = \sum_{k=1}^{K} m_k^R, \quad m^S = \sum_{k=1}^{K} m_k^S, \quad b_k^R = K + \sum_{k'=1}^{k-1} m_{k'}^R, \quad b_k^S = K + m^R + \sum_{k'=1}^{k-1} m_{k'}^S.$$

Then the multi-dimensional MC $X(t) = \{X_1(t), \ldots, X_{K+m^R+m^S}(t) : t \geqslant 0\}$ has the state space $\mathcal{S} = \mathcal{S}^A \times \mathcal{S}^R \times \mathcal{S}^S$, where $\mathcal{S}^A = \times_{k=1}^{K}\{1, \ldots, m_k^A\}$, $\mathcal{S}^R = \mathbb{Z}_{\geqslant 0}^{m^R}$, $\mathcal{S}^S = \{\boldsymbol{y} = (y_1, \ldots, y_{m^S}) \in \mathbb{Z}_{\geqslant 0}^{m^S} \mid \boldsymbol{y}\boldsymbol{e} \leqslant c\}$, and a possible state representation of the model is $\boldsymbol{x} = (x_1, \ldots, x_{K+m^R+m^S}) \in \mathcal{S}$. We let $\boldsymbol{n}(\boldsymbol{x})$ denote the number of busy servers in state $\boldsymbol{x}$; that is, $\boldsymbol{n}(\boldsymbol{x}) = \sum_{k=1}^{K} \sum_{i=1}^{m_k^S} x_{b_k^S + i}$. Set $\mathcal{S}^S$ is defined so that the number of busy servers does not exceed the number of servers, and its size is given by

$$|\mathcal{S}^S| = \sum_{i=0}^{c} \frac{(i + m^S - 1)!}{i!\,(m^S - 1)!}.$$

Note that another possible approach for modeling this system is to keep the phase of each server in a single but different dimension of the state. However, that approach leads to a larger state space as discussed in [32, 43].

Now, we give the generator matrix underlying the model in terms of matrices associated with arrival, service, and retrial. Matrix $\boldsymbol{Q}_k^A$ includes transitions associated with arrival of class k customers and is given elementwise as

$$
\boldsymbol{Q}_k^A(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases}
\boldsymbol{C}_k(x_k, i) & \text{if } i \neq x_k \text{ and } \boldsymbol{y} = \boldsymbol{x} + (i - x_k)\boldsymbol{e}_k^T \\
\boldsymbol{D}_k(x_k, i)\boldsymbol{\xi}_k(j) & \text{if } \boldsymbol{n}(x) = c \text{ and } \boldsymbol{y} = \boldsymbol{x} + (i - x_k)\boldsymbol{e}_k^T + \boldsymbol{e}_{b_k^R + j}^T \\
\boldsymbol{D}_k(x_k, i)\boldsymbol{\beta}_k(j') & \text{if } \boldsymbol{n}(x) < c \text{ and } \boldsymbol{y} = \boldsymbol{x} + (i - x_k)\boldsymbol{e}_k^T + \boldsymbol{e}_{b_k^S + j'}^T \\
0 & \text{otherwise}
\end{cases}
$$

for $i = 1, \ldots, m_k^A$, $j = 1, \ldots, m_k^R$, $j' = 1, \ldots, m_k^S$, and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$, where $\boldsymbol{e}_k$ represents the kth principal axis vector. Matrix $\boldsymbol{Q}_k^R$ includes transitions associated with retrial of class k customers and is given elementwise as

$$
\boldsymbol{Q}_k^R(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases}
x_{b_k^R + i}\boldsymbol{U}_k(i, j) & \text{if } i \neq j \text{ and } \boldsymbol{y} = \boldsymbol{x} - \boldsymbol{e}_{b_k^R + i}^T + \boldsymbol{e}_{b_k^R + j}^T \\
x_{b_k^R + i}\boldsymbol{U}_k^0(i)\boldsymbol{\beta}_k(j') & \text{if } \boldsymbol{n}(x) < c \text{ and } \boldsymbol{y} = \boldsymbol{x} - \boldsymbol{e}_{b_k^R + i}^T + \boldsymbol{e}_{b_k^S + j'}^T \\
0 & \text{otherwise}
\end{cases}
$$

for $i, j = 1, \ldots, m_k^R$, $j' = 1, \ldots, m_k^S$, and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$. Finally, matrix $\boldsymbol{Q}_k^S$ includes transitions associated with service of class k customers and is given elementwise as

$$
\boldsymbol{Q}_k^S(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases}
x_{b_k^S + i}\boldsymbol{T}_k(i, j) & \text{if } i \neq j \text{ and } \boldsymbol{y} = \boldsymbol{x} - \boldsymbol{e}_{b_k^S + i}^T + \boldsymbol{e}_{b_k^S + j}^T \\
x_{b_k^S + i}\boldsymbol{T}_k^0(i) & \text{if } \boldsymbol{y} = \boldsymbol{x} - \boldsymbol{e}_{b_k^S + i}^T \\
0 & \text{otherwise}
\end{cases}
$$

for $i, j = 1, \ldots, m_k^S$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$. Then the generator matrix underlying $X(t)$ becomes

$$
\boldsymbol{Q} = \boldsymbol{Q}^{off} + \text{diag}(-\boldsymbol{Q}^{off}\boldsymbol{e}) \quad \text{with} \quad \boldsymbol{Q}^{off} = \sum_{k=1}^{K} \left(\boldsymbol{Q}_k^A + \boldsymbol{Q}_k^R + \boldsymbol{Q}_k^S \right).
$$

Example 1. Now consider an example with $K = 2$ and $c = 2$. Let the vectors and matrices describing the arrivals, services, and retrials be given by

$$
\boldsymbol{C}_1 = \begin{bmatrix} -0.8 & 0.8 \\ 0 & -0.8 \end{bmatrix}_{2 \times 2}, \ \boldsymbol{D}_1 = \begin{bmatrix} 0 & 0 \\ 0.8 & 0 \end{bmatrix}_{2 \times 2}, \ \boldsymbol{C}_2 = \begin{bmatrix} -0.3 \end{bmatrix}_{1 \times 1}, \ \boldsymbol{D}_2 = \begin{bmatrix} 0.3 \end{bmatrix}_{1 \times 1},
$$

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}_{1\times 2}, \ \boldsymbol{U}_1 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}_{2\times 2}, \ \boldsymbol{\xi}_2 = \begin{bmatrix} 1 \end{bmatrix}_{1\times 1}, \ \boldsymbol{U}_2 = \begin{bmatrix} -0.5 \end{bmatrix}_{1\times 1},$$

$$\boldsymbol{\beta}_1 = \begin{bmatrix} 0.75 & 0.25 \end{bmatrix}_{1\times 2}, \ \boldsymbol{T}_1 = \begin{bmatrix} -1 & 0.25 \\ 0 & -0.25 \end{bmatrix}_{2\times 2}, \ \boldsymbol{\beta}_2 = \begin{bmatrix} 1 \end{bmatrix}_{1\times 1}, \ \boldsymbol{T}_2 = \begin{bmatrix} -0.5 \end{bmatrix}_{1\times 1};$$

Hence,

$$\boldsymbol{U}_1^0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{2\times 1}, \ \boldsymbol{U}_2^0 = \begin{bmatrix} 0.5 \end{bmatrix}_{1\times 1}, \ \boldsymbol{T}_1^0 = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}_{2\times 1}, \ \boldsymbol{T}_2^0 = \begin{bmatrix} 0.5 \end{bmatrix}_{1\times 1}.$$

Here, $\lambda_1 = 0.4$, $\lambda_2 = 0.3$, $\delta_1 = \delta_2 = 0.5$, $\mu_1 = 0.4$, $\mu_2 = 0.5$, $m_1^A = 2$, $m_2^A = 1$, $m_1^R = 2$, $m_2^R = 1$, $m_1^S = 2$, and $m_2^S = 1$. This is an 8-dimensional model with $m^R = 3$, $m^S = 3$, $b_1^R = 2$, $b_2^R = 4$, $b_1^S = 5$, $b_2^S = 7$, and $\boldsymbol{n}(\boldsymbol{x}) = x_6 + x_7 + x_8$. Therefore, the state space of the MC is given by $\mathcal{S} = \mathcal{S}^A \times \mathcal{S}^R \times \mathcal{S}^S$, where $\mathcal{S}^A = \{1,2\} \times \{1\}$, $\mathcal{S}^R = \mathbb{Z}_{\geq 0}^3$, $\mathcal{S}^S = \{(0,0,0),(0,0,1),(0,0,2),(0,1,0),(0,1,1),(0,2,0),(1,0,0),(1,0,1),(1,1,0),(2,0,0)\}$.

3. Ergodicity condition

With the help of Lyapunov functions, we show that the inequality

$$\sum_{k=1}^{K} \frac{\lambda_k}{\mu_k} < c \tag{1}$$

is a necessary and sufficient ergodicity condition for the multi-class MAP/PH/c queue with acyclic PH retrials considered. The Lyapunov function used to show the sufficiency of this condition will also aid us in finding a truncated state space with a given steady-state probability mass. When the arrival, service or retrial processes include phases, Lyapunov functions need to be chosen so that the phases are carefully taken into consideration. Otherwise, a necessary and sufficient ergodicity condition may not be found. In order to obtain Lyapunov functions leading to such a condition, we start with Lyapunov functions that work for simple models, and then add terms to these functions for the additional complexities of the model.

The following two lemmas introduce two vectors which include variables that will be used in the additional terms.

Lemma 1. *There exists a unique vector $\boldsymbol{u}_k \in \mathbb{R}^{m_k^A \times 1}$ for MAP with representation $(\boldsymbol{C}_k, \boldsymbol{D}_k)$ and $\lambda_k = \boldsymbol{\theta}_k \boldsymbol{D}_k \boldsymbol{e}$ such that $(\boldsymbol{C}_k + \boldsymbol{D}_k)\boldsymbol{u}_k = \lambda_k \boldsymbol{e} - \boldsymbol{D}_k \boldsymbol{e}$ and $\boldsymbol{u}_k \boldsymbol{e} = 1$.*

Proof. The reduced linear system of equations $\hat{A}_k u_k = \hat{b}_k$ with $\hat{A}_k = M_k(C_k + D_k) = (I_{m_k^A} - e_{m_k^A}e_{m_k^A}^T)(C_k + D_k)$ and $\hat{b}_k = M_k(\lambda_k e - D_k e) = (I_{m_k^A} - e_{m_k^A}e_{m_k^A}^T)(\lambda_k e - D_k e)$, where $M_k = I_{m_k^A} + e_{m_k^A}(-e_{m_k^A}^T + \theta_k)$ and $I_{m_k^A}$ denotes the identity matrix of order m_k^A, has only its last equation altered to make the equation 0 on both sides. Because $(C_k + D_k)$ is an irreducible generator matrix, this implies $\hat{A}_k$ is of rank $(m_k^A - 1)$. Thus, $(C_k + D_k)u_k = \lambda_k e - D_k e$ is consistent (see [36, Ch. 2.3]), and there exists a unique vector u_k under the normalization condition $u_k e = 1$.

Since transition rates describing arrivals in MAP depend on the phase of the process, elements of u_k will be used in additional terms to obtain an ergodicity condition based on average arrival rate λ_k instead of phase-dependent arrival rates in D_k.

Lemma 2. *There exists a unique vector $v_k \in \mathbb{R}_{\geq 0}^{m_k^S \times 1}$ for PH service distribution with representation (β_k, T_k) and $\mu_k = [-\beta_k(T_k)^{-1}e]^{-1}$ such that $v_k = -\mu_k(T_k)^{-1}e$ and $\beta_k v_k = 1$.*

Proof. Matrix $-T_k$ is a nonsingular M-matrix, $-T_k^{-1} \geq 0$ [36, p. 626], and therefore, $v_k \geq 0$ exists. This implies $\beta_k v_k = 1$ since $\beta_k v_k = -\mu_k(\beta_k(T_k)^{-1}e) = -\mu_k(-\mu_k)^{-1}$.

Since transition rates describing PH service depend on the phase of the customer, elements of v_k will be used in additional terms to obtain an ergodicity condition based on average service rate μ_k instead of phase-dependent service rates in $T_k^0 = -T_k e$.

3.1. Necessary condition

The condition in (1) is necessary for ergodicity if it can be shown that the system is non-ergodic when the condition does not hold. Fortunately, the following theorem provides a non-ergodicity condition for MCs. Note that the theorem is originally proved for discrete-time MCs (see [6, p. 22] and [27, p. 30]), but here we give a continuous-time version which is obtained by considering the embedded MC.

Theorem 1. *A MC with generator matrix Q is non-ergodic if there exists two constants $\tau, \sigma \in \mathbb{R}$ and a Lyapunov function $f : \mathcal{S} \to \mathbb{R}$ such that*

(i) $\sum_{y \in \mathcal{S}} P(x, y)|f(y) - f(x)| \leq \tau$ *for $x \in \mathcal{S}$ and*

(ii) $\sum_{y \in \mathcal{S}} P(x, y)(f(y) - f(x)) \geq 0$ *for $x \in \mathcal{R}$,*

where the sets $\mathcal{R} = \{x \in \mathcal{S} \mid f(x) > \sigma\}$ and $(\mathcal{S} \backslash \mathcal{R})$ are non-empty, and the matrix

$P \in \mathbb{R}_{\geqslant 0}^{|\mathcal{S}| \times |\mathcal{S}|}$ *is given elementwise as*

$$
P(x, y) = \begin{cases} Q(x, y)/|Q(x, x)| & \text{if } y \neq x \\ 0 & \text{otherwise} \end{cases} \quad \text{for } x, y \in \mathcal{S}.
$$

This theorem is used to prove a non-ergodicity condition for an M/M/c retrial queue with exponential retrial in [26, p. 98]. Therein, a Lyapunov function $f(x)$ linear in the infinite variables (i.e., numbers of different class customers in the orbits) is used. We have also chosen a linear Lyapunov function, but have added constant terms including elements of vectors u_k and v_k from Lemmas 1 and 2. Thus, we consider

$$
f(x) = \sum_{k=1}^{K} \frac{1}{\mu_k} \left(\sum_{i=1}^{m_k^R} x_{b_k^R + i} \right) + \sum_{k=1}^{K} \frac{u_k(x_k)}{\mu_k} + \sum_{k=1}^{K} \frac{1}{\mu_k} \left(\sum_{i=1}^{m_k^S} v_k(i) x_{b_k^S + i} \right).
$$

The first term of $f(x)$ is the initial function, whereas the other two terms are added to obtain a phase-independent condition. Note that each of the three terms is in the form of a summation of K other terms each corresponding to a different customer class.

Since Q is a generator matrix, $P(x, y) \leqslant 1$ for $x, y \in \mathcal{S}$. The value of $|f(y) - f(x)|$ is finite if $P(x, y) > 0$ for $x, y \in \mathcal{S}$. Besides, each row of the matrix P includes a finite number of nonzero elements due to the form of the particular model considered. Therefore, there exists some constant τ such that condition (i) is satisfied.

After some algebraic operations, we obtain

$$
\frac{1}{|Q(x, x)|} \sum_{y \in \mathcal{S}} Q(x, y)(f(y) - f(x)) = \frac{1}{|Q(x, x)|} \left(\sum_{k=1}^{K} \frac{\lambda_k}{\mu_k} - n(x) \right).
$$

The right-hand side of this equation is nonnegative if $\sum_{k=1}^{K} (\lambda_k/\mu_k) \geqslant c$ since $|Q(x, x)| > 0$ and $n(x) \leqslant c$ hold for $x \in \mathcal{S}$. Besides, sets $\mathcal{R}$ and $(\mathcal{S} \backslash \mathcal{R})$ are nonempty when $\sigma = \max_{x \in \mathcal{S}} \left(\sum_{k=1}^{K} u_k(x_k)/\mu_k \right)$. Therefore, condition (ii) also holds.

3.2. Sufficient condition

In coming up with a Lyapunov function to show the sufficiency of the condition in (1) for ergodicity, we benefit from the next theorem in [48]. Note that the Lyapunov function is assumed to be nonnegative in [48], but the theorem is also valid if the Lyapunov function is bounded from below (see [26, p. 97]). This theorem will also be used to bound the steady-state probability mass associated with the truncated state space from below as proposed in [18].

Theorem 2. *The MC is ergodic if and only if there exists a Lyapunov function g : $\mathcal{S} \to \mathbb{R}$ that is bounded from below and a finite set $\mathcal{C} \subset \mathcal{S}$ satisfying the three conditions*

(i) $\{\boldsymbol{x} \in \mathcal{S} \mid g(\boldsymbol{x}) \leqslant \tau\}$ *is finite for all $\tau < \infty$,*

(ii) $d(\boldsymbol{x}) \leqslant -\gamma$ *for all $\boldsymbol{x} \in \mathcal{S} \backslash \mathcal{C}$ and some $\gamma > 0$, and*

(iii) $d(\boldsymbol{x}) < \infty$ *for all $\boldsymbol{x} \in \mathcal{S}$,*

where $d(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in \mathcal{S}} \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{y}) \left(g(\boldsymbol{y}) - g(\boldsymbol{x}) \right)$ is called the drift in state $\boldsymbol{x} \in \mathcal{S}$.

We will be choosing a quadratic Lyapunov function $g(\boldsymbol{x})$ that is similar to those discussed in [8] to obtain a drift function $d(\boldsymbol{x})$ whose infinite variables have negative coefficients for $\boldsymbol{x} \in \mathcal{S}$. If customers in the orbits attempt to receive service in all PH retrial phases, adding terms with variables each corresponding to the number of customers in a service phase leads to the condition in (1). However, if the PH retrial process of a class k customer includes a phase, say i, in which no attempt is made to receive service (i.e., if $\boldsymbol{U}_k^0(i) = 0$, where $\boldsymbol{U}_k^0 = -\boldsymbol{U}_k \boldsymbol{e}$), then the coefficient of the corresponding variable, $x_{b_k^R+i}$, is positive in $d(\boldsymbol{x})$ at states with no busy servers. Therefore, the value of $g(\boldsymbol{x})$ should also depend on the number of customers in retrial phases. This can be managed by adding carefully chosen terms so that $d(\boldsymbol{x})$ at states with no busy servers become negative for a sufficiently large number of customers in the orbits. Based on this explanation, the following lemma gives the vector that includes variables to be used in the term added due to PH retrials of class k customers.

Lemma 3. *For acyclic PH retrial distribution with representation $(\boldsymbol{\xi}_k, \boldsymbol{U}_k)$, let $\hat{\boldsymbol{U}}_k = \boldsymbol{U}_k + diag(\boldsymbol{U}_k^0)$ and $\boldsymbol{\eta}_k \in \mathbb{R}^{m_k^R \times 1}$ be given elementwise as*

$$\boldsymbol{\eta}_k(i) = \begin{cases} -c/\mu_k & if \ i \in \mathcal{I}_k \\ 0 & otherwise \end{cases} \quad for \ \ i = 1, \ldots, m_k^R,$$

where $\mathcal{I}_k = \{i \in \{1, \ldots, m_k^R\} \mid \boldsymbol{U}_k^0(i) = 0\}$. Then there exists $\boldsymbol{w}_k \in \mathbb{R}^{m_k^R \times 1}$ such that $\hat{\boldsymbol{U}}_k \boldsymbol{w}_k = \boldsymbol{\eta}_k$.

Proof. Note that all elements in row i of $\hat{\boldsymbol{U}}_k$ are zero if and only if $\hat{\boldsymbol{U}}_k(i, i) = 0$. Then the result follows from the row echelon form of upper-triangular matrix $\hat{\boldsymbol{U}}_k$ obtained by interchanging all zero rows with nonzero rows below them (see [36, Ch. 2.1]), that its rank is equal to its number of nonzero rows, and $\hat{\boldsymbol{U}}_k \boldsymbol{w}_k = \boldsymbol{\eta}_k$ is consistent and has infinitely many solutions (see [36, Ch. 2.3]). Note that negative elements of $\boldsymbol{\eta}_k$ could

take a smaller value than $-c/\mu_k$ that may lead to a smaller truncated state space. We choose this value in order to bound coefficients of infinite variables from above by $(\sum_{k=1}^{K}(\lambda_k/\mu_k) - c)$ in $d(\boldsymbol{x})$ for all states.

There exist infinitely many solutions to $\hat{\boldsymbol{U}}_k \boldsymbol{w}_k = \boldsymbol{\eta}_k$. Hence, additional constraints need to be imposed. Here, we choose to set $\boldsymbol{w}_k(i)$ to 1 if row i of $\hat{\boldsymbol{U}}_k$ is zero. The elements of vector $\boldsymbol{\eta}_k$ contribute to coefficients of infinite variables corresponding to class-k retrial customers in $d(\boldsymbol{x})$; hence, its elements need to be nonpositive with at least one negative element. If PH retrials are allowed to be non-acyclic, it is a possibility that $\hat{\boldsymbol{U}}_k \boldsymbol{w}_k = \boldsymbol{\eta}_k$ is inconsistent since $\hat{\boldsymbol{U}}_k$ is a singular M-matrix. Hence, the acyclicity assumption is necessary due to the form of the chosen $g(\boldsymbol{x})$. We conjecture that the condition in (1) is also necessary and sufficient when PH retrials are non-acyclic; however, we have not managed to provide a suitable $g(\boldsymbol{x})$ for such a system.

Having defined all necessary variables, as the Lyapunov function we consider

$$g(\boldsymbol{x}) = \sum_{k'=1}^{K} \sum_{l'=1}^{K} \frac{1}{2\mu_{k'}\mu_{l'}} \left(\sum_{i=1}^{m_{k'}^R} x_{b_{k'}^R+i}\right) \left(\sum_{i=1}^{m_{l'}^R} x_{b_{l'}^R+i}\right) + \sum_{k'=1}^{K} \left(\sum_{l'=1}^{K} \frac{\boldsymbol{u}_{l'}(x_{l'})}{\mu_{k'}\mu_{l'}}\right) \left(\sum_{i=1}^{m_{k'}^R} x_{b_{k'}^R+i}\right)$$

$$+ \sum_{k'=1}^{K} \left(\sum_{l'=1}^{K} \frac{1}{\mu_{k'}\mu_{l'}} \left(\sum_{j=1}^{m_{l'}^S} \boldsymbol{v}_{l'}(j) x_{b_{l'}^S+j}\right)\right) \left(\sum_{i=1}^{m_{k'}^R} x_{b_{k'}^R+i}\right)$$

$$+ \sum_{k'=1}^{K} \sum_{i=1}^{m_{k'}^R} \boldsymbol{w}_{k'}(i) x_{b_{k'}^R+i} + \sum_{k'=1}^{K} \alpha_{k'}(\boldsymbol{x}) \left(\sum_{i=1}^{m_{k'}^S} x_{b_{k'}^S+i}\right),$$

where

$$a_k(i, \boldsymbol{x}) = \boldsymbol{w}_k(i) + \frac{1}{2\mu_k^2} + \sum_{k'=1}^{K} \frac{\boldsymbol{u}_{k'}(x_{k'})}{\mu_k \mu_{k'}} + \sum_{k'=1}^{K} \frac{1}{\mu_k \mu_{k'}} \left(\sum_{j'=1}^{m_{k'}^S} \boldsymbol{v}_{k'}(j') x_{b_{k'}^S+j'}\right) \quad \text{for} \quad i \notin \mathcal{I}_k$$

and $\alpha_k(\boldsymbol{x}) = \min_{i \notin \mathcal{I}_k} \left(a_k(i, \boldsymbol{x}) - c/(\boldsymbol{U}_k^0(i)\mu_k)\right)$ for $\boldsymbol{x} \in \mathcal{S}$ and $k = 1, \ldots, K$. The first term of $g(\boldsymbol{x})$ is the initial quadratic Lyapunov function. The second and third terms are added to obtain a phase-independent condition, and the last two terms are added due to PH retrials. Note that, $\alpha_k(\boldsymbol{x})$ is well-defined since $\mathcal{I}_k \neq \{1, \ldots, m_k^R\}$ for $k = 1, \ldots, K$ due to Definitions 2 and 3. Then $g(\boldsymbol{x})$ is a quadratic polynomial in which the coefficients of all infinite variables are finite and the coefficient of each quadratic term $(x_{b_k^R+i} x_{b_{k'}^R+i'})$ is positive for $i = 1, \ldots, m_k^R$, $i' = 1, \ldots, m_{k'}^R$, and $k, k' = 1, \ldots, K$. Hence, the function g is bounded from below and condition (i) holds.

For $\boldsymbol{n}(\boldsymbol{x}) = c$, the drift is given by

$$
d(\boldsymbol{x}) = \sum_{k=1}^{K} \frac{1}{\mu_k} \left(\sum_{k'=1}^{K} \frac{\lambda_{k'}}{\mu_{k'}} - c \right) \left(\sum_{i=1}^{m_k^R} x_{b_k^R + i} \right) - \sum_{k=1}^{K} \frac{c}{\mu_k} \left(\sum_{i \in \mathcal{I}_k} x_{b_k^R + i} \right)
$$

$$
+ \sum_{k=1}^{K} \sum_{i=1}^{m_k^A} \boldsymbol{D}_k(x_k, i) \left[\frac{\boldsymbol{u}_k(i)}{\mu_k^2} + \sum_{\substack{k'=1 \\ k' \neq k}}^{K} \frac{\boldsymbol{u}_{k'}(x_{k'})}{\mu_k \mu_{k'}} + \sum_{k'=1}^{K} \frac{1}{\mu_k \mu_{k'}} \left(\sum_{j'=1}^{m_{k'}^S} \boldsymbol{v}_{k'}(j') x_{b_{k'}^S + j'} \right) \right]
$$

$$
+ \sum_{k=1}^{K} \sum_{i=1}^{m_k^A} \boldsymbol{D}_k(x_k, i) \left[\left(\sum_{j=1}^{m_k^R} \boldsymbol{\xi}_k(j) \boldsymbol{w}_k(j) \right) + \frac{1}{2\mu_k^2} \right] - \sum_{k=1}^{K} \alpha_k(\boldsymbol{x}) \left(\sum_{i=1}^{m_k^S} x_{b_k^S + i} \boldsymbol{T}_k^0(i) \right);
$$

on the other hand, for $\boldsymbol{n}(\boldsymbol{x}) < c$ the drift is given by

$$
d(\boldsymbol{x}) = \sum_{k=1}^{K} \frac{1}{\mu_k} \left(\sum_{k'=1}^{K} \frac{\lambda_{k'}}{\mu_{k'}} - \sum_{k'=1}^{K} \sum_{i=1}^{m_{k'}^S} x_{b_{k'}^S + i} \right) \left(\sum_{i=1}^{m_k^R} x_{b_k^R + i} \right) - \sum_{k=1}^{K} \frac{c}{\mu_k} \left(\sum_{i \in \mathcal{I}_k} x_{b_k^R + i} \right)
$$

$$
+ \sum_{k=1}^{K} \sum_{i \notin \mathcal{I}_k} x_{b_k^R + i} \boldsymbol{U}_k^0(i) \left(-a_k(i, \boldsymbol{x}) + \alpha_k(\boldsymbol{x}) \right)
$$

$$
+ \sum_{k=1}^{K} \alpha_k(\boldsymbol{x}) \left(\sum_{i=1}^{m_k^A} \boldsymbol{D}_k(x_k, i) - \sum_{i=1}^{m_k^S} x_{b_k^S + i} \boldsymbol{T}_k^0(i) \right) \quad \text{for} \quad \boldsymbol{x} \in \mathcal{S}.
$$

When $\sum_{k=1}^{K} (\lambda_k / \mu_k) < c$ holds, coefficients of all infinite variables are negative; hence, condition (iii) holds. Besides, $\mathcal{C} = \{ \boldsymbol{x} \in \mathcal{S} \mid d(\boldsymbol{x}) > -\gamma \}$ is finite for any arbitrarily chosen $\gamma > 0$. Therefore, condition (ii) also holds and the model is ergodic if $\sum_{k=1}^{K} (\lambda_k / \mu_k) < c$. As discussed in [18], for some given positive $\epsilon < 1$ we have $\sum_{\boldsymbol{x} \in \mathcal{C}} \boldsymbol{\pi}(\boldsymbol{x}) \geqslant 1 - \epsilon$, where $\gamma = \sup_{\boldsymbol{x} \in \mathcal{S}} d(\boldsymbol{x}) (1/\epsilon - 1)$ and $\boldsymbol{\pi}$ is the steady-state solution.

Example 1. (cont'd) In this example,

$$
\boldsymbol{u}_1 = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}_{2 \times 1}, \quad \boldsymbol{v}_1 = \begin{bmatrix} 0.8 \\ 1.6 \end{bmatrix}_{2 \times 1}, \quad \boldsymbol{w}_1 = \begin{bmatrix} 6 \\ 1 \end{bmatrix}_{2 \times 1}, \quad \boldsymbol{u}_2 = \boldsymbol{v}_2 = \boldsymbol{w}_2 = \begin{bmatrix} 1 \end{bmatrix}_{1 \times 1}.
$$

Then $\mathcal{I}_1 = \{1\}$, $\mathcal{I}_2 = \varnothing$,

$$
a_1(2, \boldsymbol{x}) = 4.125 + 6.25 \boldsymbol{u}_1(x_1) + 5 \boldsymbol{u}_2(x_2) + 5x_6 + 10x_7 + 5x_8, \quad \alpha_1(\boldsymbol{x}) = a_1(2, \boldsymbol{x}) - 5,
$$

$$
a_2(1, \boldsymbol{x}) = 3 + 5 \boldsymbol{u}_1(x_1) + 4 \boldsymbol{u}_2(x_2) + 4x_6 + 8x_7 + 4x_8, \quad \alpha_2(\boldsymbol{x}) = a_2(1, \boldsymbol{x}) - 8.
$$

The Lyapunov function is given by

$$g(\boldsymbol{x}) = 3.125(x_3 + x_4)^2 + 5(x_3 + x_4)x_5 + 2x_5^2 + 6x_3 + x_4 + x_5$$
$$+ (6.25\boldsymbol{u}_1(x_1) + 5\boldsymbol{u}_2(x_2) + 5x_6 + 10x_7 + 5x_8)(x_3 + x_4)$$
$$+ (5\boldsymbol{u}_1(x_1) + 4\boldsymbol{u}_2(x_2) + 4x_6 + 8x_7 + 4x_8)x_5 + \alpha_1(\boldsymbol{x})(x_6 + x_7) + \alpha_2(\boldsymbol{x})x_8.$$

For $\boldsymbol{n}(\boldsymbol{x}) = 2$, the drift is given by

$$d(\boldsymbol{x}) = -6x_3 - x_4 - 0.8x_5 + \mathbb{1}_{x_1=2}(4\boldsymbol{u}_2(x_2) + 4x_6 + 8x_7 + 4x_8 + 8.55)$$
$$+ \mathbb{1}_{x_2=1}(1.5\boldsymbol{u}_1(x_1) + 1.2x_6 + 2.4x_7 + 1.2x_8 + 2.1)$$
$$+ \alpha_1(\boldsymbol{x})(-0.75x_6 - 0.25x_7) + \alpha_2(\boldsymbol{x})(-0.5x_8),$$

where $\mathbb{1}$ denotes the indicator function, and for $\boldsymbol{n}(\boldsymbol{x}) < 2$, the drift is given by

$$d(\boldsymbol{x}) = 2.5(-0.4 - x_6 - x_7 - x_8)(x_3 + x_4) + 2(-0.4 - x_6 - x_7 - x_8)x_5$$
$$+ \alpha_1(\boldsymbol{x})(\mathbb{1}_{x_1=2}\ 0.8 - 0.75x_6 - 0.25x_7) + \alpha_2(\boldsymbol{x})(\mathbb{1}_{x_2=1}\ 0.3 - 0.5x_8).$$

4. Numerical results

Once a Kronecker representation of the truncated generator matrix $\bar{\boldsymbol{Q}}$ is obtained as in [8, 19], one can employ a memory-efficient Kronecker-based iterative solver in which the nonzeros of $\bar{\boldsymbol{Q}}$ are not stored and no factorization takes place during the course of computing $\bar{\bar{\boldsymbol{\pi}}}$. In practice, this approach always saves a significant amount of memory. Another approach is to employ an LDQBD model, where level l has the state space

$$\mathcal{S}(l) = \{\boldsymbol{x} = (x_1, \ldots, x_{K+m^R+m^S}) \in \mathcal{S} \mid l = \max\{x_{K+1}, \ldots, x_{K+m^R}\}\},$$

and then use the matrix-analytic method proposed in [11]. In this method, conditional expected sojourn time matrices between two given truncation levels need to be computed and stored. For our system, the conditional expected sojourn time matrix at level l includes about $(\bar{M}^2 l^{2(m^R-1)})$ nonzeros, where $\bar{M} = |\mathcal{S}^S|(\prod_{k=1}^{K} m_k^A)m^R$. Hence, the memory allocated to store these matrices becomes extremely large. Phung-Duc et al. [42] propose an algorithm to compute the conditional expected sojourn time matrices of an LDQBD with a smaller memory usage (see [42, Algorithm. 1]). However, the stationary distribution is computed using the algorithm proposed in [11] (see [42, Algorithm. 3]) which requires computing and storing the conditional expected sojourn

time matrices (see [42, Remark. 3.5]). Hence, memory usage is expected to be large in this algorithm. On the other hand, the Kronecker-based iterative solver is expected to solve the truncated system with comparable accuracy if the value of the stopping tolerance it uses is sufficiently small.

We implemented a Kronecker solver [20] built on the Nsolve package [12] of the APNN toolbox [9]. The solver obtains the truncated state space of the model, generates the Kronecker structured matrix of the truncated model, and computes its steady-state solution by using the SOR method of Nsolve. All experiments are carried out on an Intel Core2 Duo 2.4 GHz processor with 4 GB of main memory. Iterations are stopped when infinity norm of the residual vector of the truncated model (i.e., $||\bar{\boldsymbol{\pi}}\bar{\boldsymbol{Q}}||_\infty$) becomes smaller than 10^{-15}, and the relaxation parameter of SOR is chosen as 0.9.

We considered six different models with $\epsilon = 0.2$. Table 1 reports the results of numerical experiments with these models. The first column gives the name of the model. ERL_1 is the model introduced in Example 1. ERL_2 and ERL_3 are obtained by multiplying the matrices describing the arrival processes of ERL_1 by 0.75 and 0.5, respectively. EXP_1 differs from the first model in that the retrial time of customer class 1 is exponentially distributed with average retrial rate 0.5. EXP_2 and EXP_3 are obtained by multiplying the matrices describing the arrival processes of EXP_1 by 0.75 and 0.5, respectively. The second column gives the traffic intensity $\rho = (\sum_{k=1}^{K} \lambda_k/\mu_k)c^{-1}$. The third column gives the number of states in the truncated state space. Columns $E_{1,1}$, $E_{1,2}$, and $E_{2,1}$ give the average number of class 1 customers in retrial phases 1 and 2, and the average number of class 2 customers in retrial phase 1, respectively. Retrial process of customer class 1 has a single phase in EXP_1, EXP_2, and EXP_3; hence, $E_{1,2}$ is undefined for those models. Column P_{block} provides the probability that an arriving customer finds all servers busy. Column $||\bar{\boldsymbol{\pi}}\bar{\boldsymbol{Q}}||_1$ provides the 1-norm of the residual vector of the truncated model which is an indicator of the accuracy of the solution to the truncated model. Column $||\tilde{\boldsymbol{\pi}}\boldsymbol{Q}||_1$ provides the 1-norm of the residual vector of the infinite model and is obtained from

$$||\tilde{\boldsymbol{\pi}}\boldsymbol{Q}||_1 = \sum_{\boldsymbol{x}\in\bar{\mathcal{S}}} |r(\boldsymbol{x}) - \sum_{\boldsymbol{y}\notin\bar{\mathcal{S}}} \bar{\boldsymbol{\pi}}(\boldsymbol{x})\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y})| + \sum_{\boldsymbol{x}\in\bar{\mathcal{S}}}\sum_{\boldsymbol{y}\notin\bar{\mathcal{S}}} \bar{\boldsymbol{\pi}}(\boldsymbol{x})\boldsymbol{Q}(\boldsymbol{x},\boldsymbol{y}) \quad \text{with} \quad \boldsymbol{r} = \bar{\boldsymbol{\pi}}\bar{\boldsymbol{Q}}.$$

This value is an indicator of the accuracy of the solution to the infinite model.

Relative difference between average numbers of retrial customers in ERL_i and EXP_i

TABLE 1: Numerical results

Model	ρ	$\|\bar{\mathcal{S}}\|$	$E_{1,1}$	$E_{1,2}$	$E_{2,1}$	P_{block}	$\|\|\bar{\boldsymbol{\pi}}\bar{\boldsymbol{Q}}\|\|_1$	$\|\|\tilde{\boldsymbol{\pi}}\boldsymbol{Q}\|\|_1$
ERL_1	0.8	$29,932,260$	0.2584	2.0165	2.8985	0.6784	$9e-14$	$9e-14$
ERL_2	0.6	$811,800$	0.1097	0.4422	0.6756	0.4164	$2e-14$	$2e-12$
ERL_3	0.4	$12,800$	0.0317	0.0860	0.1511	0.2048	$6e-15$	$7e-6$
EXP_1	0.8	$270,400$	3.1663		2.5486	0.6755	$8e-14$	$8e-14$
EXP_2	0.6	$25,600$	0.7154		0.6302	0.4142	$2e-14$	$2e-11$
EXP_3	0.4	$1,600$	0.1464		0.1461	0.2038	$4e-15$	$6e-6$

becomes relatively large as traffic intensity increases. This difference ranges from 0.03 (customer class 2 in ERL_3 and EXP_3) to 0.39 (customer class 1 in ERL_1 and EXP_1). Relative difference between blocking probabilities is around 0.005. In ERL_1 and EXP_1, residuals of truncated and infinite models are about the same; hence, truncation error is not larger than numerical error. In other models, truncation error is larger than numerical error. Therefore, as traffic intensity increases, choosing a smaller ϵ value does not introduce additional inaccuracy to the computed solution.

Acknowledgements

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