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Stability regions for synchronized τ-periodic orbits of coupled maps with coupling delay τ

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Motivated by the chaos suppression methods based on stabilizing an unstable periodic orbit, we study the stability of synchronized periodic orbits of coupled map systems when the period of the orbit is the same as the delay in the information transmission between coupled units. We show that the stability region of a synchronized periodic orbit is determined by the Floquet multiplier of the periodic orbit for the uncoupled map, the coupling constant, the smallest and the largest Laplacian eigenvalue of the adjacency matrix. We prove that the stabilization of an unstable τ-periodic orbit via coupling with delay τ is possible only when the Floquet multiplier of the orbit is negative and the connection structure is not bipartite. For a given coupling structure, it is possible to find the values of the coupling strength that stabilizes unstable periodic orbits. The most suitable connection topology for stabilization is found to be the all-to-all coupling. On the other hand, a negative coupling constant may lead to destabilization of τ-periodic orbits that are stable for the uncoupled map. We provide examples of coupled logistic maps demonstrating the stabilization and destabilization of synchronized τ-periodic orbits as well as chaos suppression via stabilization of a synchronized τ-periodic orbit. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4961707]

An efficient approach in chaos suppression is to stabilize an unstable periodic orbit of the system via feedback.[1] It is well known that setting a delay in the feedback may give rise to stabilization of an unstable periodic orbit with period equal to the delay.[2–4] On the other hand, chaos suppression in a network of coupled systems is an active research area.[5–9] Combining these ideas, one can aim to find a method for chaos suppression in coupled systems based on adjusting the coupling delay. Motivated by this aim, we consider diffusively coupled discrete-time dynamical systems and perform a linear stability analysis for a synchronized periodic orbit whose period is equal to the delay in the information transmission between coupled units. The linearized dynamics can be decomposed into independent modes determined by the Laplacian eigenvectors of the connection structure. This implies that the connection structure has an affect on the stability of such regular behaviors only through its Laplacian eigenvalues. For a particular case, namely, Kaneko-type[10,11] coupled maps with delay τ, stability of the synchronized τ-periodic orbits have been analyzed. A detailed investigation of the parameter region shows that the stability region of the synchronized periodic orbit is determined by the Floquet multiplier of the orbit for the uncoupled system, the coupling strength of connections, and the smallest and the largest Laplacian eigenvalues. The parameter regions are obtained where a synchronized τ-periodic orbit is stable. We construct an example where identical chaotic maps synchronize on a τ-periodic orbit after being coupled with a coupling delay τ.

I. INTRODUCTION

Stabilization of unstable periodic orbits of maps appears in different areas such as delayed feedback chaos control[2–4] and chaos suppression in coupled systems.[5–9] Combining ideas from these areas, we aim to investigate stability properties of a highly regular behavior, namely, a synchronized periodic orbit, of a coupled map system where the coupling delay is equal to the period of the orbit. We consider diffusively coupled discrete-time dynamical systems with coupling delay. We show (in Remark 1) that, for such systems, coupling delay is necessary for stabilization of a synchronized periodic orbit that is unstable for the uncoupled map. Similarly to the delayed feedback chaos control methods,[2–4] we consider periodic orbits whose period is equal to the delay; this time in the communication between different units.

It is well-known that scalar discrete-time dynamical systems

$$x(t+1) = f(x(t)), \quad x \in \mathbb{R},\ t \in \mathbb{N}, \tag{1}$$

given by the iterations of the map $f : \mathbb{R} \to \mathbb{R}$, can have a rich range of solutions, including periodic orbits $\gamma(t) = p_{t(\mathrm{mod}\,\tau)}$. We consider networks of n such systems that are evolving

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under pairwise diffusive interactions subject to an information transmission delay of $\tau \in \mathbb{N}$

$$x_i(t+1) = f(x_i(t)) + \frac{1}{d_i}\sum_{j=1}^{n} a_{ij} g\big(x_i(t), x_j(t-\tau)\big),$$
$$x_i \in \mathbb{R}, \quad i = 1, \ldots, n. \tag{2}$$

We assume that both f and g are continuously differentiable, and the interaction function $g : \mathbb{R}^2 \to \mathbb{R}$ satisfies the generalized diffusion condition

$$g(x, x) = 0, \quad \forall x \in \mathbb{R}. \tag{3}$$

The quantity $a_{ij} \geq 0$ denotes the weight of the coupling between units i and j, and $d_i = \sum_{j=1}^{n} a_{ij}$ denotes the sum of the weights of the connections to unit i. We assume that the coupling is symmetric ($a_{ij} = a_{ji}$ $\forall i, j$), the network is connected (otherwise one can consider connected components separately), and there are no isolated nodes, so that $d_i > 0$ $\forall i$. A special case of (2) that goes by the name *coupled map lattice*[12] has been studied by many authors[13–17] and is described by the equations

$$x_i(t+1) = f(x_i(t)) + \frac{\varepsilon}{d_i}\sum_{j=1}^{n} a_{ij}(f(x_j(t-\tau))$$
$$- f(x_i(t))), \quad i = 1, \ldots, n, \tag{4}$$

where ε is the coupling constant, and the connection weights are binary, i.e., $a_{ij} \in \{0, 1\}$.

A synchronized solution of the coupled system (2) is a function $\Gamma : \mathbb{N} \to \mathbb{R}^n$ of the form $\Gamma(t) = (\gamma(t), \gamma(t), \ldots, \gamma(t))^\top$, where $\gamma : \mathbb{N} \to \mathbb{R}$. We also use the notation $[\gamma] := (\gamma, \ldots, \gamma)^\top \in \mathbb{R}^n$ to denote synchronized states. By (2), all synchronized solutions $\Gamma(t) = [\gamma(t)]$ are such that γ satisfies

$$\gamma(t+1) = f(\gamma(t)) + g(\gamma(t), \gamma(t-\tau)). \tag{5}$$

When the delay τ is zero, the diffusion condition (3) yields that $\Gamma(t) = [\gamma(t)]$ is a synchronized solution of (2) if and only if $\gamma(t)$ satisfies (1). However, the stability of $\Gamma(t)$ in (2) may in general be different from the stability of $\gamma(t)$ in (1) and depends not only on the Lyapunov exponent of f but also on the network topology via the eigenvalues of the *Laplacian matrix*.[14] On the other hand, when the delay τ is nonzero, $\gamma(t)$ is in general no longer a solution of (1), except in two specific cases: The first case is when $\gamma(t)$ is constant in time; then γ is necessarily a fixed point of f – this case has been extensively studied,[18] where the stability region is found explicitly and the effect of the coupling constant and the delay is studied analytically. The second case is when $\gamma(t)$ is τ-periodic in time so that $\Gamma(t) = [\gamma(t)]$ is a τ-periodic solution of (2). This latter case forms the subject matter of the present paper.

We apply a standard linear stability analysis to the system (2) (in particular to (4)) with a well-known technique of decomposing a coupled system into independent modes that correspond different eigenvectors of the Laplacian matrix.[14,19,33] As a result, stability of a τ-periodic orbit of (4) is shown to be equivalent to the Schur stability of certain polynomials whose coefficients are functions of the Laplacian eigenvalue λ, coupling strength ε, and *scaled Floquet multiplier* β, i.e., Floquet multiplier scaled by period τ (see Eq. (18)), of the periodic orbit of the uncoupled map. We investigate these polynomials by means of mathematical analysis and algorithmic computation of their stability region via the Bistritz Tabulation method.[20] Hence, the following results are obtained on the stability of $\Gamma(t)$ as a solution of (4) and on its stability region in the parameter space $(\varepsilon, |\beta|)$.

- The coupling structure of (4) affects the stability of $\Gamma(t)$ only through its largest Laplacian eigenvalue.
- The stability region of $\Gamma(t)$ shrinks when the largest Laplacian eigenvalue is increased.
- Unstable periodic orbits with a positive Floquet multiplier cannot be stabilized, see Theorem 3.
- Unstable periodic orbits cannot be stabilized through a bipartite coupling, see Theorem 2.
- As $\tau \to \infty$, the stability region shrinks down to a minimal region, which is the region for a bipartite coupling.

We note that similar results were obtained in the paper[18] for fixed points. A similar negative result mentioned above for bipartite graphs has already been observed in a numerical study[8] in a continuous-time case. The case of $\tau \to \infty$ is studied both for delayed feedback systems in Ref. 21 and in coupled systems.[22,23] In accordance with these references, we prove that stabilization is not possible when $\tau \to \infty$. Moreover, for any connection structure as $\tau \to \infty$, stability regions coincide with the stability region of a bipartite graph, which gives a stability region that is independent from τ and is the smallest possible stability region contained in all other stability regions.

In Section II, we present a stability analysis of the synchronized periodic orbit $\Gamma(t)$ for the coupled network (2) and obtain a sufficient condition for the asymptotic stability of $\Gamma(t)$ in terms of the Laplacian eigenvalues and the derivatives of f and g at the periodic points. In Section III, we apply this condition to the coupled map lattice model (4) and obtain a sufficient condition for the asymptotic stability of $\Gamma(t)$ in terms of the coupling constant, the Laplacian eigenvalues, and the Floquet multiplier of the periodic orbit $\gamma(t)$ of (1). In Section IV, we discuss the stabilization and destabilization of $\Gamma(t)$ and chaos suppression by coupling with delay τ.

II. STABILITY ANALYSIS OF SYNCHRONIZED τ-PERIODIC ORBITS

Consider the linearization of (2) around a synchronized τ-periodic solution $\Gamma(t) = [p_{t(\mathrm{mod}\,\tau)}]$

$$\begin{aligned}\xi_i(t+1) &= f'(p_t)\xi_i(t) + \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}(\partial_1 g(p_t, p_t)\xi_i(t) \\ &\quad + \partial_2 g(p_t, p_t)\xi_j(t-\tau)) \\ &= f'(p_t)\xi_i(t) + \partial_1 g(p_t, p_t)\xi_i(t) \\ &\quad + \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}\partial_2 g(p_t, p_t)\xi_j(t-\tau),\end{aligned} \tag{6}$$

where $\xi_i(t) := x_i(t) - p_t$ and p_t should be understood as $p_{t(\mathrm{mod}\,\tau)}$. Here, ∂_1 and ∂_2 denote partial derivatives with respect to first and second arguments. We use the fact that $p_t = p_{t-\tau}$ and $d_i = \sum_{j=1}^{n} a_{ij}$. Let us define the following parameters:

$$b_k = f'(p_k) \quad \text{and} \quad c_k = \partial_2 g(p_k, p_k) = -\partial_1 g(p_k, p_k), \tag{7}$$

where the last equality follows from (3). The linear system (6) can be written in the matrix form as

$$\boldsymbol{\xi}(t+1) = (b_t - c_t) I \boldsymbol{\xi}(t) + c_t D^{-1} A \boldsymbol{\xi}(t-\tau), \tag{8}$$

where $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)^{\mathrm{T}}$, I is the identity matrix, $A = [a_{ij}]$, and $D = \mathrm{diag}\{d_1, \ldots, d_n\}$.

The (normalized) graph Laplacian is defined as

$$\mathcal{L} = I - D^{-1} A. \tag{9}$$

It is known that if the connection matrix A is symmetric, then the eigenvalues of $\mathcal{L}$ are real and the real eigenvectors of $\mathcal{L}$ form a linearly independent set.[24] For $l = 1, \ldots, n$, let λ_l and $\mathbf{v}_l$ be the eigenvalues and the eigenvectors of $\mathcal{L}$, respectively. Then,

$$D^{-1} A \mathbf{v}_l = (I - \mathcal{L}) \mathbf{v}_l = (1 - \lambda_l) \mathbf{v}_l, \quad l = 1, \ldots, n. \tag{10}$$

The minimum Laplacian eigenvalue is always zero, which corresponds to the Laplacian eigenvector $(1, \ldots, 1)$. This corresponds to the so-called longitudinal direction, namely, a direction that is parallel to the synchronization manifold. All the other eigenvalues correspond to the transversal directions. In the sequel, we will study the stability of a synchronized periodic orbit both in longitudinal and transversal directions, and therefore, stability will be checked for all Laplacian eigenvalues. Hence, we can decompose the dynamics of (6) or (8) into Laplacian eigenvectors to obtain the following τ-periodic scalar linear delay difference equation for each mode $l = 1, \ldots, n$ as:

$$\psi_l(t+1) = (b_t - c_t)\psi_l(t) + c_t(1 - \lambda_l)\psi_l(t - \tau). \tag{11}$$

This leads to the following system of first order τ-periodic difference equations:

$$\begin{bmatrix} \psi_l^{(0)}(t+1) \\ \psi_l^{(1)}(t+1) \\ \psi_l^{(2)}(t+1) \\ \vdots \\ \psi_l^{(\tau)}(t+1) \end{bmatrix} = \begin{bmatrix} (b_t - c_t) & 0 & \cdots & 0 & c_t(1-\lambda_l) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \psi_l^{(0)}(t) \\ \psi_l^{(1)}(t) \\ \psi_l^{(2)}(t) \\ \vdots \\ \psi_l^{(\tau)}(t) \end{bmatrix}, \tag{12}$$

where $\psi_l^{(k)}(t) := \psi_l(t-k)$. It is straightforward to check that the Floquet multipliers of the τ-periodic system (12) are the roots of the following polynomial:

$$p_l(s) = s^{\tau+1} - \prod_{k=0}^{\tau-1} ((b_k - c_k)s + c_k(1 - \lambda_l)). \tag{13}$$

Finally, we have the following theorem for the asymptotic stability of a synchronized τ-periodic orbit.

Theorem 1. *A synchronized τ-periodic orbit $\Gamma(t) = [p_{t(\mathrm{mod}\,\tau)}]$ of (2) is locally asymptotically stable if the roots of $p_l(s)$ given by (13) are in the open unit disc for all $l = 1, \ldots, n$, and unstable if one of the roots of $p_l(s)$ lies outside the closed unit disc.*

Remark 1. Let us show that delay is necessary for the system (2) if one wants to stabilize an unstable periodic orbit via coupling. For $\tau = 0$, (12) becomes scalar and the Floquet multipliers of a p-periodic orbit can easily be found as $\prod_{k=0}^{p-1} (b_k - c_k)\lambda_l$, for $l = 1, \ldots, n$. Note that, for the zero Laplacian eigenvalue $\lambda_l = 0$, the Floquet multiplier is $\prod_{k=0}^{p-1} b_k$, which is the Floquet multiplier of the periodic orbit for the uncoupled map. This implies that periodic orbits that are unstable for the uncoupled map cannot be stabilized via coupling of form (2) if delay is zero.

III. STABILITY OF THE SYNCHRONIZED τ-PERIODIC ORBITS OF COUPLED MAP LATTICES

In this section, we consider the coupled map lattice model (4). Applying Theorem 1, we find the set of parameters of (4) for which the asymptotic stability of a synchronized τ-periodic orbit is assured.

In this case, the connection matrix is binary and the eigenvalues $\lambda_{\min} = \lambda_1 \leq \cdots \leq \lambda_n = \lambda_{\max}$ of the graph Laplacian $\mathcal{L}$ have the following properties:[14,24,25]

- The smallest eigenvalue $\lambda_{\min}$ is zero and corresponds to the eigenvector $(1, 1, \ldots, 1)^{\top}$.
- The largest eigenvalue $\lambda_{\max}$ satisfies

$$\frac{n}{n-1} \leq \lambda_{\max} \leq 2.$$

- $\lambda_{\max} = \frac{n}{n-1}$ if and only if the connection graph is complete.
- $\lambda_{\max} = 2$ if and only if the connection graph is bipartite.

For large complete graphs, the largest eigenvalue $\lambda_{\max}$ is thus close to one. In fact, when self connections are included, $\lambda_{\max}$ becomes exactly one for any size. On the other hand, for bipartite graphs the largest eigenvalue is at its maximum possible value. This class of graphs contains many examples, such as cycles with even number of vertices, regular lattices, and trees.[24]

For a synchronized τ-periodic orbit $\Gamma(t) = [p_{t(\mathrm{mod}\,\tau)}]$, we denote its unique Floquet multiplier for the uncoupled system (1) by

$$B = \prod_{k=0}^{\tau-1} b_k = \prod_{k=0}^{\tau-1} f'(p_k). \tag{14}$$

Using (7) and (4), it can be seen that

$$\frac{c_k}{b_k} =: \varepsilon \in \mathbb{R}, \quad k = 0, \ldots, \tau - 1, \tag{15}$$

which is referred as coupling constant. For simplicity, we assume that $\varepsilon \in [-1, 1]$. Using (14) and (15), we can write

$$p_l(s) = s^{\tau+1} - B((1-\varepsilon)s + \varepsilon(1-\lambda_l))^{\tau}. \quad (16)$$

It turns out that the stability regions of $p_l(s)$ in the parameter space $(\varepsilon, B, \lambda)$ change non-monotonically with delay τ. A monotonical change in the stability region can be observed if, instead of the Floquet multiplier B, one uses the Lyapunov exponent μ of the periodic orbit $\Gamma(t)$, namely,

$$\mu = \frac{1}{\tau}\sum_{k=0}^{\tau-1} \ln|f'(p_k)| \quad (17)$$

or the modulus of the scaled Floquet multiplier β, namely,

$$|\beta| = |B|^{\frac{1}{\tau}} = e^{\mu}. \quad (18)$$

In the sequel, we use the parameter $|\beta|$ for the sake of simplicity in equations. From Eqs. (14), (17), and (18), the relation between B and $|\beta|$ can be found as

$$B = \sigma|\beta|^{\tau}, \quad (19)$$

where the parameter $\sigma \in \{-1, +1\}$ denotes the sign of the Floquet multiplier of the periodic orbit $\Gamma(t)$. Substituting (19) in (16), we have the following result.

Corollary 1. *A synchronized τ-periodic solution $\Gamma(t) = [p_{t(\mathrm{mod}\tau)}]$ of (4) is locally asymptotically stable if the roots of*

$$p(s) = s^{\tau+1} - \sigma|\beta|^{\tau}((1-\varepsilon)s + \varepsilon(1-\lambda))^{\tau} \quad (20)$$

are in the open unit disc for all $\lambda \in \{\lambda_1 \ldots \lambda_n\}$, and is unstable if p(s) has a root outside the unit disc for some $\lambda \in \{\lambda_1 \ldots \lambda_n\}$.

Let us consider the stability region of $p(s)$ in the parameter space $(\varepsilon, |\beta|, \lambda)$ for $\sigma = 1$ and for $\sigma = -1$ separately, namely, the set of points $(\varepsilon, |\beta|, \lambda)$ for which all roots of (20) are in the open unit disc. We show the following symmetry between the stability regions of $p(s)$ for $\sigma = 1$ and for $\sigma = -1$:

$$\kappa : (\sigma, \lambda) \to (-\sigma, 2-\lambda). \quad (21)$$

To show this symmetry, consider

$$\bar{p}(s) = \kappa(p(s)) = s^{\tau+1} + \sigma|\beta|^{\tau}((1-\varepsilon)s - \varepsilon(1-\lambda))^{\tau}.$$

It can be verified that $\bar{p}(-s) = -p(s)$ if τ is even and $\bar{p}(-s) = p(s)$ if τ is odd. Hence, for any τ,

$$p(s) \text{ is stable} \iff \bar{p}(s) \text{ is stable}.$$

Due to this symmetry, it is enough to check the stability of $p(s)$ for $0 \le \lambda \le 1$ and obtain the stability conditions for $1 < \lambda \le 2$ by applying the symmetry transformation κ. This also proves that for $\lambda = 1$ the stability region of $p(s)$ in $(\varepsilon, |\beta|)$ for $\sigma = 1$ is identical to the stability region for $\sigma = -1$. In fact, the roots of $p(s)$ for $\lambda = 1$ are easily seen to be $s_1 = \sigma|\beta|^{\tau}(1-\varepsilon)^{\tau}$ and $s_i = 0$ for $i = 2, \ldots, \tau+1$. Therefore, a necessary and sufficient condition for the stability of $p(s)$ for $\lambda = 1$ is

$$|\beta||1-\varepsilon| < 1. \quad (22)$$

The term $\beta(1-\varepsilon)$ can be seen as a delay-independent scaled Floquet multiplier of the coupled system with maximum modulus when $\lambda = 1$. For other values of λ, stability conditions can be splitted into delay-dependent and delay-independent ones (see Figs. 1(c) and 1(d)). Delay-dependent conditions for $p(s)$ turn out to be highly complex and seem to give no insight. In one of the simplest cases, namely, $\lambda = 0$, a delay-dependent necessary condition can be obtained from the second iteration to the Bistritz method as $|\beta| < (\tau+1)^{\frac{1}{\tau}}$. Other iterations of the Bistritz method provide extremely complex conditions due to the special structure of $p(s)$.

Applying the first condition of the Schur-Cohn criterion $(p(1) > 0)$ to $p(s)$ and $\bar{p}(s)$, the following necessary condition can be obtained for the stability of $p(s)$:

$$1 - \sigma|\beta|^{\tau}(1-\varepsilon+\varepsilon(1-\lambda))^{\tau} > 0, \quad (23)$$

$$1 + \sigma|\beta|^{\tau}(1-\varepsilon-\varepsilon(1-\lambda))^{\tau} > 0. \quad (24)$$

On the other hand, a corollary of the Gershgorin disc theorem (see Ref. 26, Theorem 5.10) implies the following sufficient condition:

$$1 - |\beta|(1-\varepsilon+|\varepsilon(1-\lambda)|) > 0. \quad (25)$$

The above necessary and sufficient conditions are used to provide some upper and lower bounds of the stability regions. In addition to these, we use an algorithmic method, namely, the Bistritz Tabulation,[20,27] to determine stability regions precisely. This method is based on a three-term recursion of symmetric polynomials generated from the main polynomial. Similar to the well-known Jury method,[28] the Bistritz tabulation method gives necessary and sufficient conditions on parameters for the stability of a polynomial, while affording significant computational savings[20] as compared to the Jury method.

A. The role of the largest Laplacian eigenvalue

Using the Bistritz tabulation method, the 3-D stability region of $p(s)$ for $\tau = 2$ is found as in Figs. 1(a) and 1(b) for $\sigma = 1$ and for $\sigma = -1$, respectively. Figs. 1(c) and 1(d) show that stability regions shrink down monotonically as τ increases.

The stability region of $\Gamma(t)$ in the parameter space $(\varepsilon, |\beta|)$ can be obtained by taking the intersection of n 2-D slices of the 3-D stability region of $p(s)$ for the Laplacian eigenvalues $\lambda = \lambda_1, \ldots, \lambda_n$. For $\tau = 3$, these 2-D slices corresponding to $\lambda = 0, 0.25, \ldots, 2$ are illustrated in Fig. 2(a) for $\sigma = 1$. Stability regions for $\sigma = -1$ can be found using the above-mentioned symmetry as in Fig. 2(b).

In order to obtain the stability region of a synchronized τ-periodic orbit $\Gamma(t)$ with $\tau = 3$, one has to take the intersection of the stability regions of $p(s)$ for $\lambda = \lambda_1, \ldots, \lambda_n$. It is straightforward to check that the stability region thus

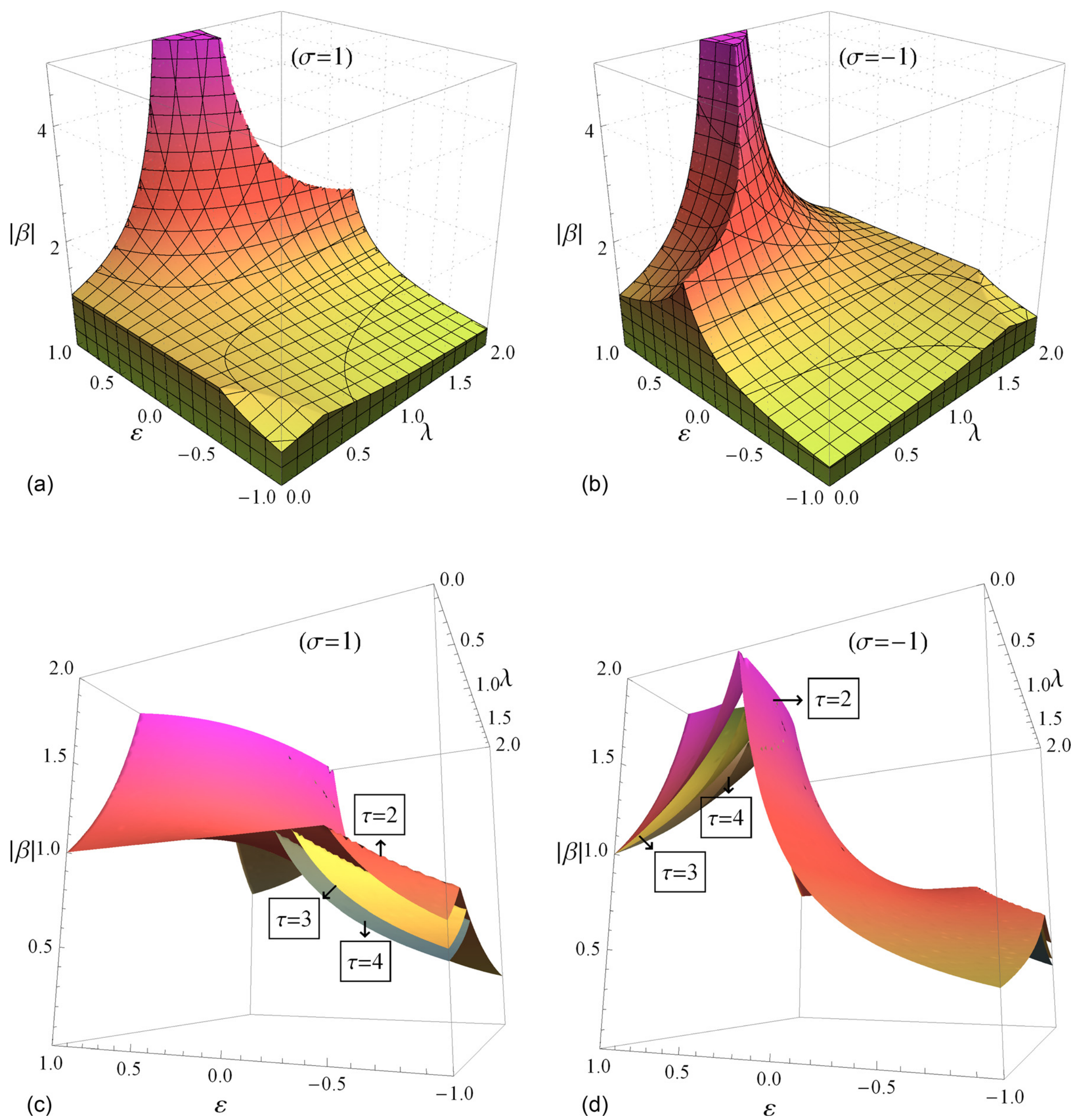

FIG. 1. The stability regions of $p(s)$ for $\tau = 2$ are shown in (a) and (b) when the Floquet multiplier is positive ($\sigma = 1$) and negative ($\sigma = -1$), respectively—these are related by the symmetry given in (21). The stability boundaries for $\tau = 2, 3, 4$ are shown in (c) and (d), for $\sigma = 1$ and $\sigma = -1$, respectively.

obtained is bounded by the curves related to the smallest and the largest Laplacian eigenvalue. We have repeated this process for different values of τ and obtained the same result, namely, the stability region of $\Gamma(t)$ depends only on the smallest and the largest Laplacian eigenvalues. However, we do not have a rigorous proof for this observation. Note that the smallest Laplacian eigenvalue is always zero, therefore, the largest Laplacian eigenvalue plays a crucial role in stability. For a general coupling structure, namely, for $\lambda_{\max} \in [1, 2]$, typical regions obtained by taking the intersection of the stability regions in Fig. 2 for $\lambda = 0$ and $\lambda = \lambda_{\max}$ are illustrated in Fig. 3.

B. Minimal stability region

Stability regions become minimal in two cases, namely, for bipartite graphs and for the case of $\tau \to \infty$. In both cases, stability regions are identical for $\sigma = 1$ and $\sigma = -1$, and given by the following inequalities:

$$|\beta| < 1 \quad \text{for } \varepsilon > 0, \tag{26}$$

$$|\beta| < \frac{1}{1 - 2\varepsilon} \quad \text{for } \varepsilon < 0. \tag{27}$$

We call this region the *minimal stability region*, which is depicted in Fig. 4. Note that β and $\beta\left(\frac{1}{1-2\varepsilon}\right)$ can be interpreted as delay-independent scaled Floquet multipliers of the uncoupled system with bipartite connection.

To see that the stability region reduces to the minimal stability region for a bipartite connection structure, consider $p(s)$ for $\lambda = 0$ and for $\lambda = 2$ and assume that $\sigma = 1$. Substituting these in (23) and (24), one gets $|\beta| < 1$ and $|\beta| < \frac{1}{1-2\varepsilon}$, which is equivalent to (26) and (27). By the symmetry (21), the same applies to the case $\sigma = -1$. On the other hand, the

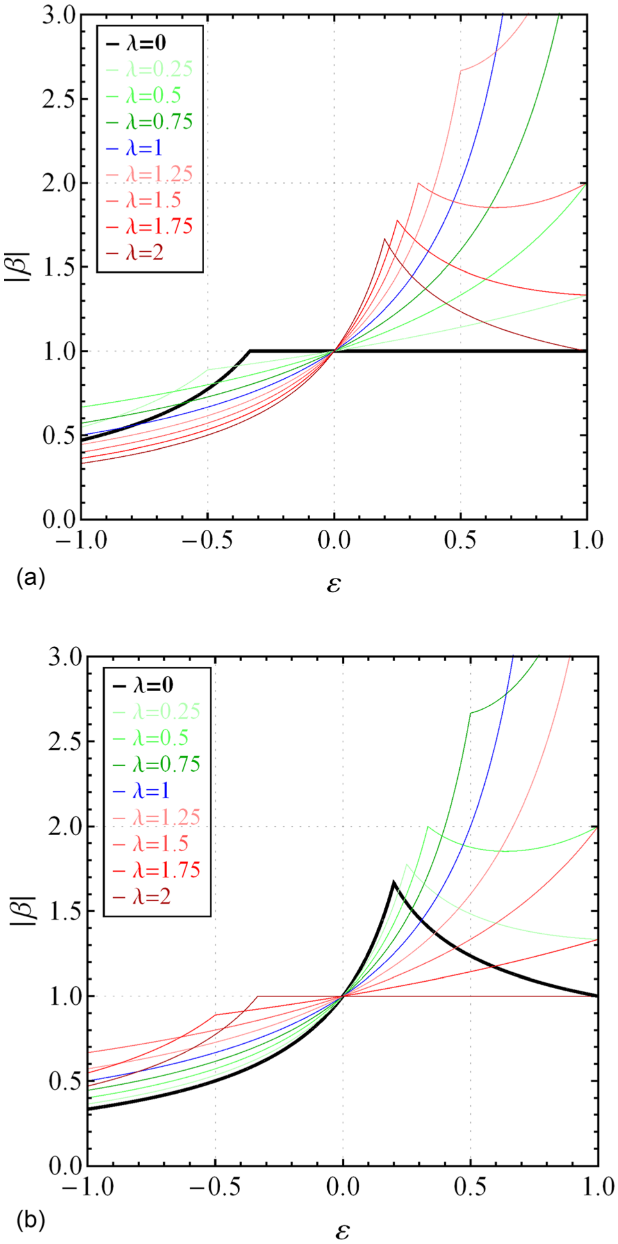

FIG. 2. The stability regions of $p(s)$ for several values of λ (for $\tau = 3$). The stability regions are the open regions under the colored curves, which are plotted for $\lambda = 0.0, 0.25, \ldots, 2.0$. (a) and (b) Stability regions when the Floquet multiplier is positive ($\sigma = 1$) and negative ($\sigma = -1$), respectively—these are related by the symmetry given in (21).

sufficient condition (25) for all Laplacian eigenvalues is equivalent to a unique condition, namely, (25) for $\lambda = 0$, which also reduces to (26) and (27). Hence, we have the following negative result for stabilization.

Theorem 2. *An unstable periodic orbit* $\Gamma(t) = p_{t(\mathrm{mod}\,\tau)}$ *cannot be stabilized via coupling of form* (4) *if the connection structure is bipartite.*

To see that the stability region is given by the minimal stability region when $\tau \to \infty$, we use the fact that for $\sigma = 1$ and for $\sigma = -1$ stability regions coincide in the limit $\tau \to \infty$. This can be seen by substituting $s = re^{i\theta}$ in $p(s)$ and observing that the magnitude equations turn out to be the same. It is known that solutions to the phase equation are uniformly distributed when $\tau \to \infty$.[29] Thus, the stability

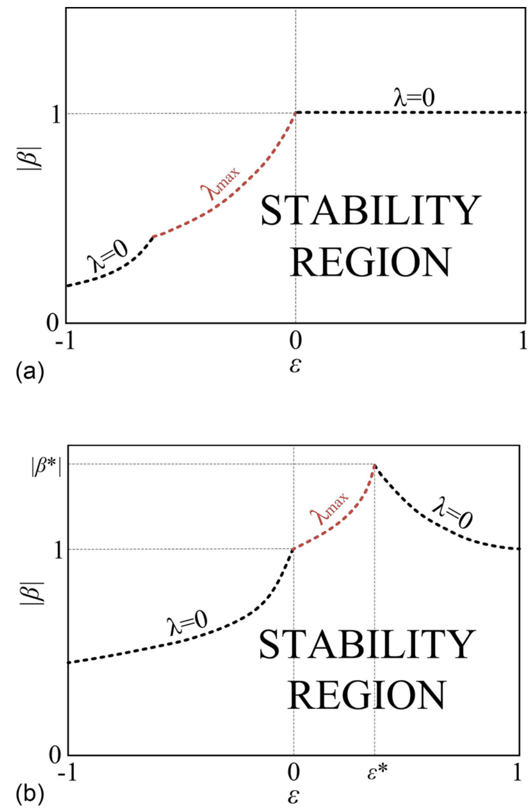

FIG. 3. Typical stability regions of the periodic orbit $\Gamma(t) = \Gamma_{t(\mathrm{mod}\tau)}$ for the system (4). The region can be obtained by taking the intersection of the stability regions in Fig. 2 for $\lambda = 0$ (in black) and for $\lambda = \lambda_{\max}$ (in red). $|\beta^*|$ is the maximum value of the modulus of the scaled Floquet multiplier for which stabilization is possible. ε^* is the coupling strength which favors stability most.

regions for $\sigma = 1$ and for $\sigma = -1$ coincide when $\tau \to \infty$. To see that these are identical to the minimal stability region, observe that for $\varepsilon > 0$ and $\sigma = 1$, both the necessary condition (23) for $\lambda = 0$ and the sufficient condition (25) reduce to (26). For the case $\varepsilon < 0$, both the necessary condition (24) for $\lambda = 0$ and the sufficient condition (25) reduce to (27) when $\sigma = -1$.

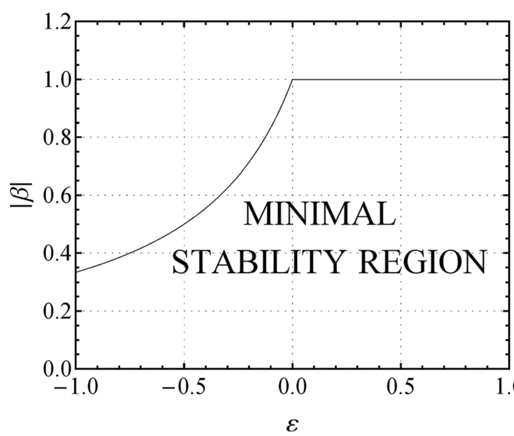

FIG. 4. Stability region of the periodic orbit $\Gamma(t) = p_{t(\mathrm{mod}\tau)}$ for the system (4) with a bipartite connection structure. The stability region is independent of τ, and it is depicted in (26) and (27).

C. Periodic orbits with a positive Floquet multiplier

It has been shown in the paper[18] that an unstable fixed point of a one-dimensional map with positive eigenvalue cannot be stabilized via coupling in the form (4). Here, we prove a similar result for a synchronized periodic orbit of a coupled map lattice with delay where the period of the orbit is equal to the delay.

Theorem 3. *An unstable periodic orbit* $\Gamma(t) = p_{t(\mathrm{mod}\,\tau)}$ *with a positive Floquet multiplier cannot be stabilized via coupling of the form* (4).

Proof. We prove the contrapositive as follows: Assume that stabilization occurs. Since $\lambda = 0$ is always an eigenvalue of the Laplacian, $p_0(s) = s^{\tau+1} - \sigma|\beta|^{\tau}((1-\varepsilon)s+\varepsilon)^{\tau}$ must be Schur stable. By the necessary condition (23), we have $1 - \sigma|\beta|^{\tau} > 0$. Since $|\beta| > 1$ by the instability assumption, one gets $\sigma = -1$.

The stability region of $p(s)$ obtained for $\lambda = 0$ and $\sigma = 1$ in Fig. 2(a) justifies Theorem 3 for $\tau = 3$.

Remark 2. Theorem 3 has important consequences. For instance, unstable periodic orbits of dyadic maps cannot be stabilized via coupling of the form (4).

D. Most stabilizing network configuration

It can be observed from Figs. 2 and 3 that the stability regions shrink as the largest eigenvalue increases. Consequently, connection structures having a small value for the largest Laplacian eigenvalue, such as the all-to-all coupling topology, favor the stability of synchronized τ-periodic orbits. It is known that, in the case of all-to-all coupling with self connections, i.e., $a_{ij} = 1, \forall i,j$, the eigenvalues of the Laplacian are $\lambda_1 = 0$ and $\lambda_k = 1$ for $k \geq 2$. Alternatively, one can consider all-to-all coupled networks without self-coupling but with a large number of nodes, for which $\lambda_1 = 0$ and $\lambda_k = n/(n-1) \cong 1, k \geq 2$. Since, in these cases, it suffices to check the stability of $p(s)$ only for $\lambda = 0$ and for $\lambda = 1$ (or for $\lambda = \frac{n}{n-1}$), the stability regions can be calculated precisely (see Fig. 5). For $\lambda = 1$, the stability region is given in (22), and for $\lambda = 0$ we use the Bistritz tabulation method to obtain the stability regions of $p_{\lambda=0}(s) = s^{\tau+1} - \sigma|\beta|^{\tau}((1-\varepsilon)s+\varepsilon)^{\tau}$.

IV. STABILIZATION/DESTABILIZATION OF SYNCHRONIZED τ-PERIODIC ORBITS AND CHAOS SUPPRESSION VIA COUPLING

In Sec. III, we have shown that unstable periodic orbits with negative Floquet multipliers can be stabilized via coupling. On the other hand, stable periodic orbits may lose stability when coupled through a negative coupling constant.

In order to illustrate the stabilization, we consider the case that favors stability most, namely, all-to-all coupling with self-connections. It can be seen from Fig. 5 that the stability regions shrink as τ increases for both $\sigma = 1$ and $\sigma = -1$. The stability region for $\sigma = -1$ (Fig. 5(b)) has a maximum value $|\beta^*|$ at a certain coupling strength ε^* (see also Fig. 3) and both $|\beta^*|$ and ε^* decrease monotonically as τ increases. Note that, as a result of Theorem 3, stabilization

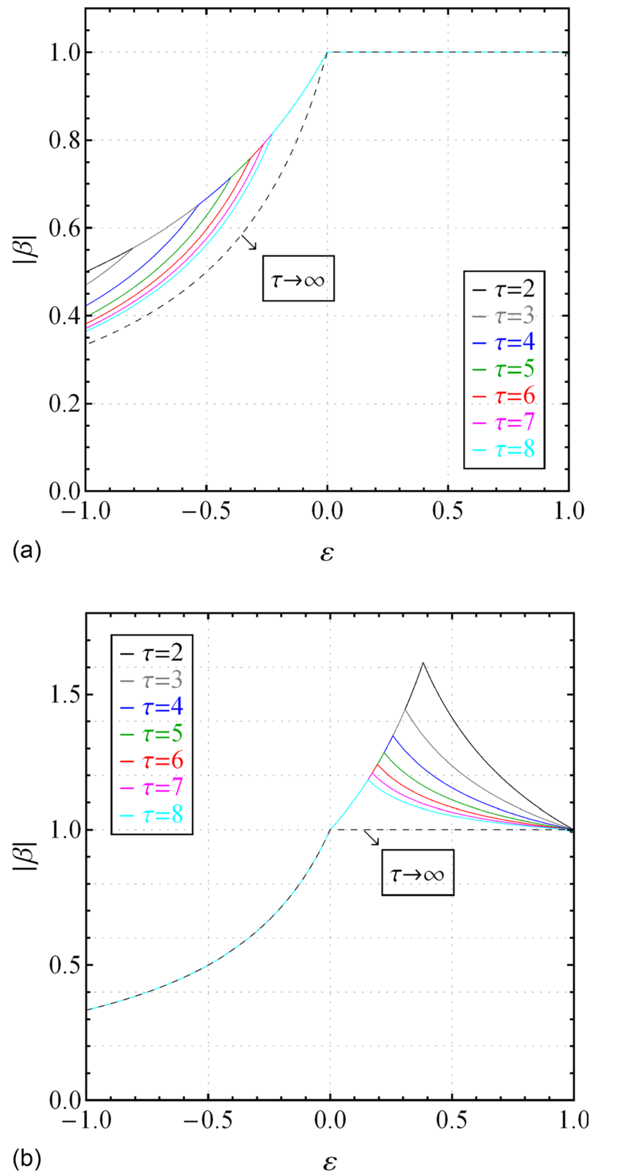

FIG. 5. Stability regions of the periodic orbit $\Gamma(t) = p_{t(\mathrm{mod}\tau)}$ for the system (4) with different delays and with an all-to-all coupled connection structure including self-connections. Stability regions are the open regions inside the colored curves. The minimal stability region ($\tau \to \infty$) is also shown as dashed line.

is not possible for positive Floquet multipliers which is seen also from Fig. 5(a). In Table I, the maximum $|\beta|$ values ($|\beta^*|$) for which stabilization is possible, and the corresponding ε^* values are given.

We demonstrate the destabilization of periodic orbits when the coupling constant is negative. It can be seen from Fig. 4 that stable τ-periodic orbits of maps may lose stability

TABLE I. Maximum modulus of the scaled Floquet multiplier $|\beta^*|$ and the corresponding values of the coupling constant ε^* for which the system (4) has a stable τ-periodic solution.

τ	2	3	4	5	6	7	8
$\lvert\beta^*\rvert$	1.605	1.435	1.338	1.276	1.236	1.205	1.178
ε^*	0.3845	0.3145	0.2605	0.2243	0.1956	0.1751	0.1575

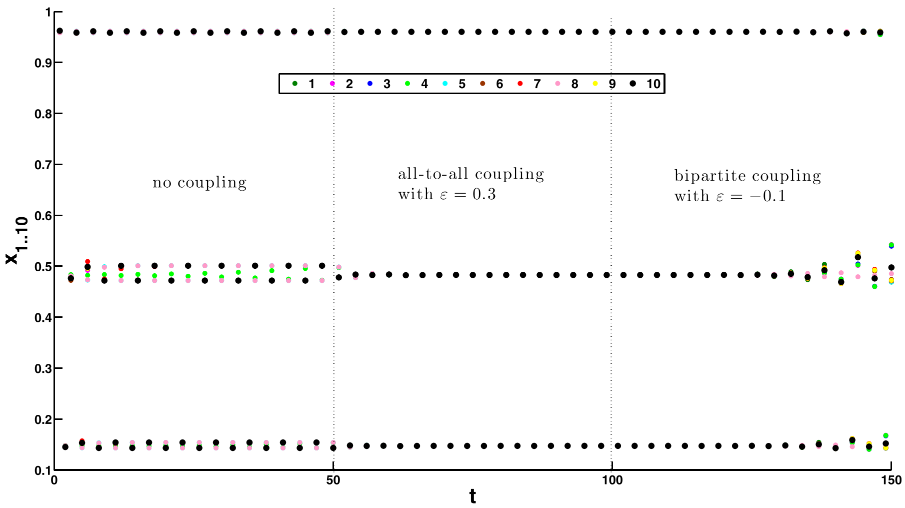

FIG. 6. Solutions of (4) for $n = 10$ and $\tau = 3$ where $f(x)$ is the logistic map with $r = 3.845$. All-to-all coupling with $\varepsilon = 0.3$ is activated at time $t = 50$, after which all maps synchronize on a stable 3-periodic orbit. At time $t = 100$, coupling is changed to a bipartite coupling (Fig. 7) with $\varepsilon = -0.1$ which leads to the instability of the 3-periodic orbit. A Gaussian noise of variance 10^{-6} is added to the state at $t = 100$ to destroy any numerical locking near the synchronous solution.

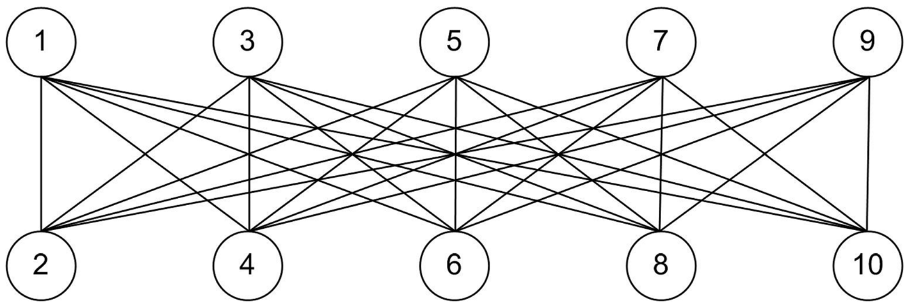

FIG. 7. A bipartite connection structure.

when the maps are connected in the form (4) with a negative coupling constant ε. The destabilization is more likely if the largest eigenvalue of the Laplacian $\lambda_{\max}$ is equal to 2, namely, the coupling structure is bipartite of which the stability region in the $(\varepsilon, |\beta|)$ plane is depicted in Fig. 4.

Example 1 (Stabilization and destabilization of a synchronized 3-periodic orbit). We consider the coupled system (4) with a delay $\tau = 3$, where f is the logistic map $f(x) = rx(1-x)$. The map f has a 3-periodic orbit, which is stable for $r \in (r_3, r_6)$,[30] where $r_3 \cong 3.8284$ is the parameter value at which the stable 3-periodic orbit appears and $r_6 \cong 3.8415$ is the value where it becomes unstable and a stable 6-periodic orbit appears via a period-doubling bifurcation.[31]

We set $r = 3.845$ and run (4) for $n = 10$ with initial conditions chosen close to the 6-periodic orbit (see Fig. 6). Note that for this value of r, the 3-periodic orbit is unstable with its Floquet multiplier being $B \cong -1.27$, namely, $\sigma = -1$ and $|\beta| \cong 1.08$. Initially, the coupling is not activated and each map converges to the 6-periodic orbit. At time $t = 50$, an all-to-all coupling (including self connections) with $\varepsilon = 0.3$ is activated which leads to the stability of a synchronized 3-periodic orbit (check Fig. 5(b) for parameters $\varepsilon = 0.3$ and $|\beta| = 1.08$). At time $t = 100$, a bipartite coupling as in Fig. 7 is activated with $\varepsilon = -0.1$, which destabilizes the synchronized 3-periodic orbit in accordance with the parameter region in Fig. 4.

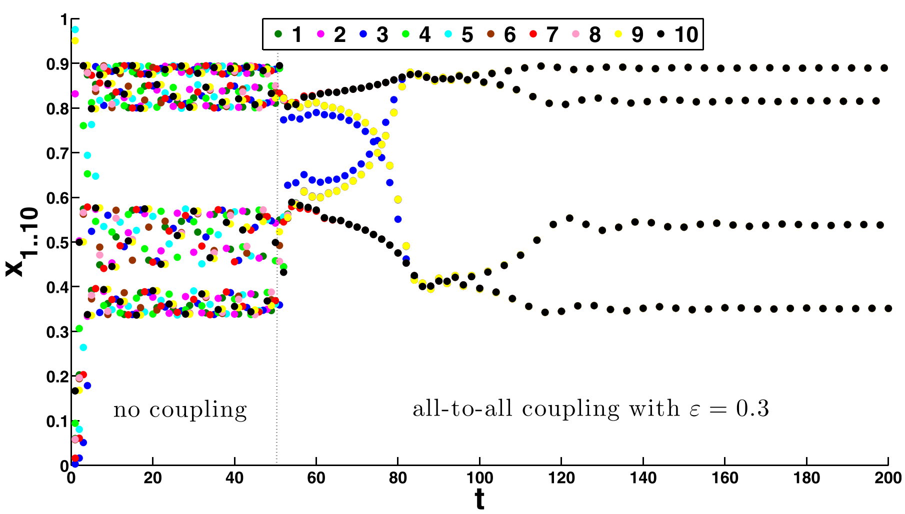

FIG. 8. A solution of (4) for $n = 10$ and $\tau = 4$ where $f(x)$ is the logistic map with $r = 3.58$. Initially, coupling is set to zero and each map converges to a chaotic attractor. All-to-all coupling with $\varepsilon = 0.3$ is activated at time $t = 50$, after which all maps synchronize on a stable 4-periodic orbit.

Example 2 (Chaos suppression). Stabilization of a synchronized τ-periodic orbit of (4) via coupling with delay τ is possible only if the modulus of the scaled Floquet multiplier ($|\beta|$) of the periodic orbit is small enough (see Table I). For instance, the logistic map with $r = 4$ has infinitely many p-periodic orbits, the Floquet multiplier of which are given as 2^p. In this case, it is not possible to stabilize τ-periodic orbits ($\tau \geq 2$) of logistic maps with $r = 4$ via delay τ. Nevertheless, when the logistic map first enters chaos at the end of the period doubling bifurcation at $r \cong 3.57$, Floquet multipliers of 2^k-periodic orbits are relatively small, which makes stabilization possible. Fig. 8 shows a simulation result for the coupled system (4) of ten logistic maps with $r = 3.58$, for which maps are chaotic with the largest Lyapunov exponent $\cong 0.109$. Initially, coupling is not activated and systems approach to their chaotic attractor independently from each other. At time $t = 50$, an all-to-all coupling with $\varepsilon = 0.3$ and $\tau = 4$ is activated which stabilizes a synchronized 4-periodic orbit ($\sigma = -1$, $|\beta| \cong 1.16$) in accordance with the stability region in Fig. 5.

V. CONCLUSION

We have analyzed the stability of synchronized periodic orbits of delay-coupled maps when the delay is equal to the period of the periodic orbit. A sufficient condition for stability is obtained in terms of the modulus of the scaled Floquet multiplier of the periodic orbit for the uncoupled map, the coupling constant, and the largest Laplacian eigenvalue. We have investigated stabilization and destabilization of periodic orbits as well as chaos suppression via coupling with delay.

Stabilization of unstable periodic orbits via delayed feedback is a popular approach in chaos control.[1,2,32] Here, we have shown that stabilization is also possible when systems are coupled to each other with coupling delays. This shows another property of delay in regulating the dynamic behaviour of coupled systems. On the other hand, stabilization has been shown to be not possible when the Floquet multiplier of the uncoupled system is positive or when the connection structure of the coupled system is bipartite.

We emphasize that the polynomial that determines the stability of a synchronized τ-periodic orbit has a special form, namely, $p(s) = s^{\tau+1} - \prod_{k=0}^{\tau-1}((b_k - c_k)s + c_k(1-\lambda))$, which reduces to the polynomial $p(s) = s^{\tau+1} - \sigma|\beta|^{\tau}((1-\varepsilon)s + \varepsilon(1-\lambda))^{\tau}$ if the coupling of form (4) is considered. A more detailed analytical investigation of these polynomials may lead to further results on the stability of such periodic orbits of coupled systems.

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