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Asymptotic Properties of Jacobi Matrices for a Family of Fractal Measures

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ABSTRACT

We study the properties and asymptotics of the Jacobi matrices associated with equilibrium measures of the weakly equilibrium Cantor sets. These family of Cantor sets were defined, and different aspects of orthogonal polynomials on them were studied recently. Our main aim is to numerically examine some conjectures concerning orthogonal polynomials which do not directly follow from previous results. We also compare our results with more general conjectures made for recurrence coefficients associated with fractal measures supported on $\mathbb{R}$.

1. Introduction

For a unit Borel measure μ with an infinite compact support on $\mathbb{R}$, using the Gram–Schmidt process for the set $\{1, x, x^2, \ldots\}$ in $L^2(\mu)$, one can find a sequence of polynomials $(q_n(\cdot; \mu))_{n=0}^{\infty}$ satisfying

$$\int q_m(x; \mu) q_n(x; \mu) \, d\mu(x) = \delta_{mn}$$

where $q_n(\cdot; \mu)$ is of degree n. Here, $q_n(\cdot; \mu))$ is called the nth orthonormal polynomial for μ. We denote its positive leading coefficient by κ_n and nth monic orthogonal polynomial $q_n(\cdot; \mu)/\kappa_n$ by $Q_n(\cdot; \mu)$. If we assume that $Q_{-1}(\cdot; \mu) := 0$ and $Q_0(\cdot; \mu) := 1$, then there are two bounded sequences $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ such that the polynomials $(Q_n(\cdot; \mu))_{n=0}^{\infty}$ satisfy a three-term recurrence relation

$$Q_{n+1}(x; \mu) = (x - b_{n+1})Q_n(x; \mu) - a_n^2 Q_{n-1}(x; \mu),$$
$$n \in \mathbb{N}_0,$$

where $a_n > 0$, $b_n \in \mathbb{R}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Conversely, if two bounded sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are given with $a_n > 0$ and $b_n \in \mathbb{R}$ for each $n \in \mathbb{N}$, then we can define the corresponding Jacobi matrix H, which is a self-adjoint bounded operator acting on $l^2(\mathbb{N})$, as the following,

$$H = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \ldots \\ a_1 & b_2 & a_2 & 0 & \ldots \\ 0 & a_2 & b_3 & a_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (1\text{--}1)$$

The (scalar valued) spectral measure μ of H for the cyclic vector $(1, 0, \ldots)^T$ is the measure that has $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ as recurrence coefficients. Due to this one-to-one correspondence between measures and Jacobi matrices, we denote the Jacobi matrix associated with μ by H_μ. For a discussion of the spectral theory of orthogonal polynomials on $\mathbb{R}$, we refer the reader to [Simon 11, Van Assche 87].

Let $c = (c_n)_{n=-\infty}^{\infty}$ be a two-sided sequence taking values on $\mathbb{C}$ and $c^j = (c_{n+j})_{n=-\infty}^{\infty}$ for $j \in \mathbb{Z}$. Then c is called almost periodic if $\{c^j\}_{j\in\mathbb{Z}}$ is precompact in $l^\infty(\mathbb{Z})$. A one-sided sequence $d = (d_n)_{n=1}^{\infty}$ is called almost periodic if it is the restriction of a two-sided almost periodic sequence to $\mathbb{N}$. Each one-sided almost periodic sequence has only one extension to $\mathbb{Z}$ which is almost periodic, see Section 5.13 in [Simon 11]. Hence, one-sided and two-sided almost periodic sequences are essentially the same objects. A Jacobi matrix H_μ is called almost periodic if the sequences of recurrence coefficients $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ for μ are almost periodic. We consider in the following sections only one-sided sequences due to the nature of our problems but, in general, for the almost periodicity, it is much more natural to consider sequences on $\mathbb{Z}$ instead of $\mathbb{N}$.

A sequence $s = (s_n)_{n=1}^{\infty}$ is called asymptotically almost periodic if there is an almost periodic sequence $d = (d_n)_{n=1}^{\infty}$ such that $d_n - s_n \to 0$ as $n \to \infty$. In this case, d is unique and it is called the almost periodic limit. See [Petersen 83, Simon 11, Teschl 00] for more details on almost periodic functions.

Several sufficient conditions on H_μ to be almost periodic or asymptotically almost periodic are given in

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[Peherstorfer and Yuditskii 03, Sodin and Yuditskii 97] for the case when ess supp(μ) (that is the support of μ excluding its isolated points) is a Parreau–Widom set (Section 3) or in particular homogeneous set in the sense of Carleson (see [Peherstorfer and Yuditskii 03] for the definition). We remark that some symmetric Cantor sets and generalized Julia sets (see [Peherstorfer and Yuditskii 03, Alpan and Goncharov 15b]) are Parreau–Widom. By [Barnsley et al. 85, Yudistkii 12], for equilibrium measures of some polynomial Julia sets, the corresponding Jacobi matrices are almost periodic. It was conjectured in [Mantica 97, Krüger and Simon 15] that Jacobi matrices for self-similar measures including the Cantor measure are asymptotically almost periodic. We should also mention that some almost periodic Jacobi matrices with applications to physics (see e.g., [Avila and Jitomirskaya 09]) have essential spectrum equal to a Cantor set.

There are many open problems regarding orthogonal polynomials on Cantor sets, such as how to define the Szegő class of measures and isospectral torus (see e.g., [Christiansen et al. 09, Christiansen et al. 11] for the previous results and [Heilman et al. 11, Krüger and Simon 15, Mantica 96, Mantica 15a, Mantica 15b] for possible extensions of the theory and important conjectures) especially when the support has zero Lebesgue measure. The family of sets that we consider here contains both positive and zero Lebesgue measure sets, Parreau–Widom and non-Parreau–Widom sets.

Widom–Hilbert factors (see Section 2 for the definition) for equilibrium measures of the weakly equilibrium Cantor sets may be bounded or unbounded depending on the particular choice of parameters. Some properties of these measures related to orthogonal polynomials were already studied in detail, but till now we do not have complete characterizations of most of the properties mentioned above in terms of the parameters. Our results and conjectures are meant to suggest some formulations of theorems for further work on these sets as well as other Cantor sets.

The plan of the article is as follows. In Section 2, we review the previous results on $K(\gamma)$ and provide evidence for the numerical stability of the algorithm obtained in Section 4 in [Alpan and Goncharov 16] for calculating the recurrence coefficients. In Section 3, we discuss the behavior of recurrence coefficients in different aspects and propose some conjectures about the character of periodicity of the Jacobi matrices. In Section 4, the properties of Widom factors are investigated. We also prove that the sequence of Widom–Hilbert factors for the equilibrium measure of autonomous quadratic Julia sets is unbounded above as soon as the Julia set is totally disconnected. In the last section, we study the local behavior of the spacing properties of the zeros of orthogonal polynomials for the equilibrium measures of weakly equilibrium Cantor sets and make a few comments on possible consequences of our numerical experiments.

For a general overview on potential theory, we refer the reader to [Ransford 95, Saff and Totik 97]. For a non-polar compact set $K \subset \mathbb{C}$, the equilibrium measure is denoted by μ_K while Cap(K) stands for the logarithmic capacity of K. The Green function for the connected component of $\overline{\mathbb{C}} \setminus K$ containing infinity is denoted by $G_K(z)$. Convergence of measures is understood as weak-star convergence. For the sup norm on K and for the Hilbert norm on $L^2(\mu)$, we use $\| \cdot \|_{L^\infty(K)}$ and $\| \cdot \|_{L^2(\mu)}$, respectively.

2. Preliminaries and numerical stability of the algorithm

Let us repeat the construction of $K(\gamma)$ which was introduced in [Goncharov 14]. Let $\gamma = (\gamma_s)_{s=1}^\infty$ be a sequence such that $0 < \gamma_s < 1/4$ holds for each $s \in \mathbb{N}$ provided that $\sum_{s=1}^\infty 2^{-s} \log(1/\gamma_s) < \infty$. Set $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$. We define $(f_n)_{n=1}^\infty$ by $f_1(z) := 2z(z-1)/\gamma_1 + 1$ and $f_n(z) := z^2/(2\gamma_n) + 1 - 1/(2\gamma_n)$ for $n > 1$. Here $E_0 := [0, 1]$ and $E_n := F_n^{-1}([-1, 1])$ where F_n is used to denote $f_n \circ \cdots \circ f_1$. Then, E_n is a union of 2^n disjoint non-degenerate closed intervals in $[0, 1]$ and $E_n \subset E_{n-1}$ for all $n \in \mathbb{N}$. Moreover, $K(\gamma) := \cap_{n=0}^\infty E_n$ is a non-polar Cantor set in $[0, 1]$ where $\{0, 1\} \subset K(\gamma)$. It is not hard to see that for each different γ we end up with a different $K(\gamma)$.

It is shown in Section 3 of [Alpan and Goncharov 16] that for all $s \in \mathbb{N}_0$ we have

$$||Q_{2^s}\left(\cdot; \mu_{K(\gamma)}\right)||_{L^2\left(\mu_{K(\gamma)}\right)} = \sqrt{(1 - 2\gamma_{s+1})\, r_s^2/4}. \quad (2\text{–}1)$$

The diagonal elements, the b_n's of $H_{\mu_{K(\gamma)}}$, are equal to 0,5 by Section 4 in [Alpan and Goncharov 16]. For the outdiagonal elements by Theorem 4.3 in [Alpan and Goncharov 16], we have the following relations:

$$a_1 = \|Q_1\left(\cdot; \mu_{K(\gamma)}\right)\|_{L^2\left(\mu_{K(\gamma)}\right)}, \quad (2\text{–}2)$$

$$a_2 = \|Q_2\left(\cdot; \mu_{K(\gamma)}\right)\|_{L^2\left(\mu_{K(\gamma)}\right)}/\|Q_1\left(\cdot; \mu_{K(\gamma)}\right)\|_{L^2\left(\mu_{K(\gamma)}\right)}. \quad (2\text{–}3)$$

If $n + 1 = 2^s > 2$ then

$$a_{n+1}$$

$$= \frac{||Q_{2^s}\left(\cdot; \mu_{K(\gamma)}\right)||_{L^2\left(\mu_{K(\gamma)}\right)}}{||Q_{2^{s-1}}\left(\cdot; \mu_{K(\gamma)}\right)||_{L^2\left(\mu_{K(\gamma)}\right)} \cdot a_{2^{s-1}+1} \cdot a_{2^{s-1}+2} \cdots a_{2^s-1}}. \quad (2\text{–}4)$$

If $n + 1 = 2^s(2k + 1)$ for some $s \in \mathbb{N}$ and $k \in \mathbb{N}$, then

$$a_{n+1}$$

$$= \sqrt{\frac{\| Q_{2^s}\left(\cdot; \mu_{K(\gamma)}\right)\|^2_{L^2\left(\mu_{K(\gamma)}\right)} - a^2_{2^{s+1}k} \cdots a^2_{2^{s+1}k-2^s+1}}{a^2_{2^s(2k+1)-1} \cdots a^2_{2^{s+1}k+1}}},$$

$$\tag{2-5}$$

If $n + 1 = (2k + 1)$ for $k \in \mathbb{N}$ then

$$a_{n+1} = \sqrt{\| Q_1\left(\cdot; \mu_{K(\gamma)}\right)\|^2_{L^2\left(\mu_{K(\gamma)}\right)} - a^2_{2k}}. \tag{2-6}$$

The relations (2–1), (2–2), (2–3), (2–4), (2–5), and (2–6) completely determine $(a_n)_{n=1}^\infty$ and naturally define an algorithm. This is the main algorithm that we use and we call it Algorithm 1. There are a couple of results for the asymptotics of $(a_n)_{n=1}^\infty$, see Lemma 4.6 and Theorem 4.7 in [Alpan and Goncharov 16].

We want to examine the numerical stability of Algorithm 1 since roundoff errors can be huge due to the recursive nature of it. Before this, let us list some remarkable properties of $K(\gamma)$ which will be considered later on. In the next theorem, one can find proofs of part (a) in [Alpan and Goncharov 14], (b) and (c) in [Alpan and Goncharov 16], (d) and (e) in [Alpan and Goncharov 15b], (f) in [Alpan et al. 16], (g) in [Goncharov 14], and (h) and (i) in [Alpan 16]. We call $W_n^2(\mu) := \frac{\| Q_n(\cdot; \mu)\|_{L^2(\mu)}}{(\mathrm{Cap}(\mathrm{supp}(\mu)))^n}$ as the nth Widom–Hilbert factor for μ.

Theorem 2.1. *For a given* $\gamma = (\gamma_s)_{s=1}^\infty$ *let* $\varepsilon_s := 1 - 4\gamma_s$. *Then the following propositions hold:*

(a) *If* $\sum_{s=1}^\infty \gamma_s < \infty$ *and* $\gamma_s \leq 1/32$ *for all* $s \in \mathbb{N}$ *then* $K(\gamma)$ *is of Hausdorff dimension zero.*

(b) *If* $\gamma_s \leq 1/6$ *for each* $s \in \mathbb{N}$ *then* $K(\gamma)$ *has zero Lebesgue measure,* $\mu_{K(\gamma)}$ *is purely singular continuous and* $\liminf a_n = 0$ *for* $\mu_{K(\gamma)}$.

(c) *Let* $\tilde{f} := (\tilde{f}_s)_{s=1}^\infty$ *be a sequence of functions such that* $\tilde{f}_s = f_s$ *for* $1 \leq s \leq k$ *for some* $k \in \mathbb{N}$ *and* $\tilde{f}_s(z) = 2z^2 - 1$ *for* $s > k$. *Then* $\cap_{n=1}^\infty \tilde{F}_n^{-1}([-1, 1]) = E_k$ *where* $\tilde{F}_n := \tilde{f}_n \circ \cdots \circ \tilde{f}_1$.

(d) $G_{K(\gamma)}$ *is Hölder continuous with exponent* $1/2$ *if and only if* $\sum_{s=1}^\infty \varepsilon_s < \infty$.

(e) $K(\gamma)$ *is a Parreau–Widom set if and only if* $\sum_{s=1}^\infty \sqrt{\varepsilon_s} < \infty$.

(f) *If* $\sum_{s=1}^\infty \varepsilon_s < \infty$ *then there is* $C > 0$ *such that for all* $n \in \mathbb{N}$ *we have*

$$W_n^2(\mu_{K(\gamma)}) = \frac{\| Q_n\left(\cdot; \mu_{K(\gamma)}\right)\|_{L^2\left(\mu_{K(\gamma)}\right)}}{(\mathrm{Cap}(K(\gamma)))^n}$$

$$= \frac{a_1 \ldots a_n}{(\mathrm{Cap}(K(\gamma)))^n} \leq Cn.$$

(g) $\mathrm{Cap}(K(\gamma)) = \exp\left(\sum_{k=1}^\infty 2^{-k} \log \gamma_k\right)$.

(h) *Let* $v_{1,1}(t) = 1/2 - (1/2)\sqrt{1 - 2\gamma_1 + 2\gamma_1 t}$ *and* $v_{2,1}(t) = 1 - v_{1,1}(t)$. *For each* $n > 1$, *let* $v_{1,n}(t) = \sqrt{1 - 2\gamma_n + 2\gamma_n t}$ *and* $v_{2,n}(t) = -v_{1,n}(t)$. *Then the zero set of* $Q_{2^s}(\cdot; \mu_{K(\gamma)})$ *is* $\{v_{i_1,1} \circ \cdots \circ v_{i_s,s}(0)\}_{i_s \in \{1,2\}}$ *for all* $s \in \mathbb{N}$.

(i) $\mathrm{supp}(\mu_{K(\gamma)}) = \mathrm{ess\,supp}(\mu_{K(\gamma)}) = K(\gamma)$. *If* $K(\gamma) = [0, 1] \setminus \cup_{k=1}^\infty (c_i, d_i)$ *where* $c_i \neq d_j$ *for all* $i, j \in \mathbb{N}$ *then* $\mu_{K(\gamma)}([0, e_i]) \subset \{m2^{-n}\}_{m,n \in \mathbb{N}}$ *where* $e_i \in (c_i, d_i)$. *Moreover for each* $m \in \mathbb{N}$ *and* $n \in \mathbb{N}$ *with* $m2^{-n} < 1$ *there is an* $i \in \mathbb{N}$ *such that* $\mu_{K(\gamma)}([0, e_i]) = m2^{-n}$.

We consider four different models depending on γ in the whole article. They are:

(1) $\gamma_s = 1/4 - (1/(50 + s)^4)$.

(2) $\gamma_s = 1/4 - (1/(50 + s)^2)$.

(3) $\gamma_s = 1/4 - (1/(50 + s)^{(5/4)})$.

(4) $\gamma_s = 1/4 - (1/50)$.

Model 1 represents an example where $K(\gamma)$ is Parreau-Widom and Model 2 gives a non-Parreau-Widom set such that $(\gamma_k)_{k=1}^\infty$ tends to $1/4$. Model 3 produces a non-Parreau–Widom $K(\gamma)$ with relatively slow growth of γ but still $G_{K(\gamma)}$ is optimally smooth. Model 4 yields a set which is neither Parreau–Widom nor the Green function for the complement of it is optimally smooth. We used Matlab in all of the experiments.

If f is a nonlinear polynomial of degree n having real coefficients with real and simple zeros $x_1 < x_2 < \cdots < x_n$ and distinct extremas $y_1 < \ldots < y_{n-1}$ where $|f(y_i)| > 1$ for $i = 1, 2, \cdots, n - 1$, we say that f is an *admissible* polynomial. Clearly, for any choice of γ, f_n is admissible for each $n \in \mathbb{N}$, and this implies by Lemma 4.3 in [Alpan and Goncharov 15b] that F_n is also admissible. By the remark after Theorem 4 and Theorem 11 in [Geronimo and Van Assche 88], it follows that the Christoffel numbers (see p. 565 in [Geronimo and Van Assche 88] for the definition) for the 2^nth orthogonal polynomial of μ_{E_n} are equal to $1/2^n$. Let $\mu^n_{K(\gamma)}$ be the measure which assigns $1/2^n$ mass to each zero of $Q_{2^n}(\cdot; \mu_{K(\gamma)})$. From Remark 4.8 in [Alpan and Goncharov 16] the recurrence coefficients $(a_k)_{k=1}^{2^n-1}$, $(b_k)_{k=1}^{2^n}$ for μ_{E_n} are exactly those of $\mu_{K(\gamma)}$. This implies that (see e.g., Theorem 1.3.5 in [Simon 11]) the Christoffel numbers corresponding to 2^nth orthogonal polynomial for $\mu_{K(\gamma)}$ are also equal to $1/2^n$.

Let

$$
H_{\mu_{K(\gamma)}^n} = \begin{pmatrix}
b_1 & a_1 & & & & \\
a_1 & b_2 & a_2 & & & \\
& a_2 & \ddots & \ddots & & \\
& & \ddots & \ddots & a_{2^n-1} & \\
& & & a_{2^n-1} & b_{2^n}
\end{pmatrix}, \qquad (2\text{-}7)
$$

where the coefficients $(a_k)_{k=1}^{2^n-1}$, $(b_k)_{k=1}^{2^n}$ are the Jacobi parameters for $\mu_{K(\gamma)}$. Then, the set of eigenvalues of $H_{\mu_{K(\gamma)}^n}$ is exactly the zero set of $Q_{2^n}(\cdot; \mu_{K(\gamma)})$. Moreover, by [Golub and Welsch 69], the square of first component of normalized eigenvectors gives one of the Christoffel numbers, which in our case is equal to $1/2^n$. For each $n \in \{1, \ldots, 14\}$, using gauss.m, we computed the eigenvalues and first component of normalized eigenvectors of $H_{\mu_{K(\gamma)}^n}$ where the coefficients are obtained from Algorithm 1. We compared these values with the zeros obtained by part (h) of Theorem 2.1 and $1/2^n$, respectively. For each n, let $\{t_k^n\}_{k=1}^{2^n}$ be the set of eigenvalues for $H_{\mu_{K(\gamma)}^n}$ and $\{q_k^n\}_{k=1}^{2^n}$ be the set of zeros where we enumerate these sets so that the smaller the index they have, the value will be smaller. Let $\{w_k^n\}_{k=1}^{2^n}$ be the set of squared first component of normalized eigenvectors. We plotted (see Figures 1 and 2) $R_n^1 := (1/2^n)(\sum_{k=1}^{2^n} |t_k^n - q_k^n|)$ and $R_n^2 := (1/2^n)(\sum_{k=1}^{2^n} |(1/2^n) - w_k^n|)$. This numerical experiment shows the reliability of Algorithm 1. One can compare these values with Figure 2 in [Mantica 15b].

3. Recurrence coefficients

It was shown (for the stretched version of this set but similar arguments are valid for this case also) in [Alpan and Goncharov 15b] that $K(\gamma)$ is a generalized polynomial Julia set (see e.g., [Brück 01, Brück and Büger 03, Büger 97] for a discussion on generalized Julia sets) if $\inf \gamma_k > 0$, that is $K(\gamma) := \partial\{z \in \overline{\mathbb{C}} : F_n(z) \to \infty$ locally uniformly$\}$. Let $J(f)$ be the (autonomous) Julia set for $f(z) = z^2 - c$ for some $c > 2$. Since $(f_n)_{n=1}^\infty$ is a sequence of quadratic polynomials, it is natural to ask that to what extent $H_{\mu_{J(f)}}$ and $H_{\mu_{K(\gamma)}}$ have similar behavior. Compare for example Theorem 4.7 in [Alpan and Goncharov 16] with Section 3 in [Bessis et al. 88].

The recurrence coefficients for $\mu_{J(f)}$ can be ordered according to their indices, see (IV.136)–(IV.138) in [Bessis 90]. We obtain similar results for $\mu_{K(\gamma)}$ in our numerical experiments in each of the four models. That is, the numerical experiments suggest that $\min_{i \in \{1, \ldots, 2^n\}} a_i = a_{2^n}$ for $n \leq 14$, and it immediately follows from (2–2) and (2–6) that $\max_{n \in \mathbb{N}} a_n = a_1$. Thus, we make the following conjecture:

Conjecture 3.1. For $\mu_{K(\gamma)}$ we have $\min_{i \in \{1, \ldots, 2^n\}} a_i = a_{2^n}$ and in particular $\liminf_{s \to \infty} a_{2^s} = \liminf_{n \to \infty} a_n$.

A non-polar compact set $K \subset \mathbb{R}$ which is regular with respect to the Dirichlet problem is called a Parreau–Widom set if $\sum_{k=1}^\infty G_K(e_k) < \infty$ where $\{e_k\}_k$ is the set of critical points, which is at most countable, of G_K. Parreau–Widom sets have positive Lebesgue measure. It is also known that (see e.g., Remark 4.8 in [Alpan and Goncharov 16]) $\liminf a_n > 0$ for μ_K provided that K is Parreau–Widom. For more on Parreau–Widom sets, we refer the reader to [Christiansen 12, Yudistkii 12].

By part (e) of Theorem 2.1, $\liminf a_n > 0$ for $\mu_{K(\gamma)}$ provided that $\sum_{s=1}^\infty \sqrt{\varepsilon_s} < \infty$. It also follows from Remark 4.8 in [Alpan and Goncharov 16] and [Dombrowski 78] that if the a_n's associated with $\mu_{K(\gamma)}$ satisfy $\liminf a_n = 0$ then $K(\gamma)$ has zero Lebesgue measure. Hence asymptotic behavior of the a_n's is also important for understanding the Hausdorff dimension of $K(\gamma)$. We computed $v_n := a_{2^n}/a_{2^{n+1}}$ (see Figures 3 and 4) for $n = 1, \ldots, 13$ in order to find for which γ's $\liminf a_n = 0$. We assume here Conjecture 3.1 is correct.

In Model 1, v_n is very close to 1 which is expected since for this case $\liminf a_n > 0$. In other models, it seems that $(v_n)_{n=1}^{13}$ seems to behave like a constant. Thus, this experiment can be read as follows: If $\sum_{s=1}^\infty \sqrt{\varepsilon_s} < \infty$ does not hold then $\liminf a_n = 0$. So, we conjecture:

Conjecture 3.2. For a given $\gamma = (\gamma_k)_{k=1}^\infty$, let $\varepsilon_k := 1 - 4\gamma_k$ for each $k \in \mathbb{N}$. Then $K(\gamma)$ is of positive Lebesgue measure if and only if $\sum_{s=1}^\infty \sqrt{\varepsilon_s} < \infty$ if and only if $\liminf a_n > 0$.

A more interesting problem is whether $H_{\mu_{K(\gamma)}}$ is almost periodic or at least asymptotically almost periodic. Since $(b_n)_{n=1}^\infty$ is a constant sequence, we only need to deal with $(a_n)_{n=1}^\infty$.

For a measure μ with an infinite compact support $\text{supp}(\mu)$, let δ_n be the normalized counting measure on the zeros of $Q_n(\cdot; \mu)$. If there is a v such that $\delta_n \to v$ then v is called the density of states (DOS) measure for H_μ. Besides, $\int_{-\infty}^x dv$ is called the integrated density of states (IDS). For $H_{\mu_{K(\gamma)}}$, the DOS measure is automatically (see Theorem 1.7 and Theorem 1.12 in [Simon 11] and also [Widom 67]) $\mu_{K(\gamma)}$. Therefore, if x is chosen from one of the gaps (by a gap of a compact set on $K \subset \mathbb{R}$ we mean a bounded component of $R \setminus K$) of $\text{supp}(\mu_{K(\gamma)})$, that is $x \in (c_i, d_i)$ (see part (i) of Theorem 2.1), then the value of the IDS is equal to $m2^{-n}$ which does not exceed 1 and also for each $m, n \in \mathbb{N}$ with $m2^{-n} < 1$ there is a gap (c_j, d_j) such that the IDS takes the value $m2^{-n}$.
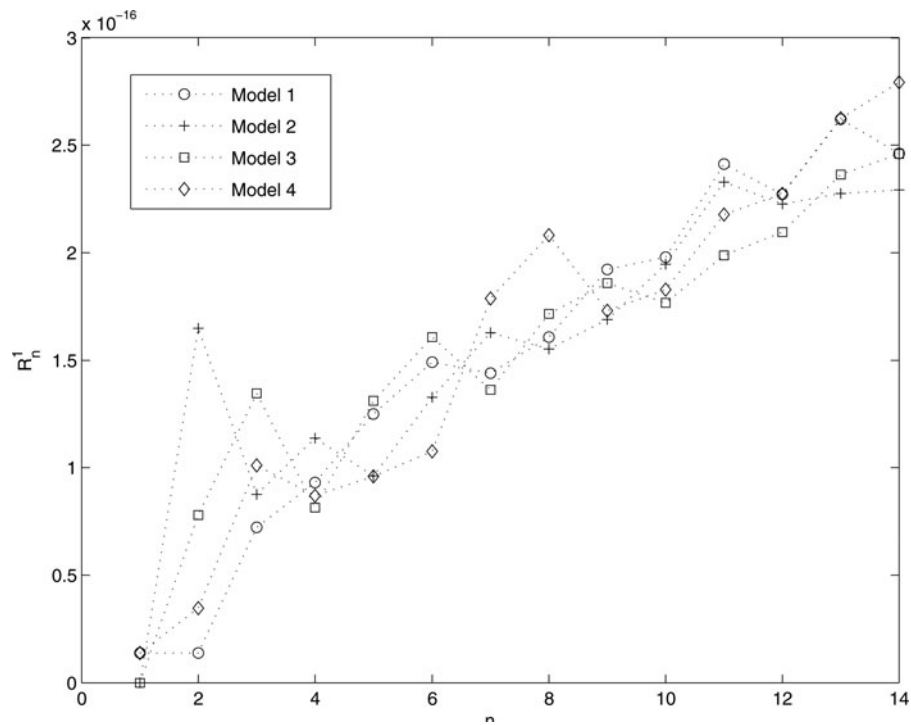
Figure 1. Errors associated with eigenvalues.

For an almost periodic sequence $c = (c_n)_{n=1}^{\infty}$, the $\mathbb{Z}$-module of the real numbers modulo 1 generated by ω satisfying

$$\left\{ \omega : \lim_{n \to \infty} \frac{1}{N} \sum_{n=1}^{N} \exp\left(2\pi i n \omega\right) c_n \neq 0 \right\}$$

is called the frequency module for c and it is denoted by $\mathcal{M}(c)$. The frequency module is always countable and c can be written as a uniform limit of Fourier series where the frequencies are chosen among $\mathcal{M}(c)$. For an almost periodic Jacobi matrix H with coefficients $a = (a_n)_{n=1}^{\infty}$ and $b = (b_n)_{n=1}^{\infty}$, the frequency module $\mathcal{M}(H)$ is the module generated by $\mathcal{M}(a)$ and $\mathcal{M}(b)$. It was shown in Theorem III.1 in [Delyon and Souillard 83] that for an almost periodic H, the values of IDS in gaps belong to $\mathcal{M}(H)$. Moreover (see e.g., Theorem 2.4 in [Geronimo 88]), an asymptotically almost periodic Jacobi matrix has
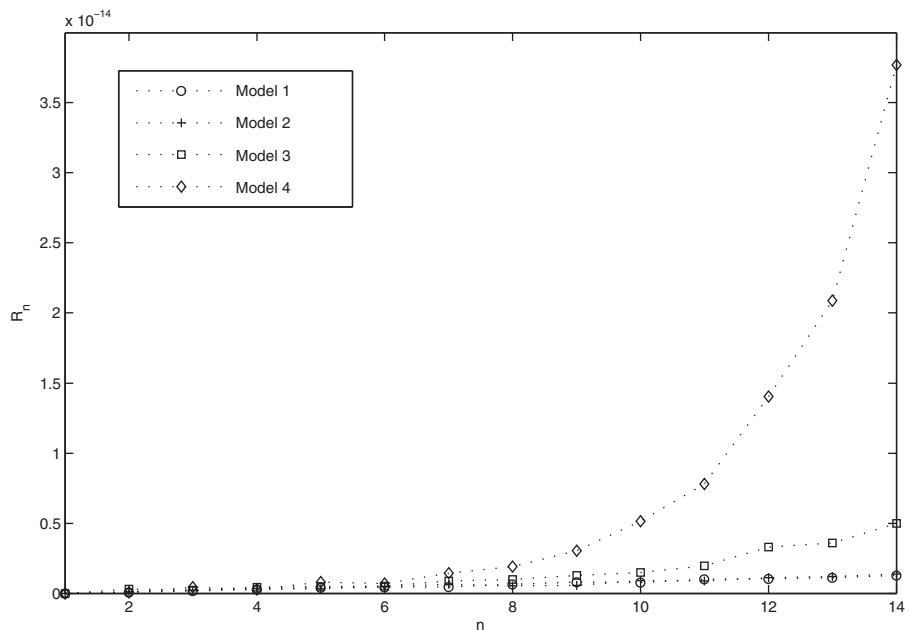

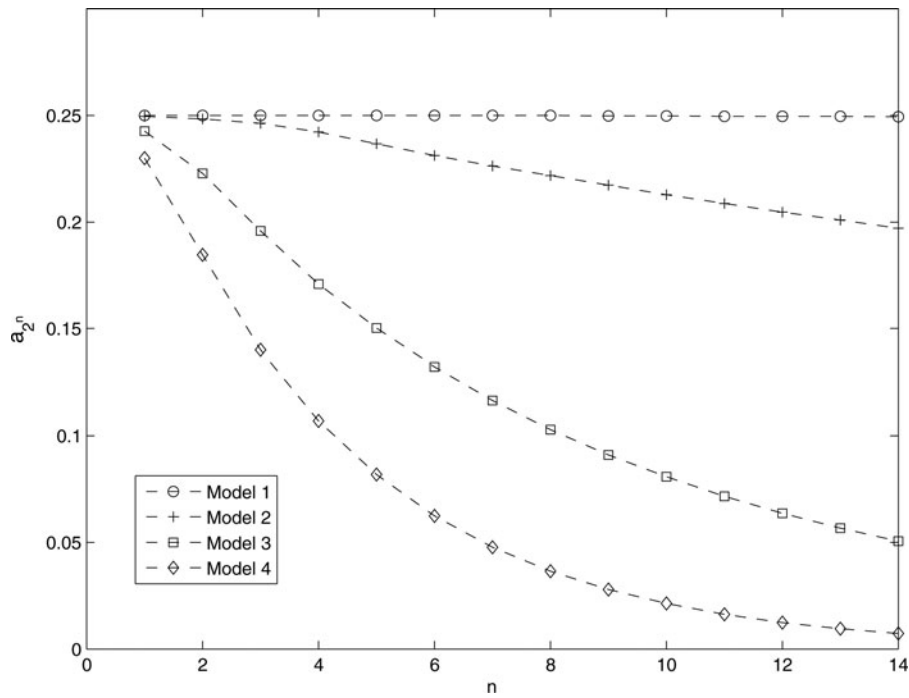**Figure 2.** Errors associated with eigenvectors.

Figure 3. The values of outdiagonal elements of Jacobi matrices at the indices of the form 2^s.

the same DOS measure with the almost periodic limit of it.

In order to examine almost periodicity of the a_n's for $\mu_{K(\gamma)}$, we computed the discrete Fourier transform $(\widehat{a}_n)_{n=1}^{2^{14}}$ for the first 2^{14} coefficients for each model where frequencies run from 0 to 1. We normalized $|\widehat{a}|^2$ dividing it by $\sum_{n=1}^{2^{14}} |\widehat{a}_n|^2$. We plotted (see Figure 5) this

normalized power spectrum while we did not plot the peak at 0, by detrending the transform.

There are only a small number of peaks in each case compared to 2^{14} frequencies which points out almost periodicity of coefficients. We consider only Model 1 here although we have similar pictures for the other models. The highest 10 peaks are at 0.5, 0.25, 0.75, 0.375, 0.625,
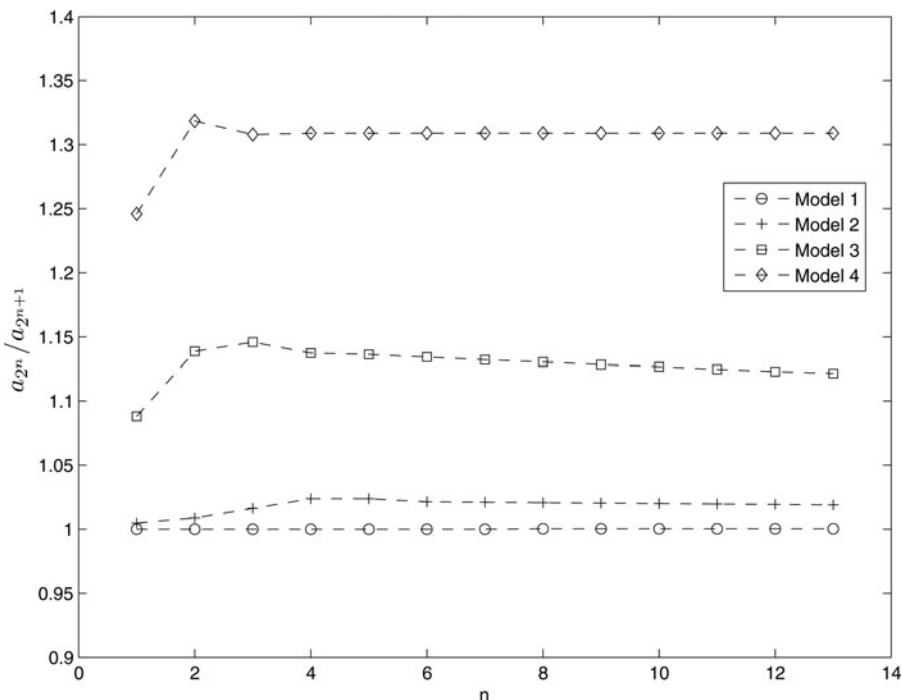

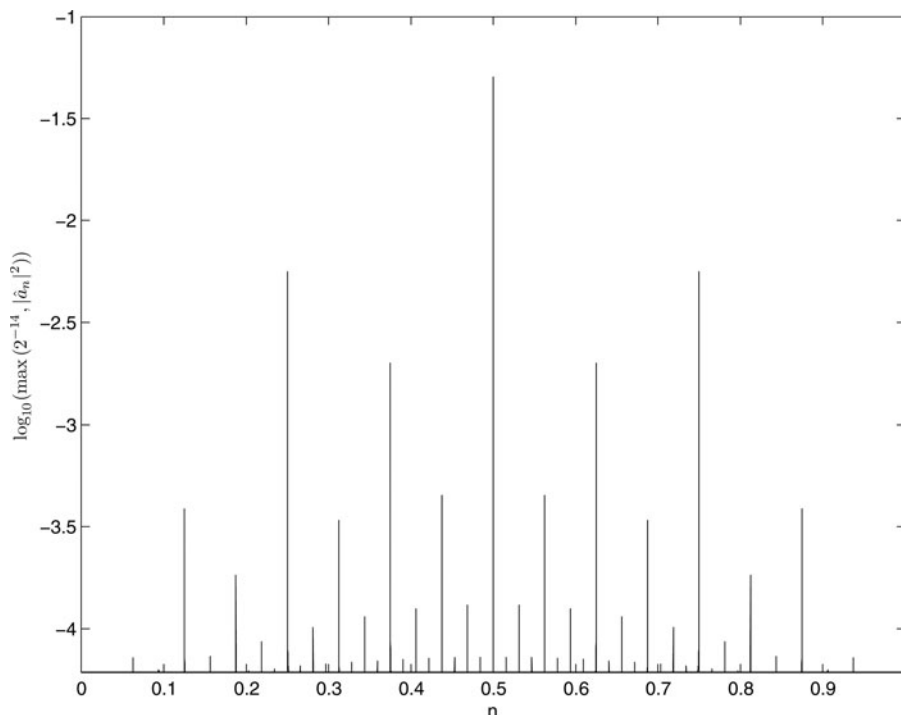**Figure 4.** The ratios of outdiagonal elements of Jacobi matrices at the indices of the form 2^s.

Figure 5. Normalized power spectrum of the a_n's for Model 1.

0.4375, 0.5625, 0.125, 0.875, 0.3125. All these values are of the form $m2^{-n}$ where $n \leq 4$. This is an important indicator of almost periodicity as these frequencies are exactly the values of IDS for $H_{\mu_{K(\gamma)}}$ in the gaps which appear earlier in the construction of the Cantor set. The following conjecture follows naturally from the above discussion.

Conjecture 3.3. For any γ, $(a_n)_{n=1}^{\infty}$ for $H_{\mu_{K(\gamma)}}$ is asymptotically almost periodic where the almost periodic limit has frequency module equal to $\{m2^{-n}\}_{m,n \in \{N_0\}}$ modulo 1.

4. Widom factors

Let $K \subset \mathbb{C}$ be a non-polar compact set. Then the unique monic polynomial T_n of degree n satisfying

$$\|T_n\|_{L^{\infty}(K)} = \min \{\|P_n\|_{L^{\infty}(K)} :$$

$$P_n \text{ complex monic polynomial of degree } n\}$$

is called the *n*th *Chebyshev polynomial* on K.

We define the *n*th Widom factor for the sup-norm on K by $W_n(K) = \|T_n\|_{L^{\infty}(K)}/(\text{Cap}(K))^n$. It is due to Schiefermayr [Schiefermayr 08] that $W_n(K) \geq 2$ if $K \subset \mathbb{R}$. It is also known that (see e.g., [Fekete 23, Szegő 24]) $\|T_n\|_{L^{\infty}(K)}^{1/n} \to \text{Cap}(K)$ as $n \to \infty$. This implies a theoretical constraint on the growth rate of $W_n(K)$, that is $(1/n) \log W_n(K) \to 0$ as $n \to \infty$. See for example [Totik 09, Totik 14, Totik and Yuditskii 15] for further discussion.

Theorem 4.4 in [Goncharov and Hatinoğlu 15] says that for each sequence $(M_n)_{n=1}^{\infty}$ satisfying $\lim_{n \to \infty} (1/n) \log M_n = 0$, there is a γ such that

$W_n(K(\gamma)) > M_n$. On the other hand, for many compact subsets of $\mathbb{C}$ (see e.g., [Andrievskii 16, Christiansen et al., Totik and Varga, Widom 69]) the sequence of Widom factors for the sup-norm is bounded. In particular, this is valid for Parreau–Widom sets on $\mathbb{R}$, see [Christiansen et al.]. It would be interesting to find (if any) a non-Parreau–Widom set K on $\mathbb{R}$ such that it is regular with respect to the Dirichlet problem and $(W_n(K))_{n=1}^{\infty}$ is bounded. Note that if K is a non-polar compact subset of $\mathbb{R}$ which is regular with respect to the Dirichlet problem, then by Theorem 4.2.3 in [Ransford 95] and Theorem 5.5.13 in [Simon 11] we have $\text{supp}(\mu_K) = K$. In this case, we have $W_n^2(\mu_K) \leq W_n(K)$ since $\|Q_n(\cdot; \mu_K)\|_{L^2(\mu_K)} \leq \|T_n\|_{L^2(\mu_K)} \leq \|T_n\|_{L^{\infty}(K)}$. Therefore, it is possible to formulate the above problem in a weaker form: Is there a non-Parreau–Widom set $K \subset \mathbb{R}$ which is regular with respect to the Dirichlet problem such that $(W_n^2(\mu_K))_{n=1}^{\infty}$ is bounded?

In [Alpan and Goncharov 15a], the authors following [Barnsley et al. 83] studied $(W_n^2(\mu_{J(f)}))_{n=1}^{\infty}$ where $f(z) = z^3 - \lambda z$ for $\lambda > 3$ and showed that the sequence is unbounded. For this particular case, the Julia set is a compact subset of $\mathbb{R}$ which has zero Lebesgue measure. It is always true for a polynomial autonomous Julia set $J(f)$ on $\mathbb{R}$ that $\text{supp}(\mu_{J(f)}) = J(f)$ since $J(f)$ is regular with respect to the Dirichlet problem by [Mañé and Da Rocha 92]. Now, let us show that $(W_n^2(\mu_{J(f)}))_{n=1}^{\infty}$ is unbounded when $f(z) = z^2 - c$ and $c > 2$. These quadratic Julia sets are zero Lebesgue measure Cantor sets on $\mathbb{R}$ and therefore

not Parreau–Widom. See [Brolin 65] for a deeper discussion on this particular family.

Theorem 4.1. *Let* $f(z) = z^2 - c$ *for* $c \geq 2$. *Then* $(W_n^2(\mu_{J(f)}))_{n=1}^{\infty}$ *is bounded if and only if $c = 2$.*

Proof. If $c = 2$ then $J(f) = [-2, 2]$. This implies that $(W_n^2(\mu_{J(f)}))_{n=1}^{\infty}$ is bounded since $J(f)$ is Parreau–Widom.

Let $c \neq 2$. Then $\lim_{n \to \infty} a_{2^n} = 0$ (see e.g., Section IV.5.2 in [Bessis 90]) where the a_n's are the recurrence coefficients for $\mu_{J(f)}$ and $\mathrm{Cap}(J(f)) = 1$ by [Brolin 65]. Since $Q_{2^{n+1}}(\cdot; \mu_{J(f)}) = Q_{2^n}^2(\cdot; \mu_{J(f)}) - c$ by Theorem 3 in [Barnsley et al. 82], we have $W_{2^n}^2(\mu_{J(f)}) = \|Q_{2^n}(\cdot; \mu_{J(f)})\|_{L^2(\mu_{J(f)})} = \sqrt{c}$ for all $n \geq 1$. Moreover,

$$W_{2^n-1}^2(\mu_{J(f)}) = \frac{W_{2^n}^2(\mu_{J(f)})}{a_{2^n}} = \frac{\sqrt{c}}{a_{2^n}}. \qquad (4\text{--}1)$$

Hence $\lim_{n \to \infty} W_{2^n-1}^2(\mu_{J(f)}) = \infty$ as $\lim_{n \to \infty} a_{2^n} = 0$. This completes the proof. $\square$

In [Alpan and Goncharov 16], it was shown that $(W_n^2(\mu_{K(\gamma)}))_{n=1}^{\infty}$ is unbounded if $\gamma_k \leq 1/6$ for all $k \in \mathbb{N}$. We want to examine the behavior of $(W_n^2(\mu_{K(\gamma)}))_{n=1}^{\infty}$ provided that $K(\gamma)$ is not Parreau–Widom. By [Alpan and Goncharov 16], $(W_{2^n}(\mu_{K(\gamma)})) \geq \sqrt{2}$ for all $n \in \mathbb{N}_0$ for any choice of γ. Hence, we also have

$$W_{2^n-1}^2(\mu_{K(\gamma)}) = W_{2^n}^2(\mu_{K(\gamma)}) \frac{\mathrm{Cap}(\mu_{K(\gamma)})}{a_{2^n}}$$

$$\geq \frac{\sqrt{2}\,\mathrm{Cap}(\mu_{K(\gamma)})}{a_{2^n}} \qquad (4\text{--}2)$$

for all $n \in \mathbb{N}$.

If we assume that Conjecture 3.1 and Conjecture 3.2 are correct then $\liminf_{n \to \infty} a_{2^n} = 0$ as soon as $K(\gamma)$ is not Parreau–Widom. If $\liminf_{n \to \infty} a_{2^n} = 0$ then $\limsup_{n \to \infty} W_{2^n-1}^2(\mu_{K(\gamma)}) = \infty$ by (4–2). Thus, the numerical experiments indicate the following:

Conjecture 4.2. $K(\gamma)$ is a Parreau–Widom set if and only if $(W_n^2(\mu_{K(\gamma)}))_{n=1}^{\infty}$ is bounded if and only if $(W_n(K(\gamma)))_{n=1}^{\infty}$ is bounded.

Let K be a union of finitely many compact nondegenerate intervals on $\mathbb{R}$ and ω be the Radon–Nikodym derivative of μ_K with respect to the Lebesgue measure on the line. Then μ_K satisfies the Szegő condition: $\int_K \omega(x) \log \omega(x)\, dx > -\infty$. This implies by Corollary 6.7 in [Christiansen et al. 11] that $(W_n^2(\mu_K))_{n=1}^{\infty}$ is asymptotically almost periodic. If K is a Parreau–Widom set, μ_K satisfies the Szegő condition by [Pommerenke 76]. We plotted (see Figure 6) the Widom–Hilbert factors for Model 1 for the first 2^{20} values, and it seems that $\limsup W_n^2(\mu_{(K(\gamma))}) \neq \sup W_n^2(\mu_{(K(\gamma))})$. For Model

1, we plotted (see Figure 7) the power spectrum for $(W_n^2(\mu_K))_{n=1}^{2^{14}}$ where we normalized $|\widehat{W}^2|^2$ dividing it by $\sum_{n=1}^{2^{14}} |\widehat{W}_n^2(\mu_K)|^2$. Frequencies run from 0 to 1 here and we did not plot the big peak at 0.

Clearly, there are only a few peaks as in (see Figure 5) which is an important indicator of almost periodicity. The highest 10 peaks are at 0.5, 0.00006103515625, 0.25, 0.75, 0.125, 0.875, 0.375, 0.625, 0.0625, 0.9375. These values are quite different than those of peaks in Figure 5. This may be an indicator of a different frequency module of the almost periodic limit. By Conjecture 4.2, $(W_n^2(\mu_{K(\gamma)}))_{n=1}^{\infty}$ is unbounded and cannot be asymptotically almost periodic if $K(\gamma)$ is not Parreau–Widom. We make the following conjecture:

Conjecture 4.3. $(W_n^2(\mu_{K(\gamma)}))_{n=1}^{\infty}$ is asymptotically almost periodic if and only if $K(\gamma)$ is Parreau–Widom. If $K(\gamma)$ is Parreau–Widom, then the almost periodic limit's frequency module includes the module generated by $\{m2^{-n}\}_{m,n \in \{\mathbb{N}_0\}}$ modulo 1.

5. Spacing properties of orthogonal polynomials and further discussion

For a measure μ having support on $\mathbb{R}$, let $Z_n(\mu) := \{x : Q_n(x; \mu) = 0\}$. For $n > 1$ with $n \in \mathbb{N}$, we define $M_n(\mu)$ by

$$M_n(\mu) := \inf_{\substack{x,x' \in Z_n(\mu) \\ x \neq x'}} |x - x'|.$$

For a given $\gamma = (\gamma_k)_{k=1}^{\infty}$, let us enumerate the elements of $Z_N(\mu_{K(\gamma)})$ by $x_{1,N} < \cdots < x_{N,N}$. The behaviors of $(M_N(\mu_{K(\gamma)}))_{N=1}^{\infty}$, in other words, the global behavior of the spacing of the zeros, were investigated in [Alpan 16]. Here, we numerically study some aspects of the local behavior of the zeros.

We consider only Model 1 since the calculations give similar results for the other models. For $N = 2^3, 2^4, \ldots, 2^{14}$, let $A_{n,N} := |x_{2n,N} - x_{2n-1,N}|$ where $n \in \{1, \ldots, N/2\}$. We computed (see Figure 8) $A_N := \max_{n,m \in \{1,\ldots,N/2\}} \frac{A_{n,N}}{A_{m,N}}$ for each such N.

$(A_{2^n})_{n=3}^{14}$ increases fast and this indicates that $(A_{2^n})_{n=2}^{\infty}$ is unbounded.

For $N = 2^{14}$ and $s = 2, s = 6$ we plotted (see Figure 9) $A_{s,N}/A_{1,N}$. These ratios tend to converge fast.

In the next conjecture, we exclude the case of small γ for the following reason: Let $\gamma = (\gamma_k)_{k=1}^{\infty}$ satisfy $\sum_{k=1}^{\infty} \gamma_k = M < \infty$ with $\gamma_k \leq 1/32$ for all $k \in \mathbb{N}$ and $\delta_k := \gamma_1 \cdots \gamma_k$. Then $A_{j,2^k} \leq \exp(16M)\delta_{k-1}$ for all $k > 1$ by Lemma 6 in [Goncharov 14]. By Lemma 4 and
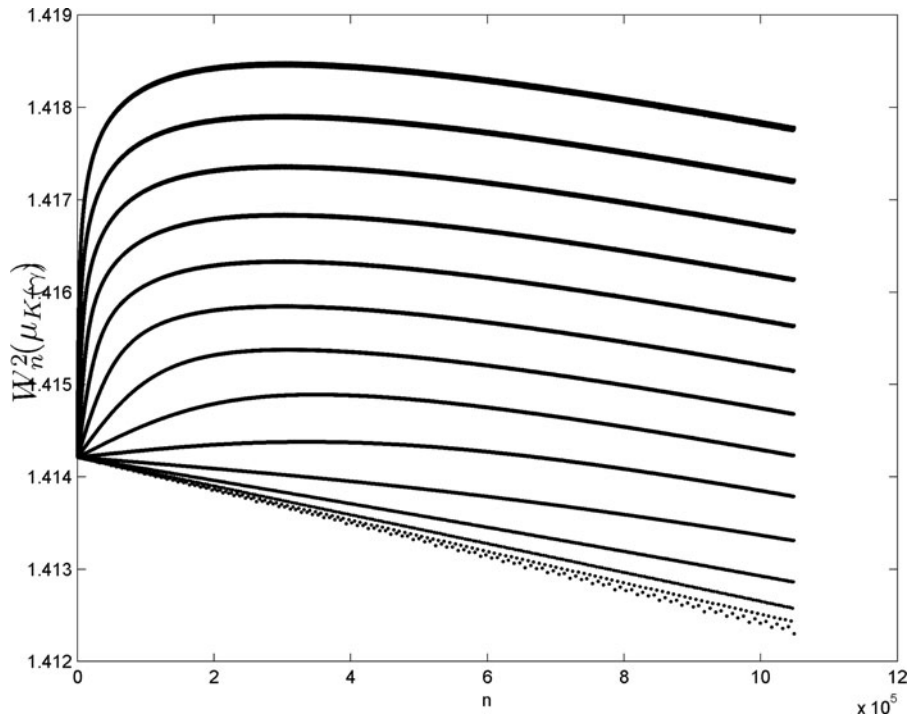
Figure 6. Widom–Hilbert factors for Model 1.

Lemma 6 in [Goncharov 14], we conversely have $A_{j,2^k} \geq (7/8)\delta_{k-1}$. Therefore, $A_{2^k} \leq (8/7)\exp(16M)$. Hence, $(A_{2^n})_{n=2}^{\infty}$ is bounded.

Conjecture 5.1. For each $\gamma = (\gamma_k)_{k=1}^{\infty}$ with $\inf_k \gamma_k > 0$, $(A_{2^k})_{k=1}^{\infty}$ is an unbounded sequence. If $s = 2^k$ for some

$k \in \mathbb{N}$, there is a $c_0 \in \mathbb{R}$ depending on k such that

$$\lim_{n \to \infty} \frac{A_{s,2^n}}{A_{1,2^n}} = c_0.$$

For the parameters $c > 3$, $H_{\mu_{J(f)}}$ is almost periodic where $f(z) = z^2 - c$, see [Bellissard et al. 82]. It was
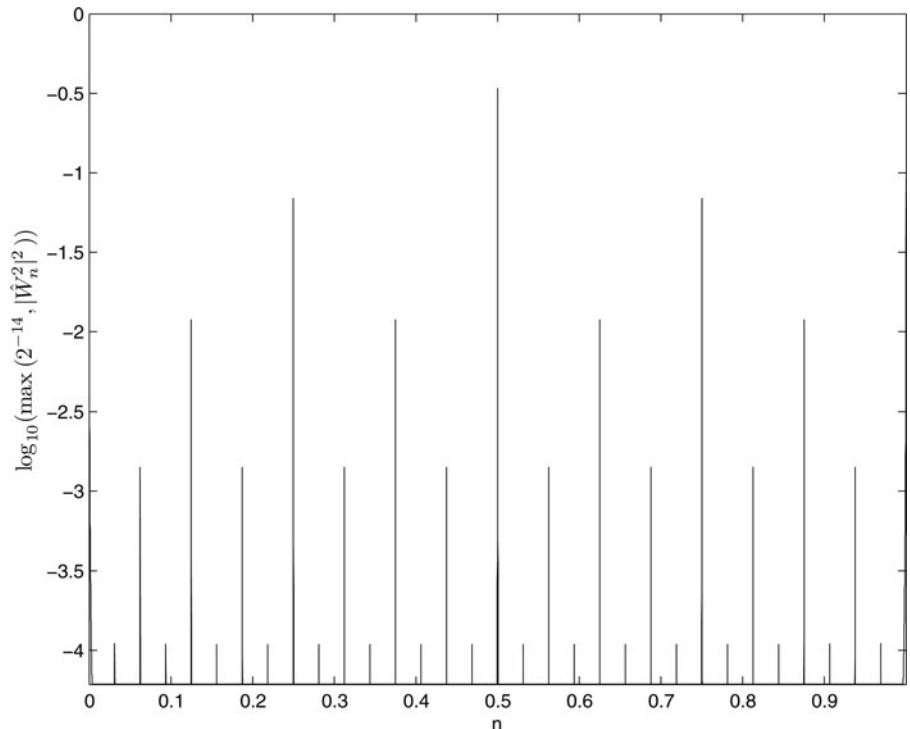

Figure 7. Normalized power spectrum of the $W_n^2(\mu_{K(\gamma)})$'s for Model 1.
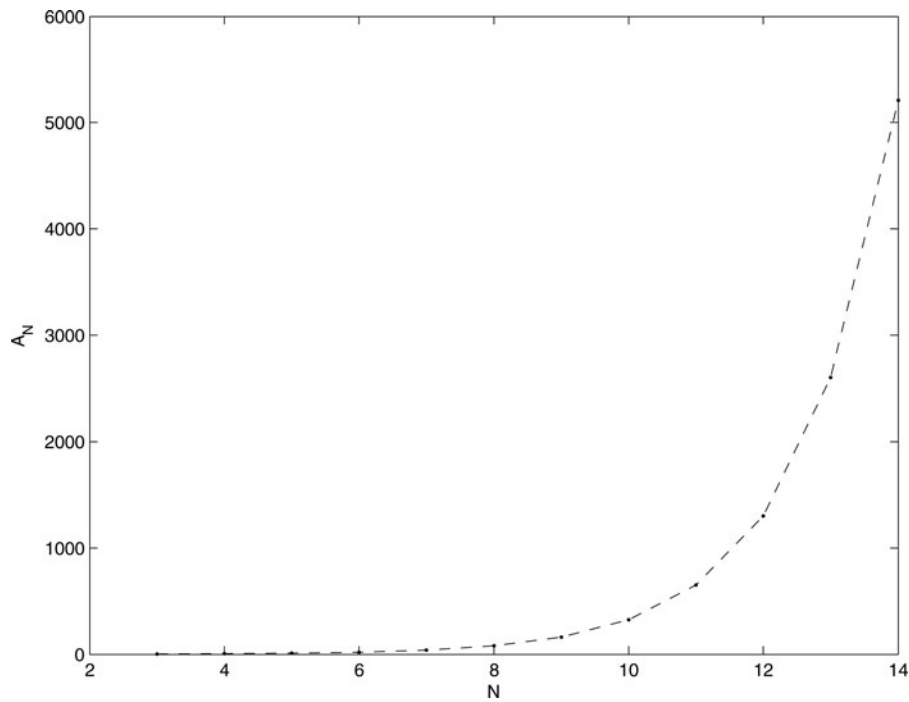
Figure 8. Maximal ratios of the distances between adjacent zeros.

conjectured in p. 123 of [Bellissard 92] (see also [Bellissard et. al 05] and [Peherstorfer et. al. 06] for later developments concerning this conjecture) that $H_{\mu_{J(f)}}$ is always almost periodic as soon as $c > 2$. Therefore, if this conjecture is true, then we have the following: $H_{\mu_{J(f)}}$ is almost periodic if and only if $J(f)$ is non-Parreau–Widom.

We did not make any distinction between asymptotic almost periodicity and almost periodicity in Sections 3 and 4 since these two cases are indistinguishable numerically. But we remark that if $\liminf a_n \neq 0$ then the asymptotics $\lim_{j\to\infty} a_{j\cdot 2^s+n} = a_n$ cease to hold immediately. We do not expect $H_{\mu_{K(\gamma)}}$ to be almost periodic for the Parreau–Widom case for that reason. For a parameter $\gamma = (\gamma_s)_{s=1}^\infty$
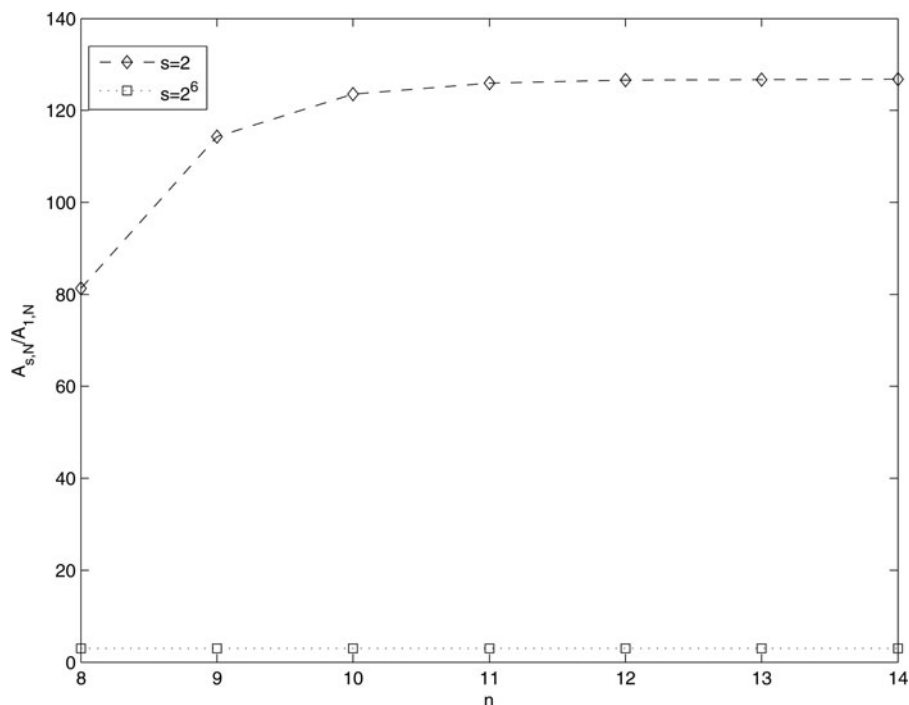

Figure 9. Ratios of the distances between prescribed adjacent zeros.

such that $\lim_{j\to\infty} a_{j\cdot 2^s+n} = a_n$ holds for each s and n it is likely that $H_{\mu_{K(\gamma)}}$ is almost periodic. These asymptotics hold only for the non-Parreau–Widom case, but it is unclear that if these hold for all parameters making $K(\gamma)$ non-Parreau–Widom.

Hausdorff dimension of a unit Borel measure μ supported on $\mathbb{C}$ is defined by $\dim(\mu) := \inf\{HD(K) : \mu(K) = 1\}$ where $HD(\cdot)$ stands for the Hausdorff dimension of the given set. Hausdorff dimension of equilibrium measures were studied for many fractals (see [Makarov 99] for an account of the previous results) and in particular for autonomous polynomials Julia sets (see e.g., [Przytycki 85]). If f is a nonlinear monic polyomial and $J(f)$ is a Cantor set then by p. 176 in[Przytycki 85] (see also p. 22 in [Makarov 99]) we have $\dim(\mu_{J(f)}) < 1$. For $K(\gamma)$, $\sum_{s=1}^{\infty} \sqrt{\varepsilon_s} < \infty$ implies that $\dim(\mu_{K(\gamma)}) = 1$ since $\mu_{(K(\gamma))}$ and the Lebesgue measure restricted to $K(\gamma)$ (see 4.6.1 in [Sodin and Yuditskii 97]) are mutually absolutely continuous. Moreover, our numerical experiments suggest that $K(\gamma)$ has zero Lebesgue measure for non-Parreau–Widom case. It may also be true that $\dim(\mu_{K(\gamma)}) < 1$ for this particular case. Hence, it is an interesting problem to find a systematic way of calculating the dimension of equilibrium measures of $K(\gamma)$ and generalized Julia sets in general.

Funding

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