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Traveling wave solutions of degenerate coupled multi-KdV equations

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Traveling wave solutions of degenerate coupled ℓ-KdV equations are studied. Due to symmetry reduction these equations reduce to one ordinary differential equation (ODE), i.e., $(f')^2 = P_n(f)$ where $P_n(f)$ is a polynomial function of f of degree $n = \ell + 2$, where $\ell \geq 3$ in this work. Here ℓ is the number of coupled fields. There is no known method to solve such ordinary differential equations when $\ell \geq 3$. For this purpose, we introduce two different types of methods to solve the reduced equation and apply these methods to degenerate three-coupled KdV equation. One of the methods uses the Chebyshev's theorem. In this case, we find several solutions, some of which may correspond to solitary waves. The second method is a kind of factorizing the polynomial $P_n(f)$ as a product of lower degree polynomials. Each part of this product is assumed to satisfy different ODEs. *Published by AIP Publishing.*
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I. INTRODUCTION

The system of degenerate coupled multi-field KdV equations is given as[1–6]

$$u_t = \frac{3}{2}uu_x + q_x^2,$$

$$q_t^2 = q^2 u_x + \frac{1}{2}uq_x^2 + q_x^3,$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$q_t^{\ell-1} = q^{\ell-1}u_x + \frac{1}{2}uq_x^{\ell-1} + v_x,$$

$$v_t = -\frac{1}{4}u_{xxx} + vu_x + \frac{1}{2}uv_x, \tag{1}$$

where $q^1 = u$ and $q^\ell = v$. In Refs. 1–7, it was shown that this system is also a degenerate KdV system of rank one. In a previous work,[8] we focused on Equation (1) for $\ell = 2$. We reduced this equation into an ODE, $(f')^2 = P_4(f)$, where $P_4(f)$ is a polynomial function of degree four. We analyzed all possible cases about the zeros of $P_4(f)$. Due to this analysis, we determined the cases when the solution is periodic or solitary. When the polynomial has one double f_2 and two simple zeros f_1 and f_3 with $f_1 < f_2 < f_3$, or one triple and one simple zeros, the solution is solitary. Other cases give periodic or non-real solutions. By using the Jacobi elliptic functions,[9] we obtained periodic solutions and all solitary wave solutions which rapidly decay to some constants, explicitly. We have also shown that there are no real asymptotically vanishing traveling wave solutions for $\ell = 2$. Indeed, we have the following theorem for the degenerate coupled ℓ-KdV equation, $\ell \geq 2$. The degenerate coupled ℓ-KdV equation can be reduced to the equation

$$(f')^2 = P_{\ell+2}(f) \tag{2}$$

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by taking ℓ functions as $u(x,t) = q^1(x,t) = f(\xi)$, $q^2(x,t) = f_2(\xi)$, ..., $v(x,t) = q^\ell = f_\ell(\xi)$, where $\xi = x - ct$ in (1). Here $P_{\ell+2}(f)$ is a polynomial of f of degree $\ell + 2$. If we apply the asymptotically vanishing boundary conditions to (2), we have

$$(f')^2 = Bf^2(f + 2c)^\ell. \tag{3}$$

Theorem 1.1. *When $\ell = odd$, we have $B > 0$, and then the degenerate coupled ℓ-KdV equation has a real traveling wave solution with asymptotically vanishing boundary conditions, but when $\ell = even$, the constant $B < 0$. Hence the equation does not have a real traveling wave solution with asymptotically vanishing boundary conditions.*

Proof. Consider the degenerate coupled ℓ-KdV equation (1). Let $u(x,t) = q^1(x,t) = f(\xi)$, $q^2(x,t) = f_2(\xi)$, ..., $v(x,t) = q^\ell = f_\ell(\xi)$, where $\xi = x - ct$. By using the first $\ell - 1$ equations, we obtain all functions $f_i(\xi)$, $i = 2, 3, \ldots, \ell$ as a polynomial of $f(\xi)$. We get

$$f_2(\xi) = Q_2(f) = -cf - \frac{3}{4}f^2 + d_1 = -\alpha_2 f^2 + A_1(f),$$

$$f_3(\xi) = Q_3(f) = \frac{3}{2}cf^2 + \frac{1}{2}f^3 + (c^2 - d_1)f + d_2$$
$$= \alpha_3 f^3 + A_2(f),$$

$$f_4(\xi) = Q_4(f) = -\frac{5}{16}f^4 - \frac{3}{2}cf^3 + \left(-\frac{9}{4}c^2 + \frac{3}{4}d_1\right)f^2 + (-c^3 + cd_1 - d_2)f + d_3$$
$$= \alpha_4 f^4 + A_3(f),$$
$$\vdots$$
$$f_\ell(\xi) = Q_\ell(f) = (-1)^{\ell+1}\alpha_\ell f^\ell + A_{\ell-1}(f),$$

where $\alpha_j > 0$ are constants, and $A_i(f)$ and $Q_j(f)$ are polynomials of f of degree i and j, $i = 1, 2, \ldots, \ell - 1$ and $j = 2, 3, \ldots, \ell$, respectively.

Now use the ℓth equation. We obtain

$$\frac{1}{4}f''' = (-1)^{\ell-1}\alpha_\ell f'f^\ell\left(1 + \frac{\ell}{2}\right) + c(-1)^{\ell-1}\ell\alpha_\ell f'f^{\ell-1} + f'A_{\ell-1} + \frac{1}{2}f\frac{\partial A_{\ell-1}(f)}{\partial x} - \frac{\partial A_{\ell-1}(f)}{\partial t}. \tag{4}$$

Integrating the above equation once, we get

$$\frac{1}{4}f'' = (-1)^{\ell-1}\frac{\alpha_\ell}{\ell+1}f^{\ell+1}\left(1 + \frac{\ell}{2}\right) + R_\ell(f).$$

By using f' as an integrating factor, we integrate once more. Finally, we obtain

$$(f')^2 = (-1)^{\ell-1}\frac{8\alpha_\ell}{(\ell+1)(\ell+2)}f^{\ell+2}\left(1 + \frac{\ell}{2}\right) + R_{\ell+1}(f) = Bf^{\ell+2} + R_{\ell+1}(f), \tag{5}$$

where $R_i(f)$ is a polynomial of f of degree i, $i = \ell, \ell + 1$. When ℓ is odd, the coefficient of $f^{\ell+2}$, that is B, is positive, and when ℓ is even, B is negative. Applying asymptotically vanishing boundary conditions, we get $(f')^2 = Bf^2(f + 2c)^\ell$, where $\text{sign}(B) = (-1)^{\ell-1}$. Hence for $\ell = $ odd, the degenerate coupled ℓ-KdV equation has a real traveling wave solution with asymptotically vanishing boundary conditions, but when $\ell = $ even, it does not. $\square$

In this work, we study Equation (1) for $\ell = 3$, which is

$$u_t = \frac{3}{2}uu_x + v_x, \tag{6}$$

$$v_t = vu_x + \frac{1}{2}uv_x + \omega_x, \tag{7}$$

$$\omega_t = -\frac{1}{4}u_{xxx} + \omega u_x + \frac{1}{2}u\omega_x, \tag{8}$$

in detail. It is clear from Theorem 1.1 that unlike the case $\ell = 2$, we have the real traveling wave solution with asymptotically vanishing boundary conditions in $\ell = 3$ case.

Here the system (6)–(8) reduces to a polynomial of degree five

$$(f')^2 = \frac{f^5}{2} + 3cf^4 + (6c^2 - 2d_1)f^3 + 4(c^3 - cd_1 + d_2)f^2 + 8d_3 f + 8d_4 = P_5(f), \tag{9}$$

where $f(\xi) = u(x,t)$, and c, d_1, d_2, d_3, d_4 are constants. When the degree of the polynomial in the reduced equation is equal to five or greater, it is almost impossible to solve them. As far as we know, there is no known method to solve these equations. We shall introduce two methods to solve such equations. The first one is based on the Chebyshev's theorem[10] which is used recently to solve the Einstein field equations for a cosmological model.[11,12] By using the Chebyshev's theorem, we give several solutions of the reduced equations for $\ell = 3$ and also for arbitrary ℓ. The second method is based on factorizing the polynomial $P_{\ell+2}(f)$ as the product of lower degree polynomials. In this way, we make use of the reduced equations of lower degrees. We have given all possible such solutions for $\ell = 3$.

The layout of our paper is as follows: In Sec. II, we study the behavior of the solutions in the neighborhood of the zeros of $P_5(f)$ and discuss the cases giving solitary wave solutions. In Sec. III, we find exact solutions of the system (6)–(8) by a new method proposed for any $\ell \geq 3$ which uses the Chebyshev's theorem and analyze the cases in which we may have solitary wave and kink-type solutions. In Sec. IV, we present an alternative method which is based on a factorization of polynomials. Particularly, we find the solutions of the system (6)–(8). Here we obtain many solutions for $\ell = 3$ including solitary wave, kink-type, periodic, and unbounded solutions. We present some of them in the text but the rest of the solutions are given in Appendices A and B.

II. GENERAL WAVES OF PERMANENT FORM FOR ($\ell = 3$)

A. Zeros of $P_5(f)$ and types of solutions

Here we will analyze the zeros of $P_5(f)$ in (9).

(i) If $f_1 = f(\xi_1)$ is a *simple zero* of $P_5(f)$, we have $P_5(f_1) = 0$. Taylor expansion of $P_5(f)$ about f_1 gives

$$(f')^2 = P_5(f_1) + P_5'(f_1)(f - f_1) + O((f - f_1)^2)$$
$$= P_5'(f_1)(f - f_1) + O((f - f_1)^2).$$

From here, we get $f'(\xi_1) = 0$ and $f''(\xi_1) = P_5'(f_1)/2$. Hence we can write the function $f(\xi)$ as

$$f(\xi) = f(\xi_1) + (\xi - \xi_1)f'(\xi_1) + \frac{1}{2}(\xi - \xi_1)^2 f''(\xi_1) + O((\xi - \xi_1)^3)$$
$$= f_1 + \frac{1}{4}(\xi - \xi_1)^2 P_5'(f_1) + O((\xi - \xi_1)^3). \tag{10}$$

Thus, in the neighborhood of $\xi = \xi_1$, the function $f(\xi)$ has a local minimum or maximum as $P_5'(f_1)$ is positive or negative, respectively, since $f''(\xi_1) = P_5'(f_1)/2$.

(ii) If $f_1 = f(\xi_1)$ is a *double zero* of $P_5(f)$, we have $P_5(f_1) = P_5'(f_1) = 0$. Taylor expansion of $P_5(f)$ about f_1 gives

$$(f')^2 = P_5(f_1) + P_5'(f_1)(f - f_1) + \frac{1}{2}(f - f_1)^2 P_5''(f_1) + O((f - f_1)^3)$$
$$= \frac{1}{2}(f - f_1)^2 P_5''(f_1) + O((f - f_1)^3). \tag{11}$$

To have the real solution f, we should have $P_5''(f_1) > 0$. From the equality (11), we get

$$f' \pm \frac{1}{\sqrt{2}} f \sqrt{P_5''(f_1)} \sim \pm \frac{1}{\sqrt{2}} f_1 \sqrt{P_5''(f_1)},$$

which gives

$$f(\xi) \sim f_1 + \alpha e^{\pm \frac{1}{\sqrt{2}} \sqrt{P_5''(f_1)} \xi}, \tag{12}$$

where α is a constant. Hence $f \to f_1$ as $\xi \to \mp\infty$. The solution f can have only one peak and the wave extends from $-\infty$ to ∞.

(iii)   If $f_1 = f(\xi_1)$ is a *triple zero* of $P_5(f)$, we have $P_5(f_1) = P_5'(f_1) = P_5''(f_1) = 0$. Taylor expansion of $P_5(f)$ about f_1 gives

$$(f')^2 = P_5(f_1) + P_5'(f_1)(f - f_1) + \frac{1}{2}(f - f_1)^2 P_5''(f_1) + \frac{1}{6}(f - f_1)^3 + O((f - f_1)^4)$$

$$= \frac{1}{6}(f - f_1)^3 P_5'''(f_1) + O((f - f_1)^4). \tag{13}$$

This is valid only if both signs of $(f - f_1)^3$ and $P_5'''(f_1)$ are same. Hence, to obtain the real solution f we have the following two possibilities:

(1) $(f - f_1) > 0$ and $P_5'''(f_1) > 0$,
(2) $(f - f_1) < 0$ and $P_5'''(f_1) < 0$.
If $(f - f_1) > 0$ and $P_5'''(f_1) > 0$, then we have

$$f' \sim \pm\frac{1}{\sqrt{6}}(f - f_1)^{3/2}\sqrt{P_5'''(f_1)},$$

which gives

$$f(\xi) \sim f_1 + \frac{4}{\left(\pm\frac{1}{\sqrt{6}}\sqrt{P_5'''(f_1)}\xi + \alpha_1\right)^2}, \tag{14}$$

where α_1 is a constant. Thus $f \to f_1$ as $\xi \to \pm\infty$.

Let $(f - f_1) < 0$ and $P_5'''(f_1) < 0$. Then

$$f' \sim \pm\frac{1}{\sqrt{6}}(f_1 - f)^{3/2}\sqrt{-P_5'''(f_1)},$$

which yields

$$f(\xi) \sim f_1 - \frac{4}{\left(\pm\frac{1}{\sqrt{6}}\sqrt{-P_5'''(f_1)}\xi + \alpha_2\right)^2}, \tag{15}$$

where α_2 is a constant. Thus $f \to f_1$ as $\xi \to \pm\infty$.

(iv)   If $f_1 = f(\xi_1)$ is a *quadruple zero* of $P_5(f)$, then we have $P_5(f_1) = P_5'(f_1) = P_5''(f_1) = P_5'''(f_1) = 0$. In this case, Taylor expansion of $P_5(f)$ about f_1 gives

$$(f')^2 = \frac{1}{24}(f - f_1)^4 P_5^{(4)}(f_1) + O((f - f_1)^5). \tag{16}$$

This is valid only if $P_5^{(4)}(f_1) > 0$. Then we have

$$f' \sim \pm\frac{1}{2\sqrt{6}}(f - f_1)^2\sqrt{P_5^{(4)}(f_1)},$$

which gives

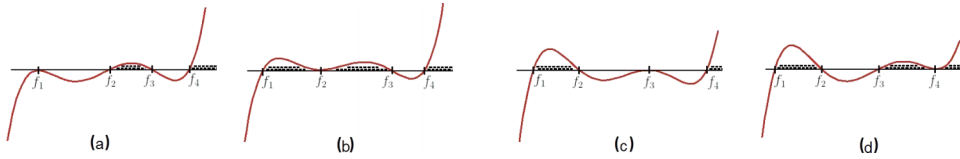$$f(\xi) \sim f_1 - \frac{1}{\pm\frac{1}{2\sqrt{6}}\sqrt{P_5^{(4)}(f_1)}\xi + \gamma_1}, \tag{17}$$

where γ_1 is a constant. Thus $f \to f_1$ as $\xi \to \pm\infty$.

(v)   If $f_1 = f(\xi_1)$ is a *zero of multiplicity 5* of $P_5(f)$, then we have $P_5(f) = (f - f_1)^5/2$. This is valid only if $f - f_1 > 0$. So we obtain the solution f as

$$f = f_1 + \left(\frac{4}{9}\right)^{1/3}\frac{1}{\left(\pm\frac{\xi}{\sqrt{2}} + m_1\right)^{2/3}}, \tag{18}$$

where m_1 is a constant. Hence $f \to f_1$ as $\xi \to \pm\infty$.

FIG. 1. Graphs of $P_5(f)$ having one double and three simple zeros.

B. All possible cases giving solitary wave solutions

We analyze all possible cases about the zeros of $P_5(f)$ that may give solitary wave solutions. Here in each cases, we will present the sketches of the graphs of $P_5(f)$. Real solutions of $(f')^2 = P_5(f) \geq 0$ occur in the shaded regions.

1. One double and three simple zeros

In Figure 1(b), f_1, f_3, and f_4 are simple zeros and f_2 is a double zero. The real solution occurs when f stays between f_1 and f_2 or f_2 and f_3. At f_1, $P_5'(f_1) = f''(\xi_1) > 0$, hence the graph of the function f is concave up at ξ_1. At double zero f_2, $f \to f_2$ as $\xi \to \pm\infty$. Hence we have a solitary wave solution with amplitude $f_1 - f_2 < 0$. Similarly at f_3, $P_5'(f_3) = f''(\xi_3) < 0$, hence the graph of the function f is concave down at ξ_3. Therefore, we also have a solitary wave solution with amplitude $f_3 - f_2 > 0$.

Now consider the graph (d) in Figure 1. For f_3 we have $P_5'(f_3) = f''(\xi_3) > 0$, thus the graph of the function is concave up at ξ_3. At double zero f_4, $f \to f_4$ as $\xi \to \pm\infty$. Hence we have a solitary wave solution with amplitude $f_3 - f_4 < 0$. In other cases, we have periodic solutions.
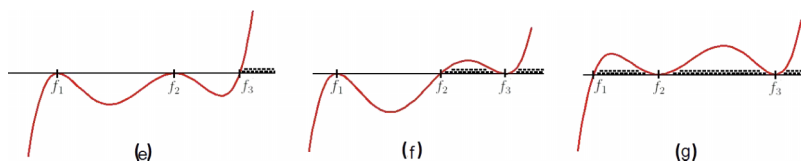
2. Two double and one simple zeros

In Figure 2(f), f_1 and f_3 are double zeros and f_2 is a simple zero. The real solution occurs when f stays between f_2 and f_3. For f_2, we have $P_5'(f_2) = f''(\xi_2) > 0$, thus the graph of the function is concave up at ξ_2. At double zero f_3, $f \to f_3$ as $\xi \to \pm\infty$. Hence we have a solitary wave solution with amplitude $f_2 - f_3 < 0$.

Now consider the graph (g) in Figure 2. Here f_2 and f_3 are double zeros and f_1 is a simple zero. The real solution occurs when f stays between f_1 and f_2 or f_2 and f_3. For f_1 we have $P_5'(f_1) = f''(\xi_1) > 0$, thus the graph of the function is concave up at ξ_1. At double zero f_2, $f \to f_2$ as $\xi \to \pm\infty$. Hence we have a solitary wave solution with amplitude $f_1 - f_2 < 0$. The other cases give kink, anti-kink type, or unbounded solutions.

3. One triple and two simple zeros

Consider the graph (h) in Figure 3. Here f_2 and f_3 are simple zeros and f_1 is a triple zero. The real solution occurs when f stays between f_1 and f_2. For f_2 we have $P_5'(f_2) = f''(\xi_2) < 0$, so the graph of the function f is concave down at ξ_2. At triple zero f_1, $f \to f_1$ as $\xi \to \pm\infty$. Hence we may have a solitary wave solution with amplitude $f_2 - f_1 > 0$.

In the graph in Figure 3(i), f_1 and f_3 are simple zeros and f_2 is a triple zero. The real solution occurs when f stays between f_1 and f_2. For f_1 we have $P_5'(f_1) = f''(\xi_1) > 0$, so the graph of the

FIG. 2. Graphs of $P_5(f)$ having two double and one simple zeros.
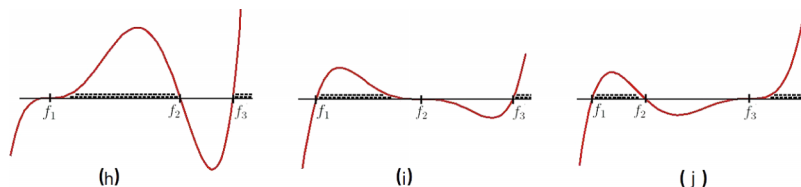
FIG. 3. Graphs of $P_5(f)$ having one triple and two simple zeros.
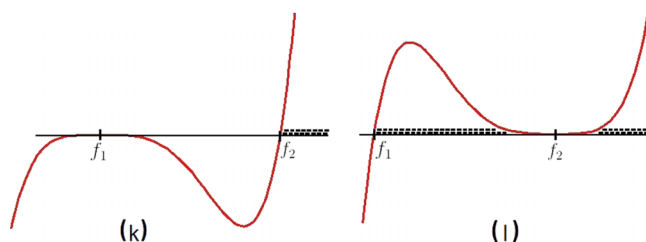
FIG. 4. Graphs of $P_5(f)$ having one quadruple and one simple zeros.
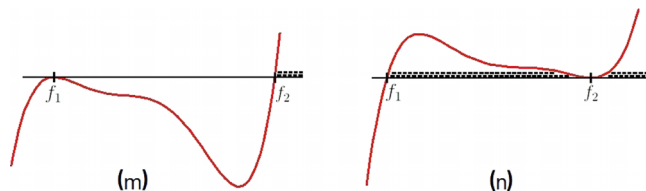
FIG. 5. Graphs of $P_5(f)$ having one double and one simple zeros.

function f is concave up at ξ_1. At triple zero f_2, $f \to f_2$ as $\xi \to \pm\infty$. Hence we may have a solitary wave solution with amplitude $f_1 - f_2 < 0$. The other case gives a periodic solution.

4. One quadruple and one simple zeros

In the graph in Figure 4(l), f_1 is a simple zero and f_2 is a quadruple zero. The real solution occurs between f_1 and f_2. At f_1 we have $P_5'(f_1) = f''(\xi_1) > 0$, so the graph of the function f is concave up at ξ_1. At quadruple zero f_2, $f \to f_2$ as $\xi \to \pm\infty$. Hence we may have a solitary wave solution with amplitude $f_1 - f_2 < 0$. The other case gives unbounded solution.

5. One double and one simple zeros

Consider the graph in Figure 5(n). Here f_1 is a simple zero and f_2 is a double zero. The real solution occurs between f_1 and f_2. At f_1 we have $P_5'(f_1) = f''(\xi_1) > 0$, so the graph of the function f is concave up at ξ_1. At double zero f_2, $f \to f_2$ as $\xi \to \pm\infty$. Hence we have a solitary wave solution with amplitude $f_1 - f_2 < 0$. The other case gives an unbounded solution.

To sum up, we can give the following proposition for $\ell = 3$ case.

Proposition 2.1. Equation (9) may admit solitary wave solutions when the polynomial function $P_5(f)$ admits (i) one double and three simple zeros, (ii) two double and one simple zeros, (iii) one triple and two simple zeros, (iv) one quadruple and one simple zeros, and (v) one double and one simple zeros.

III. EXACT SOLUTIONS BY USING THE CHEBYSHEV'S THEOREM

The Chebyshev's theorem is given as follows.[10]

Theorem 3.1. *Let a, b, c, α, and β be the given real numbers and $\alpha\beta \neq 0$. The antiderivative,*

$$I = \int x^a(\alpha + \beta x^b)^c \, dx, \tag{19}$$

is expressible by means of the elementary functions only in the three cases

$$(1) \quad \frac{a+1}{b} + c \in \mathbb{Z}, \quad (2) \quad \frac{a+1}{b} \in \mathbb{Z}, \quad (3) \quad c \in \mathbb{Z}. \tag{20}$$

The term $x^a(\alpha + \beta x^b)^c$ is called a differential binomial. Note that the differential binomial may be expressed in terms of the incomplete beta function and the hypergeometric function. Let us define $u = \beta x^b/\alpha$. Then we have

$$I = \frac{1}{b} \alpha^{\frac{a+1}{b}+c} \beta^{-\frac{a+1}{b}} B_y\left(\frac{1+a}{b}, c-1\right).$$
$$= \frac{1}{1+a} \alpha^{\frac{a+1}{b}+c} \beta^{-\frac{a+1}{b}} u^{\frac{1+a}{b}} F\left(\frac{a+1}{b}, 2-c; \frac{1+a+b}{b}; u\right), \tag{21}$$

where B_y is the incomplete beta function and $F(\tau, \kappa; \eta; u)$ is the hypergeometric function. Our aim is to transform the system (1) to $(y')^2 = \bar{P}_{\ell+2}(y)$ by taking $f(\xi) = \gamma + \bar{\alpha} y(\bar{\beta}\xi)$. We can apply the Chebyshev's theorem to this equation, if we assume that $\bar{P}_{\ell+2}(y)$ reduces to the form $\bar{P}_{\ell+2}(y) = Ay^{-2a}(\alpha + \beta y^b)^{-2c}$, where $-2a - 2c + b = \ell + 2$. For $\ell = 3$, let $u(x,t) = f(\xi) = \gamma + \bar{\alpha} y(\bar{\beta}\xi)$ in (9), then the equation becomes

$$(y')^2 = \bar{P}_5(y) = \alpha_1 y^5 + \alpha_2 y^4 + \alpha_3 y^3 + \alpha_4 y^2 + \alpha_5 y + \alpha_6, \tag{22}$$

where

$$\alpha_1 = \frac{\bar{\alpha}^3}{2\bar{\beta}^2},$$

$$\alpha_2 = \frac{1}{\bar{\beta}^2}\left(\frac{5\bar{\alpha}^2\gamma}{2} + 3\bar{\alpha}^2 c\right),$$

$$\alpha_3 = \frac{1}{\bar{\beta}^2}\left(5\bar{\alpha}\gamma^2 + 6\bar{\alpha}c^2 - 2\bar{\alpha}d_1 + 12\bar{\alpha}c\gamma\right),$$

$$\alpha_4 = \frac{1}{\bar{\beta}^2}\left(-4cd_1 + 4d_2 + 18c\gamma^2 - 6d_1\gamma + 5\gamma^3 + 4c^3 + 18c^2\gamma\right),$$

$$\alpha_5 = \frac{1}{\bar{\alpha}\bar{\beta}^2}\left(8d_3 + 12c\gamma^3 + \frac{5\gamma^4}{2} + 18c^2\gamma^2 - 8cd_1\gamma + 8c^3\gamma + 8d_2\gamma - 6d_1\gamma^2\right),$$

$$\alpha_6 = \frac{1}{\bar{\alpha}^2\bar{\beta}^2}\left(8d_3\gamma + 6c^2\gamma^3 + \frac{\gamma^5}{2} + 8d_4 - 4cd_1\gamma^2 - 2d_1\gamma^3 + 4c^3\gamma^2 + 4d_2\gamma^2\right).$$

To apply the Chebyshev's theorem 3.1 we assume that $\bar{P}_5(y)$ reduces to the following form:

$$\bar{P}_5(y) = Cy^{-2a}(\alpha + \beta y^b)^{-2c}, \tag{23}$$

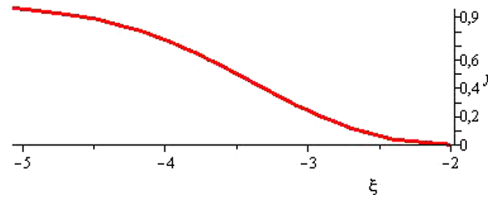where a, b, and c are the constants in Theorem 3.1, $b - 2a - 2c = 5$, and C is a constant.

Here we present the cases mentioned in Proposition 2.1. Other cases are given in Appendix A.

(1) Let $\bar{P}_5(y) = y(\alpha + \beta y^2)^2$. This form corresponds to the case of one simple or one simple and two double zeros. We have $y' = \pm y^{1/2}(\alpha + \beta y^2)$, so

$$\int y^{-1/2}(\alpha + \beta y^2)^{-1} dy = \pm\xi + A. \tag{24}$$

Here $a = -1/2$, $b = 2$, and $c = -1$. Hence

$$(1) \quad \frac{a+1}{b} + c = -3/4 \notin \mathbb{Z}, \quad (2) \quad \frac{a+1}{b} = 1/4 \notin \mathbb{Z}, \quad (3) \quad c = -1 \in \mathbb{Z}.$$

FIG. 6. Graph of a solution of $(y')^2 = y(-1 + y^2)^2$.

For (3), from (24), by letting $u = \beta y^b / \alpha$, we obtain

$$2\alpha^{-3/4}\beta^{-1/4}u^{1/4}F\left(\frac{1}{4}, 3; \frac{5}{4}; u\right) = \pm\xi + A. \tag{25}$$

Here choosing $\alpha = -1$, $\beta = 1$, the integration constant $A = 2$, and the plus sign in (24), we have

$$\operatorname{arctanh}(\sqrt{y}) + \arctan(\sqrt{y}) = -(\xi + 2) \tag{26}$$

and the graph of this solution for $\xi \leq -2$ is given in Figure 6.

This is a kink-type solution.

(2) Let $\bar{P}_5(y) = y(\alpha + \beta y)^4$. This form corresponds to the case of one simple and one quadruple zeros. We have $y' = \pm y^{1/2}(\alpha + \beta y)^2$, so

$$\int y^{-1/2}(\alpha + \beta y)^{-2} dy = \pm\xi + A. \tag{27}$$

Here $a = -1/2$, $b = 1$, and $c = -2$. Hence

$$(1) \quad \frac{a+1}{b} + c = -3/2 \notin \mathbb{Z}, \quad (2) \quad \frac{a+1}{b} = 1/2 \notin \mathbb{Z}, \quad (3) \quad c = -2 \in \mathbb{Z}.$$
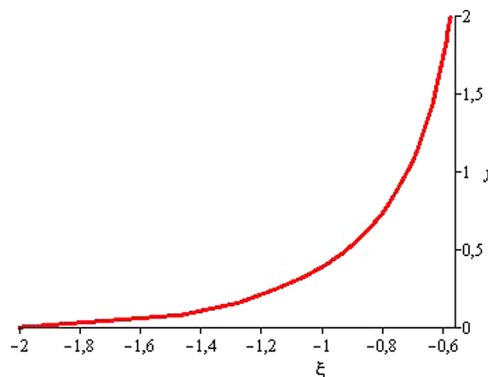
For (3), from (27), by letting $u = \beta y^b / \alpha$, we obtain

$$2\alpha^{-3/2}\beta^{-1/2}u^{1/2}F\left(\frac{1}{2}, 4; \frac{3}{2}; u\right) = \frac{\sqrt{y}}{\alpha(\beta y + \alpha)} + \frac{\arctan\left(\frac{\beta\sqrt{y}}{\sqrt{\beta\alpha}}\right)}{\alpha\sqrt{\beta\alpha}} = \pm\xi + A. \tag{28}$$

Here choosing $\alpha = 1$, $\beta = 1$, the integration constant $A = 2$, and the plus sign in (27), we have

$$\frac{\sqrt{y}}{1 + y} + \arctan(\sqrt{y}) = \xi + 2 \tag{29}$$

and the graph of this solution for $-2 \leq \xi < \pi/2 - 2$ is given in Figure 7.

FIG. 7. Graph of a solution of $(y')^2 = y(1 + y)^4$.

(3) Let $\bar{P}_5(y) = y^2(\alpha + \beta y^3)$. This form corresponds to the case of one simple and one double zeros. We have $y' = \pm y(\alpha + \beta y^3)^{1/2}$, so

$$\int y^{-1}(\alpha + \beta y^3)^{-1/2}dy = \pm\xi + A. \tag{30}$$

Here $a = -1$, $b = 3$, and $c = -1/2$. Hence

(1) $\dfrac{a+1}{b} + c = -1/2 \notin \mathbb{Z}$, (2) $\dfrac{a+1}{b} = 0 \in \mathbb{Z}$, (3) $c = -1/2 \notin \mathbb{Z}$.

For (2), from (30), we obtain

$$y = \Big(\frac{\alpha}{\beta}\Big(\tanh^2\Big(\frac{3}{2}\sqrt{\alpha}(A \pm \xi)\Big) - 1\Big)\Big)^{1/3}. \tag{31}$$

Here choosing $\alpha = 4$, $\beta = 4$, the integration constant $A = 2$, and the plus sign in (30), we obtain

$$y(\xi) = -(\operatorname{sech}(3\xi + 6))^{2/3} \tag{32}$$

and the graph of this solution is given in Figure 8.

This is clearly a solitary wave solution.

(4) Let $\bar{P}_5(y) = y^2(\alpha + \beta y)^3$. This form corresponds to the case of one double and one triple zeros. We have $y' = \pm y(\alpha + \beta y)^{3/2}$, so

$$\int y^{-1}(\alpha + \beta y)^{-3/2}dy = \pm\xi + A. \tag{33}$$

Here $a = -1$, $b = 1$, and $c = -3/2$. Hence

(1) $\dfrac{a+1}{b} + c = -3/2 \notin \mathbb{Z}$, (2) $\dfrac{a+1}{b} = 0 \in \mathbb{Z}$, (3) $c = -3/2 \notin \mathbb{Z}$.

For (2), from (33), we obtain

$$\frac{2}{\alpha\sqrt{\beta y + \alpha}} - \frac{2\operatorname{arctanh}\Big(\frac{\sqrt{\beta y+\alpha}}{\sqrt{\alpha}}\Big)}{\alpha^{3/2}} = \pm\xi + A. \tag{34}$$

Here choosing $\alpha = 1$, $\beta = 1$, $A = 2$, and the plus sign in (33), we have

$$\frac{2}{\sqrt{y+1}} - 2\operatorname{arctanh}(\sqrt{y+1}) = \xi + 2 \tag{35}$$

and the graph of this solution is given in Figure 9.

(5) Let $\bar{P}_5(y) = y^4(\alpha + \beta y)$. This form corresponds to the case of one simple and one quadruple zeros. We have $y' = \pm y^2(\alpha + \beta y)^{1/2}$, so

$$\int y^{-2}(\alpha + \beta y)^{-1/2}dy = \pm\xi + A. \tag{36}$$
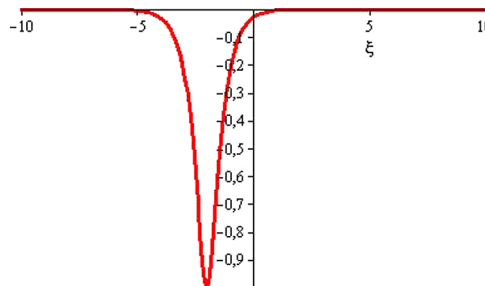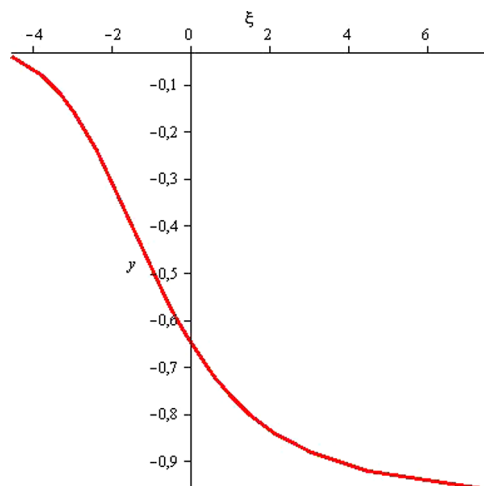

FIG. 8.   Graph of a solution of $(y')^2 = y^2(4 + 4y^3)$.

FIG. 9.   Graph of a solution of $(y')^2 = y^2(1+y)^3$.

Here $a = -2$, $b = 1$, and $c = -1/2$. Hence

$$(1) \quad \frac{a+1}{b} + c = -3/2 \notin \mathbb{Z}, \quad (2) \quad \frac{a+1}{b} = -1 \in \mathbb{Z}, \quad (3) \quad c = -1/2 \notin \mathbb{Z}.$$
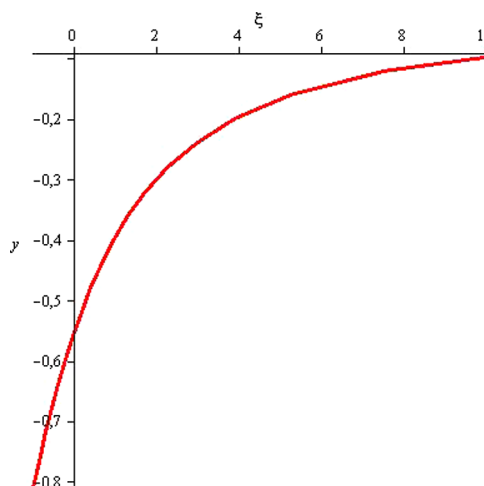
For (2), from (36), by letting $u = \beta y^b/\alpha$, we obtain

$$-\alpha^{-3/2}\beta u^{-1} F\left(-1, \frac{5}{2}; 0; u\right) = -\frac{\sqrt{\beta y + \alpha}}{\alpha y} + \frac{\beta \operatorname{arctanh}\left(\frac{\sqrt{\beta y + \alpha}}{\sqrt{\alpha}}\right)}{\alpha^{3/2}} = \pm\xi + A. \tag{37}$$

Taking $\alpha = 1$, $\beta = 1$, $A = 2$, and the plus sign in (36), we get

$$\operatorname{arctanh}(\sqrt{y+1}) - \frac{\sqrt{y+1}}{y} = \xi + 2 \tag{38}$$

and the graph of this solution for $\xi \geq -2$ is given in Figure 10.

FIG. 10.   Graph of a solution of $(y')^2 = y^4(1+y)$.

IV. AN ALTERNATIVE METHOD TO SOLVE $(f')^2 = P_n(f)$, WHEN $n \geq 5$: FACTORIZATION OF POLYNOMIALS

When we have $(f')^2 = P_n(f)$ for the reduced equation, it becomes quite difficult to solve such equations for $n \geq 5$. For this purpose, we shall introduce a new method which is based on the factorization of the polynomial $P_n(f) = P_{\ell+2}(f)$, $n \geq 5$. Let the polynomial $P_{\ell+2}(f)$ have $\ell + 2$ real roots, i.e., $P_{\ell+2}(f) = B \prod_{i=1}^{\ell+2}(f - f_i)$, B is a constant. Define a new function $\rho(\xi)$, so that $f = f(\rho(\xi))$. Hence we have $(f')^2 = \left(\frac{df}{d\rho}\right)^2\left(\frac{d\rho}{d\xi}\right)^2$. By taking

$$\left(\frac{df}{d\rho}\right)^2 = \kappa \prod_{i=1}^{N_1}(f - f_i),$$

$$\left(\frac{d\rho}{d\xi}\right)^2 = \mu \prod_{i=1}^{N_2}(f - f_i),$$

where $N_1 + N_2 = \ell + 2$ and $B = \kappa\mu$, we get a system of ordinary differential equations. Solving this system gives the solution of the degenerate coupled ℓ-KdV equation.

For illustration, we start with a differential equation where we know the solution. Consider

$$(y')^2 = \alpha(y - y_1)(y - y_2)(y - y_3)(y - y_4). \tag{39}$$

Let $y = y(\rho(\xi))$ so

$$\left(\frac{dy}{d\rho}\right)^2\left(\frac{d\rho}{d\xi}\right)^2 = \alpha(y - y_1)(y - y_2)(y - y_3)(y - y_4).$$

Taking

$$\left(\frac{dy}{d\rho}\right)^2 = \kappa(y - y_1)(y - y_2), \tag{40}$$

$$\left(\frac{d\rho}{d\xi}\right)^2 = \mu(y - y_3)(y - y_4), \tag{41}$$

where $\kappa\mu = \alpha$. We start with solving Equation (40). We have

$$\frac{dy}{\sqrt{(y - y_1)(y - y_2)}} = \frac{dy}{b\sqrt{\left(\frac{y-a}{b}\right)^2 - 1}} = \pm\sqrt{\kappa}d\rho,$$

where $a = (y_1 + y_2)/2$ and $b = (y_1 - y_2)/2$. Let $(y - a)/b = \cosh z$. This gives us $dz = \pm\sqrt{\kappa}d\rho$. After taking the integral, we obtain

$$\text{arccosh}\left(\frac{y - a}{b}\right) = \pm\sqrt{\kappa}\rho + C_1, \quad C_1 \quad \text{constant},$$

so that

$$y = h(\rho) = a + b\cosh(\sqrt{\kappa}\rho + C_2), \quad C_2 \quad \text{constant}. \tag{42}$$

Now we insert this solution into Eq. (41). Let $\cosh(\sqrt{\kappa}\rho + C_2) = v$. Hence we get

$$\int \frac{dv}{\sqrt{\kappa}b\sqrt{v^2 - 1}\sqrt{(v - A)^2 - B^2}} = \pm\sqrt{\mu}\xi + C_3, \tag{43}$$

where $A = (y_3 + y_4 - y_1 - y_2)/(y_1 - y_2)$, $B = (y_3 - y_4)/(y_1 - y_2)$, and C_3 is a constant. Solving (43) and using (42) gives

$$y = \frac{y_1(y_3 - y_2) + y_2(y_1 - y_3)\text{sn}^2((1/2)\sqrt{\kappa(y_3 - y_2)(y_4 - y_1)}(\sqrt{\mu}\xi + C_3), k)}{(y_3 - y_2) + (y_1 - y_3)\text{sn}^2((1/2)\sqrt{\kappa(y_3 - y_2)(y_4 - y_1)}(\sqrt{\mu}\xi + C_3), k)}, \tag{44}$$

which was obtained in Ref. 8. Here sn is the Jacobi elliptic function and k is the elliptic modulus satisfying $k^2 = (f_2 - f_4)(f_1 - f_3)/(f_2 - f_3)(f_1 - f_4)$.

Now we apply our method to the case when the polynomial $P_n(f)$ is of degree $n = 5$. We have several possible cases, but here we shall give the case when the polynomial (9) has five real roots. Other cases are presented in Appendix B.

Case 1. If (9) has five real roots, we can write it in the form

$$(f')^2 = \alpha(f - f_1)(f - f_2)(f - f_3)(f - f_4)(f - f_5), \tag{45}$$

where f_1, f_2, f_3, f_4, and f_5 are the zeros of the polynomial function $P_5(f)$. Now define a new function $\rho(\xi)$, so that $f = f(\rho(\xi))$. Hence (45) becomes

$$(f')^2 = \left(\frac{df}{d\rho}\right)^2\left(\frac{d\rho}{d\xi}\right)^2 = \alpha(f - f_1)(f - f_2)(f - f_3)(f - f_4)(f - f_5). \tag{46}$$

(i) Take

$$\left(\frac{df}{d\rho}\right)^2 = -(f - f_1)(f - f_2)(f - f_3)(f - f_4), \tag{47}$$

$$\left(\frac{d\rho}{d\xi}\right)^2 = -\alpha(f - f_5). \tag{48}$$

In Ref. 8, we have found the solutions of Eq. (47). One of the solutions is

$$f = h(\rho) = \frac{f_4(f_3 - f_1) + f_3(f_1 - f_4)\mathrm{sn}^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)}\rho, k)}{(f_3 - f_1) + (f_1 - f_4)\mathrm{sn}^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)}\rho, k)}, \tag{49}$$

where k is the elliptic modulus satisfying $k^2 = (f_3 - f_2)(f_4 - f_1)/(f_3 - f_1)(f_4 - f_2)$. We use this solution in Equation (48) and get

$$\int_0^\rho \frac{d\hat{\rho}}{\sqrt{f_5 - h(\hat{\rho})}} = \int_0^\rho \sqrt{\frac{A + B\mathrm{sn}^2(\omega\hat{\rho}, k)}{C + D\mathrm{sn}^2(\omega\hat{\rho}, k)}}d\hat{\rho} = \pm\sqrt{\alpha}\xi + C_1, \quad C_1 \quad \text{constant}, \tag{50}$$

where

$$A = f_3 - f_1, \ B = f_1 - f_4, \ C = (f_5 - f_4)(f_3 - f_1), \ D = (f_5 - f_3)(f_1 - f_4),$$

and $\omega = (1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)}$. For particular values, $f_1 = f_2 = -1$, $f_3 = 3$, $f_4 = 0$, $f_5 = 2$, $\alpha = 1$, $C_1 = 0$, and choosing the plus sign in (50), we get the graph of the solution $f(\xi)$ given in Figure 11.
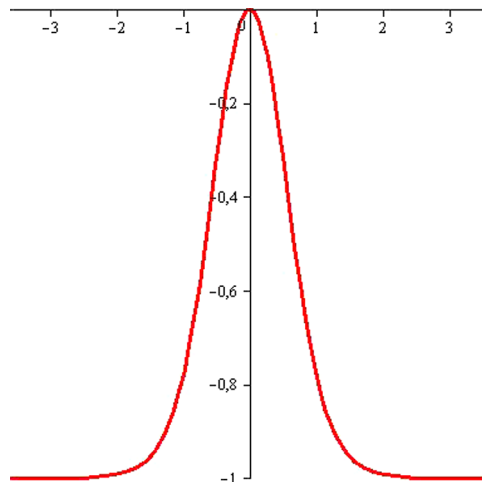

FIG. 11. Graph of the solution (49) with (50) for particular parameters.

This is a solitary wave solution.

(ii) Take

$$\left(\frac{df}{d\rho}\right)^2 = \kappa(f - f_1)(f - f_2)(f - f_3), \tag{51}$$

$$\left(\frac{d\rho}{d\xi}\right)^2 = \mu(f - f_4)(f - f_5), \tag{52}$$

where $\kappa\mu = \alpha$. Eq. (51) has a solution

$$f = h(\rho) = (f_2 - f_1)\mathrm{sn}^2((1/2)\sqrt{f_3 - f_1}(\sqrt{\kappa}\rho + C_1), k) + f_1, \tag{53}$$

where k satisfies $k^2 = (f_1 - f_2)/(f_1 - f_3)$ and C_1 is a constant. We use this solution in Eq. (52) and get

$$\int_0^\rho \frac{d\hat{\rho}}{\sqrt{(h(\hat{\rho}) - f_4)(h(\hat{\rho}) - f_5)}} = \int_0^\rho \frac{d\hat{\rho}}{\sqrt{A\mathrm{sn}^4(\omega\hat{\rho} + \delta, k) + B\mathrm{sn}^2(\omega\hat{\rho} + \delta, k) + C}} = \pm\sqrt{\mu}\xi + C_2, \tag{54}$$

where C_2 is a constant and

$$A = (f_2 - f_1)^2, \, B = (f_2 - f_1)(2f_1 - f_4 - f_5), \, C = (f_1 - f_4)(f_1 - f_5), \, \omega = (1/2)\sqrt{\kappa(f_3 - f_1)},$$

and $\delta = (1/2)\sqrt{f_3 - f_1}C_1$. For particular values, $f_1 = 1$, $f_2 = 2$, $f_3 = 3$, $f_4 = -1$, $f_5 = -2$, $\kappa = \mu = 1$, $C_1 = 0$, and choosing the plus sign in (54), we get the graph of the solution $f(\xi)$ given in Figure 12.

This solution is periodic.

(iii) Take

$$\left(\frac{df}{d\rho}\right)^2 = \kappa(f - f_1)(f - f_2), \tag{55}$$

$$\left(\frac{d\rho}{d\xi}\right)^2 = \mu(f - f_3)(f - f_4)(f - f_5), \tag{56}$$

where $\kappa\mu = \alpha$. Consider first Equation (55).

We have

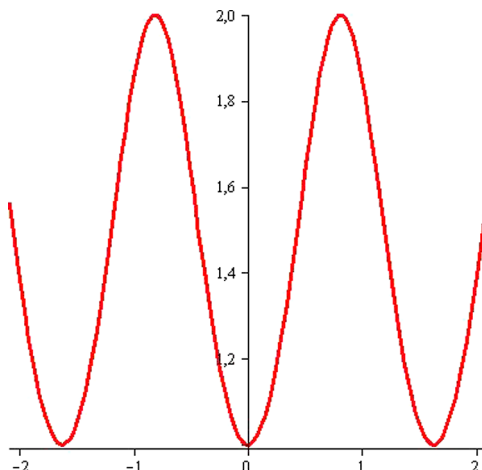$$\frac{df}{\sqrt{(f - f_1)(f - f_2)}} = \pm\sqrt{\kappa}d\rho. \tag{57}$$

FIG. 12. Graph of the solution (53) with (54) for particular parameters.

This equality can be reduced to

$$\frac{df}{b\sqrt{\left(\frac{f-a}{b}\right)^2 - 1}} = \pm\sqrt{\kappa}d\rho, \tag{58}$$

where $a = (f_1 + f_2)/2$ and $b = (f_1 - f_2)/2$. By making the change of variables $(f - a)/b = u$, we get

$$\int \frac{du}{\sqrt{u^2 - 1}} = \pm\sqrt{\kappa}\rho + C_1, \tag{59}$$

where C_1 is a constant. Thus, we obtain

$$\text{arccosh}u = \text{arccosh}\left(\frac{f - a}{b}\right) = \pm\sqrt{\kappa}\rho + C_1, \tag{60}$$

which yields

$$f = h(\rho) = a + b\cosh(\sqrt{\kappa}\rho + C_1), \quad a = (f_1 + f_2)/2, \quad b = (f_1 - f_2)/2. \tag{61}$$

Now we use this result in (56),

$$\int_0^\rho \frac{d\hat{\rho}}{\sqrt{(h(\hat{\rho}) - f_3)(h(\hat{\rho}) - f_4)(h(\hat{\rho}) - f_5)}} =$$
$$= \int_0^\rho \frac{d\hat{\rho}}{\sqrt{A\cosh^3(\sqrt{\kappa}\hat{\rho} + C_1) + B\cosh^2(\sqrt{\kappa}\hat{\rho} + C_1) + C\cosh(\sqrt{\kappa}\hat{\rho} + C_1) + D}}$$
$$= \pm\sqrt{\mu}\xi + C_2, \tag{62}$$

where C_2 is a constant and

$$A = b^3, \ B = b^2(3a - f_3 - f_4 - f_5), \ C = b\{(a - f_3)(a - f_4) + (a - f_3)(a - f_5) + (a - f_4)(a - f_5)\},$$

and $D = (a - f_3)(a - f_4)(a - f_5)$.

For particular values, $f_1 = 5$, $f_2 = 1$, $f_3 = f_4 = -3$, $f_5 = -4$, $\kappa = 4$, $\mu = 1$, $C_1 = C_2 = 0$, and choosing the plus sign in (62), we get the graph of the solution $f(\xi)$ given in Figure 13.

(iv) Take

$$\left(\frac{df}{d\rho}\right)^2 = \kappa(f - f_1), \tag{63}$$

$$\left(\frac{d\rho}{d\xi}\right)^2 = \mu(f - f_2)(f - f_3)(f - f_4)(f - f_5), \tag{64}$$
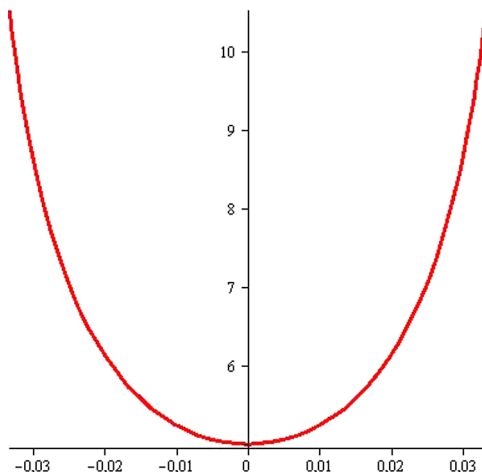

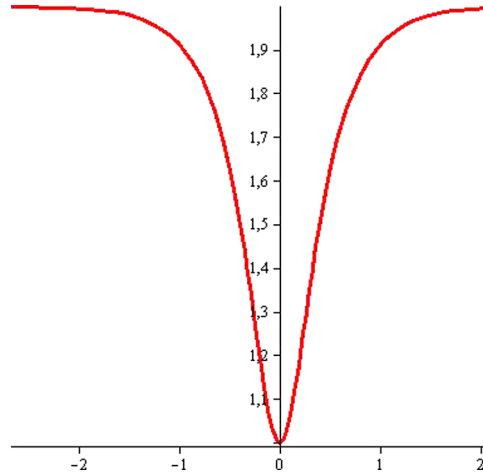FIG. 13. Graph of the solution (61) for particular parameters.

FIG. 14. Graph of the solution (66) for particular parameters.

where $\kappa\mu = \alpha$. Consider Eq. (63). We have

$$\frac{df}{\sqrt{f - f_1}} = \pm\sqrt{\kappa}d\rho. \tag{65}$$

Integrating both sides gives

$$f = h(\rho) = \left(\pm\frac{\sqrt{\kappa}}{2}\rho + C_1\right)^2 + f_1, \quad C_1 \quad \text{constant}. \tag{66}$$

We use this result in Equation (64),

$$\int_0^\rho \frac{d\hat{\rho}}{\sqrt{(h(\hat{\rho}) - f_2)(h(\hat{\rho}) - f_3)(h(\hat{\rho}) - f_4)(h(\hat{\rho}) - f_5)}} =$$

$$= \int_0^\rho \frac{d\hat{\rho}}{\sqrt{(\frac{\kappa}{4}\hat{\rho}^2 \pm \sqrt{\kappa}C_1\hat{\rho} + A)(\frac{\kappa}{4}\hat{\rho}^2 \pm \sqrt{\kappa}C_1\hat{\rho} + B)(\frac{\kappa}{4}\hat{\rho}^2 \pm \sqrt{\kappa}C_1\hat{\rho} + C)(\frac{\kappa}{4}\hat{\rho}^2 \pm \sqrt{\kappa}C_1\hat{\rho} + D)}}$$

$$= \pm\sqrt{\mu}\xi + C_2, \tag{67}$$

where C_2 is a constant and

$$A = C_1^2 + f_1 - f_2, \, B = C_1^2 + f_1 - f_3, \, C = C_1^2 + f_1 - f_4, \, D = C_1^2 + f_1 - f_5.$$

For particular values, $f_1 = 1$, $f_2 = f_3 = 2$, $f_4 = 3$, $f_5 = 4$, $\kappa = 4$, $\mu = 1$, $C_1 = C_2 = 0$, and choosing the plus sign in (67), we get the graph of the solution $f(\xi)$ given in Figure 14.

This is a solitary wave solution.

V. CONCLUSION

We study the degenerate ℓ-coupled KdV equations. We reduce these equations into an ODE of the form $(f')^2 = P_{\ell+2}(f)$, where $P_{\ell+2}$ is a polynomial function of f of degree $\ell + 2$, $\ell \geq 3$. We give a general approach to solve the degenerate ℓ-coupled equations by introducing two new methods that one of them uses the Chebyshev's theorem and the other one is an alternative method, based on the factorization of $P_{\ell+2}(f)$, $\ell \geq 3$. Particularly, for the degenerate three-coupled KdV equations, we obtain solitary-wave, kink-type, periodic, or unbounded solutions by using these methods.

ACKNOWLEDGMENTS

APPENDIX A: OTHER CASES USING CHEBYSHEV'S THEOREM FOR DEGENERATE THREE-COUPLED KdV EQUATION

Here we present other cases that we use the Chebyshev's theorem to solve degenerate three-coupled KdV equation.

(1) Let $\bar{P}_5(y) = y(\alpha + \beta y^4)$. This form corresponds to the case of one simple zero or three simple zeros. We have $y' = \pm y^{1/2}(\alpha + \beta y^4)^{1/2}$, so

$$\int y^{-1/2}(\alpha + \beta y^4)^{-1/2}dy = \pm\xi + A.$$

Here $a = -1/2$, $b = 4$, and $c = -1/2$. So we have

$$(1)\ \frac{a+1}{b} + c = -3/8 \notin \mathbb{Z}, \quad (2)\ \frac{a+1}{b} = 1/8 \notin \mathbb{Z}, \quad (3)\ c = -1/2 \notin \mathbb{Z}.$$

Hence we cannot obtain a solution through the Chebyshev's theorem.

(2) Let $\bar{P}_5(y) = y^3(\alpha + \beta y^2)$. This form corresponds to the case of one triple or two simple and one triple zeros. We have $y' = \pm y^{3/2}(\alpha + \beta y^2)^{1/2}$, so

$$\int y^{-3/2}(\alpha + \beta y^2)^{-1/2}dy = \pm\xi + A.$$

Here $a = -3/2$, $b = 2$, and $c = -1/2$. So we have

$$(1)\ \frac{a+1}{b} + c = -3/4 \notin \mathbb{Z}, \quad (2)\ \frac{a+1}{b} = -1/4 \notin \mathbb{Z}, \quad (3)\ c = -1/2 \notin \mathbb{Z}.$$

Hence we cannot obtain a solution through the Chebyshev's theorem.

(3) Let $\bar{P}_5(y) = y^3(\alpha + \beta y)^2$. This form corresponds to the case of one double and one triple zeros. We have $y' = \pm y^{3/2}(\alpha + \beta y)$, so

$$\int y^{-3/2}(\alpha + \beta y)^{-1}dy = \pm\xi + A. \tag{A1}$$

Here $a = -3/2$, $b = 1$, and $c = -1$. Hence

$$(1)\ \frac{a+1}{b} + c = -3/2 \notin \mathbb{Z}, \quad (2)\ \frac{a+1}{b} = -1/2 \notin \mathbb{Z}, \quad (3)\ c = -1 \in \mathbb{Z}.$$

For (3), from (A1), by letting $u = \beta y^b/\alpha$, we obtain

$$-2\alpha^{-3/2}\beta^{1/2}u^{-1/2}F\left(-\frac{1}{2}, 3; \frac{1}{2}; u\right) = -\frac{2}{\alpha\sqrt{y}} + \frac{2\sqrt{\beta}\arctan\left(\frac{\sqrt{\beta y}}{\sqrt{\alpha}}\right)}{\alpha^{3/2}} = \pm\xi + A. \tag{A2}$$

(4) Let $\bar{P}_5(y) = (\alpha + \beta y)^5$. This form corresponds to the case of a zero with multiplicity five. We have $y' = \pm(\alpha + \beta y)^{5/2}$, so

$$\int (\alpha + \beta y)^{-5/2}dy = \pm\xi + A. \tag{A3}$$

Here $a = 0$, $b = 1$, and $c = -5/2$. Hence

$$(1)\ \frac{a+1}{b} + c = -3/2 \notin \mathbb{Z}, \quad (2)\ \frac{a+1}{b} = 1 \in \mathbb{Z}, \quad (3)\ c = -5/2 \notin \mathbb{Z}.$$

For (2), from (A3), by letting $u = \beta y^b/\alpha$, we obtain

$$\alpha^{-3/2}\beta^{-1}uF\left(1, \frac{9}{2}; 2; u\right) = -\frac{2}{3\beta(\beta y + \alpha)^{3/2}} = \pm\xi + A. \tag{A4}$$

Here we get

$$y = \frac{1}{\beta}\left[\left(\frac{4}{9\beta^2(\pm\xi + A)^2}\right) - \alpha\right]. \tag{A5}$$

(5) Let $\bar{P}_5(y) = (\alpha + \beta y^5)$. This form corresponds to the case of one simple zero. We have $y' = \pm(\alpha + \beta y^5)^{1/2}$, so

$$\int (\alpha + \beta y^5)^{-1/2} dy = \pm\xi + A.$$

Here $a = 0$, $b = 5$, and $c = -1/2$. We have

(1) $\dfrac{a+1}{b} + c = -3/10 \notin \mathbb{Z}$, (2) $\dfrac{a+1}{b} = 1/5 \notin \mathbb{Z}$, (3) $c = -1/2 \notin \mathbb{Z}$.

Hence we cannot obtain a solution through the Chebyshev's theorem.

APPENDIX B: OTHER CASES USING ALTERNATIVE METHOD FOR DEGENERATE THREE-COUPLED KdV EQUATION

Here we present the other cases for the method of factorization of the polynomial $P_5(f)$.

Case 2. If (9) has three real roots, we can write it in the form

$$(f')^2 = \alpha(f - f_1)(f - f_2)(f - f_3)(af^2 + bf + c), \tag{B1}$$

where f_1, f_2, and f_3 are the zeros of the polynomial function $P_5(f)$ of degree five and $b^2 - 4ac < 0$. Now define a new function $\rho(\xi)$, so that $f = f(\rho(\xi))$. Hence (B1) becomes

$$(f')^2 = \Big(\frac{df}{d\rho}\Big)^2\Big(\frac{d\rho}{d\xi}\Big)^2 = \alpha(f - f_1)(f - f_2)(f - f_3)(af^2 + bf + c). \tag{B2}$$

(i) Take

$$\Big(\frac{df}{d\rho}\Big)^2 = \kappa(f - f_1)(f - f_2)(f - f_3), \tag{B3}$$

$$\Big(\frac{d\rho}{d\xi}\Big)^2 = \mu(af^2 + bf + c), \tag{B4}$$

where $\kappa\mu = \alpha$. Consider the first equation above. We have

$$\frac{df}{\sqrt{(f - f_1)(f - f_2)(f - f_3)}} = \pm\sqrt{\kappa}\,d\rho. \tag{B5}$$

We know from Case 1(ii) that Eq. (B3) has a solution

$$f = h(\rho) = (f_2 - f_1)\mathrm{sn}^2((1/2)\sqrt{f_3 - f_1}(\sqrt{\kappa}\rho + C_1)) + f_1, \tag{B6}$$

where C_1 is a constant. We use this solution in (B4), and we get

$$\int_0^\rho \frac{d\hat{\rho}}{\sqrt{ah^2(\hat{\rho}) + bh(\hat{\rho}) + c}} = \int_0^\rho \frac{d\hat{\rho}}{\sqrt{A\,\mathrm{sn}^4(\omega\hat{\rho} + \delta) + B\,\mathrm{sn}^2(\omega\hat{\rho} + \delta) + C}} = \pm\sqrt{\mu}\xi + C_2, \tag{B7}$$

where C_2 is a constant and

$$A = a(f_2 - f_1)^2, \ B = (2af_1 + b)(f_2 - f_1), \ C = (af_1^2 + bf_1 + c), \ \omega = (1/2)\sqrt{\kappa(f_3 - f_1)},$$

and $\delta = (1/2)\sqrt{f_3 - f_1}\,C_1$.

(ii) Take

$$\Big(\frac{df}{d\rho}\Big)^2 = \kappa(f - f_1)(f - f_2), \tag{B8}$$

$$\Big(\frac{d\rho}{d\xi}\Big)^2 = \mu(f - f_3)(af^2 + bf + c), \tag{B9}$$

where $\kappa\mu = \alpha$. Consider first Equation (B8). It has a solution

$$f = h(\rho) = a_1 + b_1\cosh(\sqrt{\kappa}\rho + C_1), \quad a_1 = (f_1 + f_2)/2, \quad b_1 = (f_1 - f_2)/2, \tag{B10}$$

as it is found in Case 1(iii). We use this solution in (B9) and get

$$
\int_0^\rho \frac{d\hat{\rho}}{\sqrt{(h(\hat{\rho}) - f_3)(ah^2(\hat{\rho}) + bh(\hat{\rho}) + c)}} =
$$
$$
= \int_0^\rho \frac{d\hat{\rho}}{\sqrt{A\cosh^3(\sqrt{\kappa}\hat{\rho} + C_1) + B\cosh^2(\sqrt{\kappa}\hat{\rho} + C_1) + C\cosh(\sqrt{\kappa}\hat{\rho} + C_1) + D}}
$$
$$
= \pm\sqrt{\mu}\xi + C_2, \tag{B11}
$$

where C_2 is a constant and

$$
A = ab_1^3,\ B = b_1^2(3aa_1 + b - f_3),\ C = b_1(3aa_1^2 - 2aa_1f_3 + 2ba_1 - f_3b + c),
$$

and $D = (a_1 - f_3)(a_1^2a + ba_1 + c)$.

(iii) Take

$$
\left(\frac{df}{d\rho}\right)^2 = \kappa(f - f_1), \tag{B12}
$$

$$
\left(\frac{d\rho}{d\xi}\right)^2 = \mu(f - f_2)(f - f_3)(af^2 + bf + c), \tag{B13}
$$

where $\kappa\mu = \alpha$. Consider Equation (B12). It has a solution

$$
f = h(\rho) = \left(\pm\frac{\sqrt{\kappa}}{2}\rho + C_1\right)^2 + f_1,\quad C_1 \quad \text{constant}, \tag{B14}
$$

as it is obtained in Case 1(iv). We use this result in Eq. (B13)

$$
\int_0^\rho \frac{d\hat{\rho}}{\sqrt{(h(\hat{\rho}) - f_2)(h(\hat{\rho}) - f_3)(ah^2(\hat{\rho}) + bh(\hat{\rho}) + c)}} =
$$
$$
= \int_0^\rho \frac{d\hat{\rho}}{\sqrt{(\frac{\kappa}{4}\hat{\rho}^2 \pm \sqrt{\kappa}C_1\hat{\rho} + A)(\frac{\kappa}{4}\hat{\rho}^2 \pm \sqrt{\kappa}C_1\hat{\rho} + B)(\frac{a\kappa^2}{16}\hat{\rho}^4 \pm \frac{a\kappa^{3/2}}{2}C_1\hat{\rho}^3 + C\hat{\rho}^2 + D\hat{\rho} + E)}}
$$
$$
= \pm\sqrt{\mu}\xi + C_2, \tag{B15}
$$

where C_2 is a constant and

$$
A = C_1^2 + f_1 - f_2,\ B = C_1^2 + f_1 - f_3,\ C = \kappa(b + 6aC_1^2 + 2af_1)/4
$$

and

$$
D = \pm\sqrt{\kappa}C_1(2af_1 + 2aC_1^2 + b),\ E = (f_1 + C_1^2)(af_1 + aC_1^2 + b) + c.
$$

(iv) Take

$$
\left(\frac{df}{d\rho}\right)^2 = \kappa(af^2 + bf + c), \tag{B16}
$$

$$
\left(\frac{d\rho}{d\xi}\right)^2 = \mu(f - f_1)(f - f_2)(f - f_3), \tag{B17}
$$

where $\kappa\mu = \alpha$. Consider the first equation above. We have

$$
\int_0^f \frac{d\hat{f}}{\sqrt{a}\sqrt{\left(\frac{\hat{f} + \frac{b}{2a}}{W}\right)^2 + 1}} = \pm\sqrt{\kappa}\rho + C_1,
$$

where $W = \sqrt{4ac - b^2}/2a$. Let $\left(f + \frac{b}{2a}\right)/W = \tan\theta$. Hence the above equality becomes

$$
\int_0^\theta \sec\hat{\theta}d\hat{\theta} = \sqrt{a}(\pm\sqrt{\kappa}\rho + C_1),
$$

which gives

$$
\ln|\sec\theta + \tan\theta| = \sqrt{a}(\pm\sqrt{\kappa}\rho + C_1).
$$

Hence after some simplifications and taking the integration constant zero, we get

$$f = h(\rho) = \pm\sqrt{4ac - b^2}\sinh(\sqrt{a}\kappa\rho)/2a. \tag{B18}$$

We use this result in (B17),

$$\int_0^\rho \frac{d\hat{\rho}}{\sqrt{(h(\hat{\rho}) - f_1)(h(\hat{\rho}) - f_2)(h(\hat{\rho}) - f_3)}} =$$

$$= \int_0^\rho \frac{d\hat{\rho}}{\sqrt{A\sinh^3(\sqrt{a}\kappa\hat{\rho}) + B\sinh^2(\sqrt{a}\kappa\hat{\rho}) + C\sinh(\sqrt{a}\kappa\hat{\rho}) + D}}$$

$$= \pm\sqrt{\mu}\xi + C_2, \tag{B19}$$

where C_2 is a constant and

$$A = \pm(4ac - b^2)^{-3/2}/8a^3, B = (b^2 - 4ac)(f_1 + f_2 + f_3)/4a^2,$$

and

$$C = \pm\sqrt{4ac - b^2}(f_1 f_3 + f_2 f_3 + f_1 f_2)/2a, D = -f_1 f_2 f_3.$$

Case 3. If (9) has just one real root, we can write it in the form

$$(f')^2 = \alpha(f - f_1)(f^4 + a_3 f^3 + a_2 f^2 + a_1 f + a_0), \tag{B20}$$

where f_1 is the zero of the polynomial function $P_5(f)$ of degree five and the constants a_i, $i = 0, 1, 2, 3$ are so that $f^4 + a_3 f^3 + a_2 f^2 + a_1 f + a_0 \neq 0$ for real f.

Now define a new function $\rho(\xi)$, so that $f = f(\rho(\xi))$. Hence (B20) becomes

$$(f')^2 = \left(\frac{df}{d\rho}\right)^2\left(\frac{d\rho}{d\xi}\right)^2 = \alpha(f - f_1)(f^4 + a_3 f^3 + a_2 f^2 + a_1 f + a_0). \tag{B21}$$

(i) Take

$$\left(\frac{df}{d\rho}\right)^2 = \kappa(f - f_1), \tag{B22}$$

$$\left(\frac{d\rho}{d\xi}\right)^2 = \mu(f^4 + a_3 f^3 + a_2 f^2 + a_1 f + a_0), \tag{B23}$$

where $\kappa\mu = \alpha$. Consider the first equation above. It has a solution

$$f = h(\rho) = \left(\pm\frac{\sqrt{\kappa}}{2}\rho + C_1\right)^2 + f_1, \quad C_1 \quad \text{constant}, \tag{B24}$$

as it is obtained in Case 1(iv). We use this result in (B23) and get

$$\int_0^\rho \frac{d\hat{\rho}}{\sqrt{h^4(\hat{\rho}) + a_3 h^3(\hat{\rho}) + a_2 h^2(\hat{\rho}) + a_1 h(\hat{\rho}) + a_0}} = \pm\sqrt{\mu}\xi + C_2, \tag{B25}$$

where C_2 is a constant. Here we do not get a simpler expression than the original equation. Hence it is meaningless to use the method of factorization of the polynomial for this case.

(ii) Take

$$\left(\frac{df}{d\rho}\right)^2 = \kappa(f^4 + a_3 f^3 + a_2 f^2 + a_1 f + a_0), \tag{B26}$$

$$\left(\frac{d\rho}{d\xi}\right)^2 = \mu(f - f_1), \tag{B27}$$

where $\kappa\mu = \alpha$. Here for simplicity, we will use the form $f^4 + bf^2 + c$ instead of the polynomial function of degree four above. Since $f^4 + bf^2 + c = (f^2 + b/2)^2 - b^2/4 + c$, we will take $c > b^2/4$ to get an irreducible polynomial. If $c = b^2/4$, then take $b > 0$. From $\left(\frac{df}{d\rho}\right)^2 = \kappa(f^4 + bf^2 + c)$,

$$\int_0^f \frac{d\hat{f}}{\sqrt{\hat{f}^4 + b\hat{f}^2 + c}} = \pm\sqrt{\kappa}\rho + C_1, \quad C_1 \quad \text{constant}.$$

We have

$$f(\rho) = \frac{\sqrt{2c}}{\sqrt{-b + \sqrt{-4c + b^2}}} \mathrm{sn}((\sqrt{2}/2)(C_1 \pm \sqrt{\kappa}\rho)\sqrt{-b + \sqrt{-4c + b^2}}, k),$$

where $k = \frac{\sqrt{2}}{2}\sqrt{\frac{-2c + b^2 + b\sqrt{-4c + b^2}}{c}}$. Then we use this function f in (B27) and get

$$\int_0^\rho \frac{\sqrt{A}d\hat{\rho}}{\sqrt{\sqrt{2c}\,\mathrm{sn}((A\sqrt{2}/2)(C_1 \pm \sqrt{\kappa}\rho), k) - Af_1}} = \pm\sqrt{\mu}\xi + C_2, \quad C_2 \quad \text{constant,} \qquad (B28)$$

where $A = \sqrt{-b + \sqrt{-4c + b^2}}$.

Now take $c = b^2/4$ with $b > 0$. In this case, we have

$$\int_0^f \frac{d\hat{f}}{\hat{f}^2 + b/2} = \pm\sqrt{\kappa}\rho + C_1, \quad C_1 \quad \text{constant,}$$

which gives the solution

$$f = h(\rho) = \pm\frac{\sqrt{2b}}{2}\tan\left(\frac{\sqrt{2b}}{2}(\sqrt{\kappa}\rho + \tilde{C}_1)\right). \qquad (B29)$$

Now use Equation (B27). We have

$$\int_0^\rho \frac{d\hat{\rho}}{\sqrt{\pm\frac{\sqrt{2b}}{2}\tan\left(\frac{\sqrt{2b}}{2}(\sqrt{\kappa}\hat{\rho} + \tilde{C}_1)\right) - f_1}} = \pm\sqrt{\mu}\xi + C_2,$$

where C_2 is a constant. Solving the above equation gives

$$\frac{\pm 4}{\sqrt{\kappa}b(A_2^2 + A_3^2)A_3}\left\{\frac{\sqrt{2b}}{4}A_3^2\ln\left[\frac{(A_1 - A_2)^2 + A_3^2}{(A_1 + A_2)^2 + A_3^2}\right] - b\arctan\left(\frac{2A_1A_3}{A_3^2 - A_1^2 + A_2^2}\right)\right\} = \pm\sqrt{\mu}\xi + C_2,$$

where

$$A_1 = \sqrt{\pm 2\sqrt{2b}\tan\left(\frac{\sqrt{2b}}{2}(\sqrt{\kappa}\rho + \tilde{C}_1)\right) - 4f_1}, A_2 = \sqrt{\sqrt{4f_1^2 + 2b} - 2f_1},$$

and $A_3 = \sqrt{\sqrt{4f_1^2 + 2b} + 2f_1}$.

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