

POINCARÉ DUALITY IN MODULAR COINVARIANT RINGS

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ABSTRACT. We classify the modular representations of a cyclic group of prime order whose corresponding rings of coinvariants are Poincaré duality algebras. It turns out that these algebras are actually complete intersections. For other representations we demonstrate that the dimension of the top degree of the coinvariants grows at least linearly with respect to the number of summands of dimension at least four in the representation.

1. INTRODUCTION

Let V be a finite dimensional representation of a finite group G over a field k . The representation is called modular if the characteristic of k divides the order of G . Otherwise it is called non-modular. The induced action on V^* extends naturally to $k[V] := S(V^*)$ by graded algebra automorphisms. We let $k[V]^G := \{f \in k[V] \mid g(f) = f \ \forall g \in G\}$ denote the subalgebra of invariant polynomials in $k[V]$. A classical problem is to characterize the representations whose invariant rings are polynomial. By the famous result of Chevalley [3] and Shephard and Todd [17], a non-modular representation $k[V]^G$ is polynomial if and only if G is generated by reflections. In the modular case Serre [12] proved that G still has to be a reflection group if $k[V]^G$ is polynomial, but there are modular reflection groups with non-polynomial invariants. In this paper we study the ring of coinvariants which is the quotient ring

$$k[V]_G := k[V]/I$$

where $I := k[V]_+^G \cdot k[V]$ is the Hilbert ideal of V in $k[V]$ generated by invariants of positive degree. Since G is finite, $k[V]_G$ is a finite dimensional vector space. Coinvariants provide information about the invariants and often play an important role in the construction of the invariant ring. The polynomial property of the invariants is also encoded in the coinvariants. In the non-modular case $k[V]^G$ is polynomial if and only if $k[V]_G$ satisfies Poincaré duality. For representations in characteristic zero, this result is due to Steinberg [20] and Kane [8]. For non-modular representations over finite fields this result was proven by Lin [10] and for general non-modular representations it was proven by Dwyer and Wilkerson [5]. This equivalence does not hold in the modular case as first noted by Smith in [18] where he showed that the coinvariant ring of a four dimensional representation of C_2 is a Poincaré duality algebra but the corresponding invariants do not generate a polynomial ring. Nevertheless, in the same source it is shown that when $\dim V = 2$

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equivalence in the non-modular case carries over and when $\dim V = 3$ a weaker implication holds in the following way: If G is a p -group and $k[V]_G$ is a Poincaré duality algebra, then the Hilbert ideal I of V is generated by three elements, i.e., $k[V]_G$ is a complete intersection. In [16] it is proven that, with the exception of the regular representation, all coinvariant rings of the modular indecomposable representations of the Klein four group are complete intersections (while that of the regular representation is not a Poincaré duality algebra). More examples of Poincaré duality coinvariant rings are provided in [5] where the invariant rings are not even Cohen-Macaulay. And as noted in [5, Remark 4.5] in all these examples the coinvariants are in fact complete intersections.

Before we describe our work we recall the representation theory of a cyclic group C_p of prime order p over a field k of characteristic p . Over k , there are exactly p indecomposable C_p -representations V_1, \dots, V_p and each indecomposable representation V_i is afforded by a Jordan block of dimension i with 1's on the diagonal ([1, p. 38]). Our main result in this note is the following classification theorem.

Theorem 1. *Assume the notation of the previous paragraph. For a modular representation $V = \sum_{1 \leq j \leq l} V_{n_j}$ of C_p with $n_j > 1$, the ring of coinvariants $k[V]_{C_p}$ is a Poincaré duality algebra if and only if either $V = lV_2$ or $V = V_3 + (l-1)V_2$, where l is a positive integer.*

As in the previously studied examples, Poincaré duality coinvariant rings that we list in the theorem are actually complete intersections. We feel that the accumulation of these findings gives support for Conjecture 8. Also in the process of the proof of Theorem 1 we show that, when V is not lV_2 or $V_3 + (l-1)V_2$, the dimension of the top degree piece of $k[V]_{C_p}$ grows at least linearly with respect to the number of summands in V of dimension at least three or four (see Proposition 5 for the precise statement). In the final section we demonstrate that for each V there is a separating set of invariants whose corresponding quotient ring is a complete intersection.

For more background on modular invariant theory we refer the reader to [2] and [4].

2. POINCARÉ DUALITY IN MODULAR COINVARIANT RINGS

A graded ring $R = \bigoplus_{i=0}^d R^i$ with $R^0 = k$ is called a *Poincaré duality algebra* provided

- (1) $\dim_k(R^d) = 1$;
- (2) the bilinear form $R^i \otimes_k R^{d-i} \xrightarrow{a \otimes b \mapsto ab} R^d$ induces a perfect pairing, i.e., $a \in R^i$ is 0 if and only if $a \cdot b = 0$ for all $b \in R^{d-i}$.

Note that for Artinian rings, satisfying Poincaré duality is equivalent to being Gorenstein. Before specializing to a cyclic group of prime order, we prove a couple of general results.

Lemma 2. *Let V and W be representations of G . Fix a term order on $k[V \oplus W]$. Let M be a monomial in $k[V]$ that is not a lead term in the Hilbert ideal of V . Then M is not a lead term in the Hilbert ideal of $V \oplus W$.*

Proof. Dual to the inclusion $V \rightarrow V \oplus W$, there is a G -equivariant surjection $\pi : k[V \oplus W] \twoheadrightarrow k[V]$ which is given by the restriction $f \mapsto f|_V$. Assume on the contrary that a monomial M in $k[V]$ is a lead term in the Hilbert ideal of $V \oplus W$.

Then there exists $g_i \in k[V \oplus W]$ and $f_i \in k[V \oplus W]_+^G$ such that the lead term of $\sum f_i g_i$ is M . But if the lead term of a polynomial survives after restricting to V , then it stays the lead term after this restriction. So it follows that M is also the lead term of the polynomial $\sum \pi(f_i) \pi(g_i)$. But this gives a contradiction as $\pi(f_i)$ is either zero or is in $k[V]_+^G$ and so $\sum \pi(f_i) \pi(g_i)$ is in the Hilbert ideal of V . \square

The maximal degree of a polynomial in a minimal homogeneous generating set for $k[V]^G$ is known as the Noether number of V and is denoted by $\beta(G, V)$. We also denote the largest degree in which $k[V]_G$ is non-zero by $\text{topdeg } k[V]_G$.

Remark 3. In the modular case, $\text{topdeg } k[V]_G$ gives an upper bound for the maximal degree of an indecomposable transfer, i.e., an invariant of the form $\text{Tr}(f) := \sum_{g \in G} g(f)$. To see this, take a minimal homogeneous generating set f_1, \dots, f_t for $k[V]$ as a module over $k[V]^G$. By the graded Nakayama Lemma, $\text{topdeg } k[V]_G$ is equal to the maximum of the degrees of these module generators. Let $f \in k[V]$ be an arbitrary homogeneous element whose degree is strictly larger than $\text{topdeg } k[V]_G$. Write $f = \sum g_i f_i$, where $g_i \in k[V]^G$ are homogeneous. Then $\text{Tr}(f) = \sum g_i \text{Tr}(f_i)$. Therefore $\text{Tr}(f)$ is expressible in terms of invariants of strictly smaller degree since $\text{Tr}(c) = 0$ for any constant $c \in k$.

For the rest of this section, we write $V = \sum_{1 \leq j \leq l} V_{n_j}$ for an arbitrary representation of C_p and assume that $n_1 \geq n_2 \geq \dots \geq n_l$. Since trivial summands do not contribute to the coinvariant ring we also assume that $n_l > 1$. We identify $k[V]$ with $k[X_{i,j} \mid 1 \leq i \leq n_j, 1 \leq j \leq l]$ and assume that a fixed generator σ of C_p acts on the variables by $\sigma(X_{i,j}) = X_{i,j} + X_{i-1,j}$ for $1 < i \leq n_j$ and $\sigma(X_{1,j}) = X_{1,j}$. We use graded reverse lexicographic order on $k[V]$ with $X_{i_1,j_1} < X_{i_2,j_2}$ for $i_1 < i_2$ and $X_{i_1,j_1} < X_{i_2,j_2}$ if $j_1 < j_2$. The number $\beta(C_p, V)$ has been computed for all such V in [6]. In [2, §7.7] another computation of $\beta(C_p, V)$ is given which does not use the monotonicity of the Noether number. As an intermediate result it is shown that $\text{topdeg } k[V]_{C_p}$ and $\beta(C_p, V)$ actually coincide. Conversely, it is also possible to compute $\text{topdeg } k[V]_{C_p}$ as a corollary to the monotonicity and results in [6].

Proposition 4. *Assume the convention of the previous paragraph. We have*

$$\text{topdeg } k[V]_{C_p} = \begin{cases} l(p-1) + p - 2 & \text{if } n_1 > 3; \\ l(p-1) + 1 & \text{if } n_1 \leq 3 \text{ and } V \neq lV_2. \end{cases}$$

Proof. Assume that $n_1 > 3$. Then by [6, 3.3] $\text{topdeg } k[V]_{C_p}$ is bounded above by $l(p-1) + p - 2$. On the other hand, $k[V]_{C_p}$ is generated by transfers, orbit products of the variables $X_{n_j,j}$ for $1 \leq j \leq l$ and invariants of degree at most $l(p-1) - (\dim V - l)$ by [7, 2.12]. Note that orbit products of the variables have degree p and $\beta(C_p, V) = l(p-1) + p - 2$ by [6, 1.1]. Since $l(p-1) + p - 2$ is strictly larger than $l(p-1) - (\dim V - l)$, it follows that the maximum degree of an indecomposable transfer is $l(p-1) + p - 2$. So by the previous remark $\text{topdeg } k[V]_{C_p} \geq l(p-1) + p - 2$ as well. The second case is treated similarly as follows. We have $\beta(C_p, V) = l(p-1) + 1$ by [6, 1.1]. The description of the generating set for $k[V]_{C_p}$ in [7, 2.12] again gives that there is an indecomposable transfer of degree $l(p-1) + 1$. So from the previous remark we get $\text{topdeg } k[V]_{C_p} \geq l(p-1) + 1$. The reverse inequality follows from [15, 2.8]. \square

We show that each summand in V with large enough dimension contributes to the dimension of the top degree of $k[V]_{C_p}$.

Proposition 5. (1) Assume that $n_1 > 3$ and let l' be the number of summands in V whose dimensions are at least four. Then

$$\dim_k k[V]_{C_p}^{l(p-1)+p+2} \geq l'.$$

(2) Assume that $n_1 \leq 3$ and let l' denote the number of summands in V whose dimension is three. Assume further that $l' \neq 0$. Then

$$\dim_k k[V]_{C_p}^{l(p-1)+1} \geq l'.$$

Proof. By Lemma 6, $X_{n_i-1,i}^{p-2}X_{n_i,i}^{p-1}$ is not a lead term in the Hilbert ideal of V_{n_i} for $1 \leq i \leq l'$. Then by Lemma 2 it follows that $X_{n_i-1,i}^{p-2}X_{n_i,i}^{p-1}$ is not a lead term in the Hilbert ideal of V for $1 \leq i \leq l'$. So from [14, Remark 4] we get that $X_{n_i-1,i}^{p-2} \prod_{1 \leq j \leq l} X_{n_j,j}^{p-1}$ is not a lead term in the Hilbert ideal of V for $1 \leq i \leq l'$. Hence the first assertion follows.

Secondly, by [15, 2.6] $X_{2,i}X_{3,i}^{p-1}$ is not a lead term in the Hilbert ideal of V_{n_i} for $1 \leq i \leq l'$. Then as in the first case, using Lemma 2 and [14, Remark 4] we get that $X_{2,i} \prod_{1 \leq j \leq l} X_{n_j,j}^{p-1}$ is not a lead term in the Hilbert ideal of V for $1 \leq i \leq l'$. \square

Lemma 6. Let $V = V_{n_1}$ and $n_1 > 3$. Then the monomial $X_{n_1-1,1}^{p-2}X_{n_1,1}^{p-1}$ is not a lead term in the Hilbert ideal of V .

Proof. For simplicity put n for n_1 and X_i for $X_{i,1}$. We proceed by induction on n . A reduced Gröbner basis for the Hilbert ideal of V_4 has been computed in [15, 3.2]. We see from there that $X_3^{p-2}X_4^{p-1}$ is not a lead term in the Hilbert ideal of V_4 . Now let $n > 4$. To avoid ambiguity identify V_{n-1}^* with Y_1, \dots, Y_{n-1} where Y_1 is the fixed point. Dual to the inclusion $V_{n-1} \rightarrow V_n$, there is surjection $\pi : k[V_n] = k[X_1, \dots, X_n] \twoheadrightarrow k[V_{n-1}] = k[Y_1, \dots, Y_{n-1}]$ given by $\pi(X_i) = Y_{i-1}$ for $2 \leq i \leq n$ and $\pi(X_1) = 0$. By induction $Y_{n-2}^{p-2}Y_{n-1}^{p-1}$ is not a lead term in the Hilbert ideal of V_{n-1} . But π maps the Hilbert ideal of V_n into the Hilbert ideal of V_{n-1} , so $X_{n-1}^{p-2}X_n^{p-1}$ is not a lead term in the Hilbert ideal of V_n either because $\pi(X_{n-1}^{p-2}X_n^{p-1}) = Y_{n-2}^{p-2}Y_{n-1}^{p-1}$ and $\text{LT}(\pi(f)) = \pi(\text{LT}(f))$ for f not in the kernel of π . \square

We are now ready to prove Theorem 1.

Proof of Theorem 1. We have $k[V]_{C_p} = k[X_{2,1}, \dots, X_{2,l}]/(X_{2,1}^p, \dots, X_{2,l}^p)$ if $V = lV_2$ by [15, 2.6]. From the same source we also have

$$k[V]_{C_p} = k[X_{2,1}, X_{3,1}, X_{2,2}, \dots, X_{2,l}]/(X_{2,1}^2, X_{3,1}^p, X_{2,2}^p, \dots, X_{2,l}^p),$$

if $V = V_3 + (l-1)V_2$. Notice that these algebras are complete intersections and so in particular they are Poincaré duality algebras. We show that $k[V]_{C_p}$ is not a Poincaré duality algebra for all remaining representations.

Assume first $n_1 \geq 4$. Then by the proof of Proposition 5, $X_{n_1-1,1}^{p-2} \prod_{1 \leq j \leq l} X_{n_j,j}^{p-1}$ is not a lead term in the Hilbert ideal of V . Note that the degree of this element is the top degree in $k[V]_{C_p}$ by Proposition 4, and hence it is a socle element. We show that $k[V]_{C_p}$ is not a Poincaré duality algebra by producing another monomial which is a socle element. Since $X_{1,j}$ for $1 \leq j \leq l$ span the invariant linear forms in $k[V]^{C_p}$, $X_{2,1}$ is not a lead term of an invariant polynomial. Among the monomials in $k[V]$ that is divisible by $X_{2,1}$ consider a monomial of maximal degree that is not a lead term in the Hilbert ideal of V . Call this monomial M . The class of M

is non-zero in the coinvariants. Moreover, since $n_1 \geq 4$ the divisibility condition by $X_{2,1}$ implies that $M \neq X_{n_1-1,1}^{p-2} \prod_{1 \leq j \leq l} X_{n_j,j}^{p-1}$. Let Y be any variable in $k[V]$. Then by maximality of the degree, MY is a leading term in the Hilbert ideal of V . Therefore its class in the coinvariants is equal to a linear combination of classes of monomials that are not lead terms in the Hilbert ideal and smaller in the order. By the maximality of divisibility by $X_{2,1}$, none of these monomials are divisible by $X_{2,1}$. By our ordering, the only variable that is smaller than $X_{2,1}$ is $X_{1,1}$. But the class of this variable in the coinvariants is zero. It follows that the class of MY is zero in the coinvariants. Equivalently, the class of M is a socle element in $k[V]_{C_p}$. Since the monomials that are not lead terms in the Hilbert ideal of V are linearly independent in $k[V]_{C_p}$, the dimension of the socle of $k[V]_{C_p}$ is at least two.

Finally, if $V = l'V_3 + (l-l')V_2$ with $l' \geq 2$, then the dimension of the top degree of $k[V]_{C_p}$ is strictly larger than one by Propositions 4 and 5. \square

Remark 7. From the proof of Theorem 1, one can see that $k[V]_{C_p}$ is a Poincaré duality algebra if and only if it is a complete intersection.

In general, an Artinian Poincaré duality algebra (i.e., an Artinian graded Gorenstein algebra) is not necessarily a complete intersection. For instance, the graded ring $R = \frac{k[X,Y,Z]}{(X^2-Y^2, Y^2-Z^2, XY, YZ, ZX)}$ is a Poincaré duality algebra but not a complete intersection ([11, p. 172]). Remark 7 says that the modular coinvariant rings of C_p have a special feature. The following example indicates that modular coinvariant rings of the alternating groups also share the same special feature.

Example 1 (Alternating groups). Let W denote the n -dimensional natural representation of the symmetric group S_n (and hence a representation of the alternating group A_n) over a field k . In [19], it is proved that the Hilbert ideal of A_n is the same as the one of S_n if and only if $\text{char}(k)$ divides the order of A_n . It is well known that the Hilbert ideal of S_n is generated by n elementary symmetric polynomials e_1, \dots, e_n (regardless of the characteristic of k) and depending on the characteristic, the Hilbert ideal of A_n has one more generator¹ which we denote by Δ . When $\text{char}(k)$ divides $|A_n|$, the ring of coinvariants $k[W]_{A_n}$ is the complete intersection $k[X_1, \dots, X_n]/(e_1, \dots, e_n)$ and hence is a Poincaré duality algebra. When $\text{char}(k)$ does not divide $|A_n|$, then $k[W]_{A_n} = k[X_1, \dots, X_n]/(e_1, \dots, e_n, \Delta)$ which is an almost complete intersection; hence not a Poincaré duality algebra by [9].

Based on our results and previously studied examples, we would like to propose a conjecture.

Conjecture 8. *Let G be a p -group and let V be a modular representation of G over a field k . Then $k[V]_G$ is a Poincaré duality algebra if and only if $k[V]_G$ is a complete intersection.*

3. COINVARIANTS WITH RESPECT TO A SEPARATING SET

Recall the following definition from [4, 2.3.8].

Definition 9. Let V be a representation of a group G over a field k . A subset $S \subseteq k[V]^G$ is called a *separating set* if for any two points $u, v \in V$ we have: If there

¹If $\text{char } k \neq 2$, then $\Delta = \prod_{1 \leq i < j \leq n} (x_i - x_j)$; if $\text{char}(k) = 2$, then Δ is the orbit sum $\sum_{\sigma \in A_n} x_{\sigma(1)}^0 \cdots x_{\sigma(n)}^{n-1}$.

exists an invariant $f \in k[V]^G$ with $f(u) \neq f(v)$, then there exists $s \in \mathcal{S}$ such that $s(u) \neq s(v)$.

Remark 10. Let G be a group and V be a representation of G over a field k , where k is a field of characteristic $p > 0$. Let \mathcal{S} be a separating set of $k[V]^G$ and \mathcal{T} be an arbitrary subset of \mathcal{S} . Set \mathcal{T}' to be $\{t^{p^{e(t)}} \mid t \in \mathcal{T}\}$ where $e(t)$ is a positive integer depending on t . Then

$$(\mathcal{S} \setminus \mathcal{T}) \cup \mathcal{T}'$$

is also a separating set, since $t(u) \neq t(v)$ if and only if $t^{p^{e(t)}}(u) \neq t^{p^{e(t)}}(v)$ for all $u, v \in V$.

Definition 11. Let G be a group and V be a representation of G over k , where k is a field of characteristic $p > 0$. Let \mathcal{S} be a separating set in $k[V]^G$. We will call the ideal generated by \mathcal{S} in $k[V]$ the *separating Hilbert ideal associated to \mathcal{S}* and denote it by $\mathcal{H}(\mathcal{S})$. We will call the quotient ring $k[V]/\mathcal{H}(\mathcal{S})$ the ring separating coinvariants associated to \mathcal{S} .

Clearly, if one chooses \mathcal{S} to be a generating set of invariants, then $\mathcal{H}(\mathcal{S})$ is the Hilbert ideal and $k[V]/\mathcal{H}(\mathcal{S})$ is the ring of coinvariants $k[V]_G$.

In general, there are many choices of separating sets; e.g. Remark 10. It turns out that, for each indecomposable modular representation of C_p , there is a separating set such that the corresponding ring separating coinvariants is a Poincaré duality algebra. We use the simplified notation from Lemma 6 for the indecomposable modular representation V_n of dimension n with $k[V_n] = k[X_1, \dots, X_n]$. Also set $N(f) = \prod_{g \in G} g(f)$ for $f \in k[V]$.

Proposition 12. Let V_n be an indecomposable modular representation of the cyclic group C_p over a field k . For $n \geq j \geq 2$, set

$$\mathcal{T}_j := \{N(X_j)^{p^{j-2}}, \text{Tr}(X_j X_i^{p^{j-1}})^{p^{j-2}} \mid 2 \leq i \leq j-1\}$$

and $\mathcal{S} := \{X_1\} \cup (\bigcup_{j=2}^n \mathcal{T}_j)$. Then

- (1) \mathcal{S} is a separating set of $k[V_n]^{C_p}$; and
- (2) $k[V_n]/\mathcal{H}(\mathcal{S})$ is a complete intersection.

Proof. (1) is an immediate consequence of [13, Theorem 3] and Remark 10.

(2). Set $\mathcal{S}_j = \{X_1\} \cup (\bigcup_{\ell=2}^j \mathcal{T}_\ell)$ and let $\mathcal{S}_j \cdot k[V_n]$ denote the ideal of $k[V_n]$ generated by \mathcal{S}_j . We will use induction on j to show that

$$(1) \quad \mathcal{S}_j \cdot k[V_n] = (X_1, \dots, X_\ell^{p^{\ell-1}}, \dots, X_j^{p^{j-1}}).$$

When $j = 2$, we have

$$N(X_2)^{p^{2-2}} = N(X_2) \equiv X_2^p \pmod{X_1}.$$

Hence

$$\mathcal{S}_2 \cdot k[V_n] = (X_1, X_2^p),$$

which proves the case when $j = 2$. Assume that we have proved (1) for \mathcal{S}_{j-1} . Then we have

$$N(X_j)^{p^{j-2}} \equiv X_j^{p^{j-1}} \pmod{\mathcal{S}_{j-1} \cdot k[V_n]}$$

and

$$\text{Tr}(X_j X_i^{p^{j-1}})^{p^{j-2}} \in \mathcal{S}_{j-1} \cdot k[V_n] = (X_1, X_2^p, \dots, X_{j-1}^{p^{j-2}}) \quad \text{for } 2 \leq i \leq j-1.$$

Therefore, (1) holds; in particular, $\mathcal{S} \cdot k[V_n] = (X_1, X_2^p, \dots, X_\ell^{p^{\ell-1}}, \dots, X_n^{p^{n-1}})$. Hence,

$$\frac{k[V_n]}{\mathcal{H}(\mathcal{S})} = \frac{k[X_1, \dots, X_n]}{(X_1, X_2^p, \dots, X_\ell^{p^{\ell-1}}, \dots, X_n^{p^{n-1}})}$$

which is a complete intersection. \square

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