PHYSICAL REVIEW E **93**, 032142 (2016)

Fluctuation-dissipation and energy properties of a finite bath

A. Carcaterra*
Department of Mechanical and Aerospace Engineering, University of Rome, La Sapienza, Via Eudossiana, 18, 00184, Rome, Italy

A. Akay†
Department of Mechanical Engineering, Bilkent University, 06800 Bilkent, Ankara, Turkey
(Received 24 April 2015; published 25 March 2016)

This paper expands a recent proposal by the authors to rederive the Langevin equation for a test particle in a finite-size thermal bath using a perturbation approach that yields a cascade of Langevin-type equations. Such an approach produces a different viewpoint for the fluctuation-dissipation duality by expressing them on similar scales. General properties of energy sharing between the test particle and the bath are outlined, investigating the resonant and nonresonant conditions.

DOI: 10.1103/PhysRevE.93.032142

I. INTRODUCTION

Advances in fluctuation theorems continue to be reported in different contexts ranging from classical to quantum dynamics and contribute to our understanding of nonequilibrium thermodynamics (viz. [1] and references in it). With increasing emphasis on nanoscale devices, thermal baths have also gained attention with a focus on fluctuation and dissipation in finite baths.

The motion of nanoscale devices is affected, and in some cases disturbed, by Brownian motion of molecules in their environment. Frequently, sensors and energy harvesting devices utilize micro- and nanoresonators. Examples include use of tunable particles as local probes [2] and use of micro and nanoresonators interacting optically with the electromagnetic field [3,4] to produce modulators, filters, delayers, and switches. Moreover, new devices that use deformable optical cavities [5], or wave-pump lasers to interact with microresonators [6] rely on opto-mechanical interactions in nanoresonators [7,8]. Interactions of a resonator in a bath can involve capturing of energy form environment and filtering the noise associated with fluctuations while also sensing. The interaction between a single mechanical resonator and a bath also forms the theoretical basis to model damping processes, for example, that of metallic resonators carrying free electrons [9]. The paradigm of the interaction between a single test particle and a molecular bath as described by Langevin's equation is used not only to investigate energy exchange between a simple oscillator and a complex system, but also to study such diverse topics as swarm robotics and the analysis of individuals belonging to a complex population [10].

Micro- and nanoresonators with characteristic dimensions between 10^{-1} and 10^{4} micron, with corresponding natural frequencies in the kHz to GHz range [11–13], can be tuned to bath fluctuations that may contain frequencies from the radio-wave or near-infrared regime up to ultraviolet range. The very small size of the environment with a limited bandwidth makes the number of the particles contained in the bath "countable," straining the approximation about infinite number of particles in the standard Langevin solutions. The approach developed in this paper does not employ such an approximation.

In Langevin equations, the fluctuations in a bath are represented stochastically and the dissipation by a reduced variable. In other words, fluctuations represent microscopic behavior through probabilistic descriptions, whereas dissipation is usually characterized as a macroscopic behavior. Conventionally, under equilibrium conditions, dissipation is expressed in terms of autocorrelation of fluctuations by invoking equipartitioning (viz., [14]). In such approaches, dynamics of particles in a heat bath are often modeled using independent linear oscillators attached to a test particle [15–22]. Assuming a large number of oscillators, $N \to \infty$, and an appropriate choice of frequency distribution, such as that given by Caldeira-Leggett, the fluctuation can be shown to be Markovian, satisfying the assumptions upon which Langevin equation is derived, viz. [18,20].

When the bath has a finite number of oscillators, and thus a finite bandwidth, fluctuation and dissipation conditions deviate from that described by the idealized Langevin equation ($N \to \infty$). For example, fluctuations are no longer Markovian and exhibit recurrence except for special frequency distributions (viz., [23]) and the kernel of the dissipative term cannot be represented by an integral. Our approach here is to recast the Langevin equation through a perturbation of its classical form, as we had briefly reported earlier [24]. The perturbation approach projects the fluctuation and dissipation terms at similar scales to avoid mixing the micro- and macroscales discussed above. Using the resulting equations, we consider a heat bath that consists of a finite number of oscillators in which a test particle is placed and examine the energy distribution and transfer between the test particle and the bath.

The remainder of the paper is organized as follows. In Sec. II we apply perturbation to Langevin equation, including the forcing and dissipation terms in it. In Sec. III, the dissipation and fluctuation terms are modified for finite-size baths. Modifications show that the purely dissipative character of the infinite bath is now accompanied by a reactive component, along with corresponding changes in the fluctuation expression. In Sec. IV, we derive the expression for the energy of the test particle using the results derived for a finite bath and examine the effects of bandwidth size and the location of the test particle frequency relative to the bandwidth using electromagnetic

*a.carcaterra@dma.ing.uniroma1.it
†akay@bilkent.edu.tr

analogies. Corresponding effects on energy distribution in the bath are examined in Sec. V.

II. LANGEVIN PERTURBATIONS FOR THE FINITE BATH, $\epsilon = 1/N$

Consider the Langevin equation for a test particle denoted by its coordinate $q(t)$, mass M, and natural frequency Ω in a finite bath of N particles,

$$M\ddot{q}(t) + M\Omega^2 q(t) + \int_0^t \Gamma(\tau)\dot{q}(t-\tau)d\tau = \Pi(t), \tag{1}$$

together with the equations of the bath particles, described by x_i,

$$\ddot{x}_i(t) + \omega_i^2 x_i(t) = \omega_i^2 q(t), \quad i = 1,\ldots,N, \tag{2}$$

and

$$\Gamma(t) = \sum_{i=1}^{N} m_i\omega_i^2 \cos\omega_i t, \quad \Pi(t) = \sum_{i=1}^{N} m_i\omega_i^2\left[x_i(0)\cos\omega_i t + \frac{\dot{x}_i(0)}{\omega_i}\sin\omega_i t\right], \quad t \geqslant 0, \tag{3}$$

where the test particle is initially at rest. We note here that in the original and subsequent widespread formulations of the Langevin equation, an integral form of Γ is utilized to describe dissipation and yet the summation is retained when expressing the fluctuations $\Pi(t)$. We show below that a perturbation-based analysis can make fluctuation and dissipation expressions more self-consistent and also allows examination of energy distribution in the bath.

Introducing the perturbation expansions for the general case of a large number of particles, N, in the bath,

$$\begin{aligned}
\Gamma(t) &= \Gamma_0(t) + \epsilon\Gamma_1(t) + \epsilon^2\Gamma_2(t) + \cdots, \\
\Pi(t) &= \Pi_0(t) + \epsilon\Pi_1(t) + \epsilon^2\Pi_2(t) + \cdots, \qquad t \geqslant 0, \\
q(t) &= q_0(t) + \epsilon q_1(t) + \epsilon^2 q_2(t) + \cdots \qquad (i = 1,\ldots,N), \\
x_i(t) &= x_{i0}(t) + \epsilon x_{i1}(t) + \epsilon^2 x_{i2}(t) + \cdots,
\end{aligned} \tag{4}$$

where the subscript zero represents the value that corresponds to the infinite size case, reached as $\epsilon = 1/N$ tends to zero while retaining the finite bandwidth of the spectrum. Specifically, Γ_0 and Π_0 represent the case for $N = \infty$ ($\epsilon = 0$) of Eq. (3) for a finite bandwidth, leading to the corresponding integral forms:

$$\begin{aligned}
\Gamma_0(t) &= \int_{\omega_{\min}}^{\omega_{\max}} \frac{dm}{d\omega}\omega^2 \cos\omega t\, d\omega, \\
\Pi_0(t) &= \int_{\omega_{\min}}^{\omega_{\max}} \frac{dm}{d\omega}\omega^2\left[x_\omega(0)\cos\omega t + \frac{\dot{x}_\omega(0)}{\omega}\sin\omega t\right]d\omega,
\end{aligned} \qquad t \geqslant 0. \tag{5}$$

In Eq. (5), $x_\omega(0)$ and $\dot{x}_\omega(0)$ represent the continuous forms of the initial conditions $x_i(0)$ and $\dot{x}_i(0)$, respectively, and $dm/d\omega$ represents the mass spectral density of the bath, which has a finite total mass m_b. The integration limits represent the lowest and the highest frequencies of the bath. Substitution of the expansions in Eq. (4) into Eq. (1) produces the perturbation Langevin equation [24]:

$$\begin{aligned}
&M\ddot{q}_0(t) + M\Omega^2 q_0(t) + \int_0^t \Gamma_0(\tau)\dot{q}_0(t-\tau)d\tau = \Pi_0(t) \quad (k = 0), \\
&M\ddot{q}_k(t) + M\Omega^2 q_k(t) + \int_0^t \Gamma_0(\tau)\dot{q}_k(t-\tau)d\tau = \Pi_k(t) - \sum_{j=1}^{k}\int_0^t \Gamma_j(\tau)\dot{q}_{k-j}(t-\tau)d\tau \quad (k = 1,2,3,\ldots).
\end{aligned} \tag{6}$$

Additionally for the bath we have

$$\ddot{x}_{ik}(t) + \omega_i^2 x_{ik}(t) = \omega_i^2 q_k(t) \quad (k = 1,2,3,\ldots, \quad i = 1,\ldots,N). \tag{7}$$

Explicit representations of Eqs (6) and (7) for orders $k = 0$ and $k = 1$ produce

$$M\ddot{q}_0(t) + M\Omega^2 q_0(t) + \int_0^t \Gamma_0(\tau)\dot{q}_0(t-\tau)d\tau = \Pi_0(t), \tag{8}$$

$$\ddot{x}_{i0}(t) + \omega_i^2 x_{i0}(t) = \omega_i^2 q_0(t), \tag{9}$$

$$M\ddot{q}_1(t) + M\Omega^2 q_1(t) + \int_0^t \Gamma_0(\tau)\dot{q}_1(t-\tau)d\tau = \Pi_1(t) - \int_0^t \Gamma_1(\tau)\dot{q}_0(t-\tau)\,d\tau, \tag{10}$$

$$\ddot{x}_{i1}(t) + \omega_i^2 x_{i1}(t) = \omega_i^2 q_1(t). \tag{11}$$

Equation (8)–(11) are valid for both transient and asymptotic conditions and, together with their solutions described later, represent the main result of this paper.

Asymptotic expansion of Eq. (5) with respect to time shows that the zeroth-order terms $\Gamma_0(t), \Pi_0(t)$ asymptotically vanish as $t \to \infty$ [24,25] as does $q_0(t)$, indicating energy transfer from the test particle to bath particles and to higher-order fluctuations of the test particle. Consequently, (i) in the long-time range q_0 does not contribute to $q(t)$, but does so only to the transient part and yet (ii) q_0 contributes to both transient and asymptotic evolution of $x_{i0}(t)$ since there is no damping in the equation of bath for any order.

Note that even Eq. (8) does not represent the conventional form of Langevin equation since $\Pi_0(t)$ in Eq. (8) is an integral and vanishes at long times. Nevertheless, these equations are essential in the study of bath energy distribution. Equation (8) yields q_0 and its substitution in Eq. (9) produces x_{i0}. Then by substituting for $\dot{q}_0$ in Eq. (10) we can solve for q_1 and x_1. In these calculations, x_{i0} is the first nonzero contribution to $x_i(t)$ and q_1 is the first nonzero contribution to $q(t)$ in long times.

Equations (6) and (7) form the basis for a generalized form of the Langevin equation, which we simplify by limiting the analysis to long-time behavior of the test particle where we note that for $k \geqslant 2$, the fluctuations on the right-hand side depend also on dissipation through the terms Γ_j even in the long times, in addition to the initial conditions of the bath.

The following sections show analytical solutions to Eqs. (6) and (7) for long times in a finite-size bath with a finite bandwidth. Consistent with the fluctuation-dissipation concept we (i) derive a relationship between the fluctuations and the dissipation, (ii) find closed-form expressions for the energy of the test particle and for the oscillators of the bath, and (iii) illustrate different energy sharing scenarios.

III. FLUCTUATION AND DISSIPATION

A. Dissipation

As noted above, in the long-time limit, q_0 vanishes, making $q \approx \epsilon q_1$. Similarly, $\Pi \approx \epsilon \Pi_1$ and $\Gamma \approx \epsilon \Gamma_1$, since Π_0 and Γ_0 also vanish in the long-time limit.

In order to examine the response q in the long time, we multiply both sides of Eq. (10) by ϵ:

$$M\epsilon\ddot{q}_1(t) + M\Omega^2\epsilon q_1(t) + \int_0^t \Gamma_0(t-\tau)\epsilon\dot{q}_1(\tau)d\tau = \epsilon\Pi_1(t) - \int_0^t \epsilon\Gamma_1(t-\tau)\dot{q}_0(\tau)d\tau. \quad (12)$$

Substitution for the long-time approximations from above leads to

$$M\ddot{q}(t) + M\Omega^2 q(t) + \int_0^t \Gamma_0(t-\tau)\dot{q}(\tau)d\tau = \Pi(t) - \int_0^t \Gamma(t-\tau)\dot{q}_0(\tau)\,d\tau. \quad (13)$$

The last term in Eq. (13) represents an additional fluctuation source compared to the conventional forms of Langevin equation. Since, as shown in the Appendix, the last term vanishes with respect to $\Pi(t)$ in the long time, it can be neglected simplifying (13):

$$M\ddot{q}(t) + M\Omega^2 q(t) + \int_0^t \Gamma_0(t-\tau)\dot{q}(\tau)d\tau = \Pi(t). \quad (14)$$

The Fourier transform $\mathcal{F}$ of Eq. (14) yields

$$[M(\Omega^2-\omega^2)+j\omega[D(\omega)+jR(\omega)]]Q(\omega) = \mathcal{F}\{\Pi(t)\}, \quad (15)$$

with the dissipative and reactive parts of the convolution given as

$$D(\omega) = \mathrm{Re}\{\mathcal{F}\{\Gamma_0\}\}, \quad (16)$$

$$R(\omega) = \mathrm{Im}\{\mathcal{F}\{\Gamma_0\}\}. \quad (17)$$

Although in physical systems, the spectral density $dm/d\omega$ is defined for $\omega > 0$, it can be folded symmetrically with respect to the frequency origin making it an even function about $\omega = 0$. With this assumption, the expressions for D and R are related by the Hilbert transfrom $\mathcal{H}$ [26] (Kramers-Kronig relations):

$$D(\omega) = \frac{\pi}{2}\frac{dm}{d\omega}\omega^2, \quad (18)$$

$$R(\omega) = -\mathcal{H}\left\{\frac{\pi}{2}\frac{dm}{d\omega}\omega^2\right\}. \quad (19)$$

Finally, the finite mass of the bath implies that D satisfies the condition

$$m_b = \frac{2}{\pi}\int_{\omega_{\min}}^{\omega_{\max}} \frac{D(\omega)}{\omega^2}\,d\omega. \quad (20)$$

B. Fluctuation

In the absence of other sources, fluctuations in the bath are due to the oscillation of the bath particles. Using Eq. (3), the fluctuation force can be expressed in terms of the initial conditions of the particles in the bath:

$$\Pi = \sum_{i=-N}^{N} A_i e^{j\omega_i t}, \quad (21)$$

with

$$A_i = \frac{1}{2}m_i\omega_i^2\left[x_i^2(0) + \frac{\dot{x}_i^2(0)}{\omega_i^2}\right]^{1/2} e^{j\phi_i},$$

where $A_{-i} = A_i^*$.

The spectrum $e_i(0)$ of the energy E_b of the bath associated with its initial conditions can be expressed as

$$e_i(0) = \frac{1}{2\Delta\omega}m_i\omega_i^2\left[x_i^2(0) + \frac{\dot{x}_i^2(0)}{\omega_i^2}\right],$$

which, together with the dissipation function, leads to the expression

$$|A_i|^2 = \frac{1}{\pi}D(\omega_i)e_i(0)(\Delta\omega)^2, \quad (22)$$

where $\Delta\omega = (\omega_{\max} - \omega_{\min})/N$ is the average frequency spacing. Expression (22) represents the fluctuation-dissipation relation.

For the above derivation, we have not imposed any randomness to the initial conditions. However, if $e_i(0)$ is a random vector, using ensemble averages we can express fluctuation-dissipation relationship (22) in terms of power spectral density:

$$S_\Pi(\omega_i) = \frac{\overline{|A_i|^2}}{\Delta\omega} = \frac{1}{\pi} D(\omega_i)\overline{e_i(0)}\Delta\omega. \tag{23}$$

Equation (23) represents a rather general form of the fluctuation-dissipation theorem relating the fluctuation spectrum to dissipation through the initial energy distribution.

As a special case, if energy equipartitioning is assumed within the bath, i.e., $e_i(0) = e$, with temperature $T \propto e\,\Delta\omega/k_B$, where k_B is the Boltzmann's constant, Eq. (23) reduces to $S_\Pi(\omega) \propto k_B T\, D(\omega)$. In this case the autocorrelation takes the form $\langle \Pi(t)\Pi(t+\tau)\rangle \propto \frac{1}{\pi} k_B T\, \Gamma(\tau)$.

The form of FDT expressed in Eq. (23) is used later in the paper when examining different energy sharing scenarios between the bath and the test particle. Inspection of Eq. (23) shows that in an infinite bath, but with a finite bandwidth and finite mass, as $N \to \infty$, $(\Delta\omega) \to 0$ and the fluctuations of the test particle vanish, demonstrating that fluctuations appear only under the hypothesis of a finite bath (finite N). In contrast, the associated dissipation is described by a continuous (integral) representation. A rationale for this twofold nature of the problem is revealed from the perturbation approach presented here. In the next section, we will examine solutions of Eqs. (8)–(11) for different conditions.

IV. ENERGY OF THE TEST PARTICLE

Here we examine the influence of the initial energy of the bath on the test particle and the means of energy conveyance from the bath to the particle. Using Eq. (15), the spectral density of the fluctuation $q(t)$ at first order can be found as the solution to Eq. (14):

$$S_q(\omega_i) = \frac{S_\Pi(\omega_i)}{\left|M\left(\Omega^2 - \omega_i^2\right) - \omega_i R(\omega_i) + j\omega_i D(\omega_i)\right|^2}. \tag{24}$$

Substitution for $S_\Pi(\omega_i)$ from Eq. (23) produces

$$S_q(\omega_i) = \frac{\Delta\omega}{\pi} \frac{\overline{e_i(0)} D(\omega_i)}{\left|M\left(\Omega^2 - \omega_i^2\right) - \omega_i R(\omega_i) + j\omega_i D(\omega_i)\right|^2}. \tag{25}$$

The corresponding asymptotic energy of the test particle is

$$E_q = 2M \sum_{i=1}^{N} \omega_i^2 S_q(\omega_i)\Delta\omega. \tag{26}$$

With the introduction of expression (25), the energy expression (26) for a test particle takes the form

$$E_q = \frac{2M\Delta\omega}{\pi} \times \sum_{i=1}^{N} \frac{\omega_i^2\, \overline{e_i(0)} D(\omega_i)}{\left|M\left(\Omega^2 - \omega_i^2\right) - \omega_i R(\omega_i) + j\omega_i D(\omega_i)\right|^2} \Delta\omega. \tag{27}$$

Under the perturbation hypothesis invoked here for small $\epsilon = 1/N$ (e.g., for small values of $\Delta\omega \neq 0$), consistent with the first-order analysis employed, the contributions of orders $(\Delta\omega)^2$ and higher can be neglected and the summation can be approximated by an integral:

$$E_q = \frac{2M\Delta\omega}{\pi} \int_{\omega_{\min}}^{\omega_{\max}} \frac{\omega^2\, \overline{e_\omega(0)} D(\omega)}{|M(\Omega^2 - \omega^2) - \omega R(\omega) + j\omega D(\omega)|^2} d\omega = \frac{2M\Delta\omega}{\pi} \int_{\omega_{\min}}^{\omega_{\max}} \frac{\overline{e_\omega(0)} D(\omega)}{|Z(\omega)|^2} d\omega, \tag{28}$$

where

$$Z(\omega) = (\omega D(\omega) + j[-M(\Omega^2 - \omega^2) + \omega R(\omega)])/\omega. \tag{29}$$

E_q now displays the product of particle response and bath spectrum [Eq. (23)]. $Z(\omega)$ represents the combined impedance of the uncoupled particle and the bath through $R(\omega)$ and $D(\omega)$. It is the coupling of the particle and the bath that gives rise to energy exchange at frequencies other than the uncoupled particle resonance. This property is the basis for energy exchange even when the uncoupled resonant frequency of the particle falls outside of the bath bandwidth, as discussed later.

Analogous to a simple oscillator, the test particle behavior can be examined through the energy expression above in terms of familiar nondimensional terms:

$$E_q = \frac{2}{\pi} \int_{\tilde\omega_{\min}}^{\tilde\omega_{\max}} \overline{e_\omega(0)}\, \Delta\omega \frac{\tilde\omega^2 \zeta(\tilde\omega)}{|(1 - \tilde\omega^2) - \tilde\omega\chi(\tilde\omega) + j\tilde\omega\zeta(\tilde\omega)|^2} d\tilde\omega, \tag{30}$$

where $\chi(\tilde\omega) = R(\omega)/M\Omega$ and $\zeta(\tilde\omega) = D(\omega)/M\Omega$ respectively describe the normalized values of reactive component and the damping ratio, and the normalized frequency is $\tilde\omega = \omega/\Omega$.

For a direct comparison with an infinite bath, we assume an initial energy in the bath with a Caldeira-Leggett distribution, $e_i(0) = e$, which implies a constant value for $D(\omega)$ within the bandwidth and can be described by a rectangular window function $\mathcal{W}$:

$$D(\omega) = C_D \mathcal{W}(\omega_{\min}, \omega_{\max}).$$

The one-sided bandwidth is the physical representation of the symmetrically folded bandwidth $D = C_D \mathcal{W}(-\omega_{\max}, -\omega_{\min}) + C_D \mathcal{W}(\omega_{\min}, \omega_{\max})$. From Eq. (19) follows a general expression for the reactive part of a finite bandwidth [27]

$$\chi(\tilde\omega) = -\frac{\zeta}{\pi}\left[\ln\left|\frac{\tilde\omega - \tilde\omega_{\min}}{\tilde\omega - \tilde\omega_{\max}}\right| - \ln\left|\frac{\tilde\omega + \tilde\omega_{\min}}{\tilde\omega + \tilde\omega_{\max}}\right|\right], \tag{31}$$

with

$$\zeta = \frac{\pi}{2}\frac{Nm}{M}\frac{\tilde\omega_{\min}\tilde\omega_{\max}}{\tilde\omega_{\max} - \tilde\omega_{\min}} = \frac{\pi}{2}\frac{m_b}{M}\frac{\tilde\omega_{\max}}{\rho - 1}, \quad \rho = \frac{\tilde\omega_{\max}}{\tilde\omega_{\min}},$$

where $\zeta = C_D/M\Omega$ was evaluated using Eq. (20) to obtain $C_D = (\pi/2)m_b(\tilde\omega_{\min}\tilde\omega_{\max})/(\tilde\omega_{\max} - \tilde\omega_{\min})$.

With the hypotheses that $e_i(0) = e$ constant, the energy integral (30) becomes

$$J_q = \frac{E_q}{\overline{e_\omega(0)}\,\Delta\omega} = \frac{E_q}{E_b/N} = \frac{2\zeta}{\pi} \int_{\tilde\omega_{\min}}^{\tilde\omega_{\max}} \frac{\tilde\omega^2}{|(1 - \tilde\omega^2) - \tilde\omega\chi(\tilde\omega) + j\tilde\omega\zeta|^2} d\tilde\omega, \tag{32}$$

which depends only on the nondimensional parameters: ζ, $\tilde{\omega}_{\min}$, $\tilde{\omega}_{\max}$, and m_b/M, where only three are independent. In this discussion we consider only ζ, $\tilde{\omega}_{\min}$, $\tilde{\omega}_{\max}$, and the mass ratio m_b/M follows as $m_b/M = (2/\pi)\zeta(\tilde{\omega}_{\max} - \tilde{\omega}_{\min})/(\tilde{\omega}_{\min}\tilde{\omega}_{\max})$. In the next section, we investigate the relevant physical scenarios based on $J_q(\zeta,\tilde{\omega}_{\min},\tilde{\omega}_{\max})$.

A. Infinite bandwidth

For $\tilde{\omega}_{\max} \gg 1$ and $\tilde{\omega}_{\min} \approx 0$, the reactive component disappears, $\chi \approx 0$, leaving a purely resistive bath. For such an extended bath, expression for the energy absorbed by the test particle is then obtained from Eq. (32):

$$J_q = \frac{2}{\pi}\zeta \int_{\tilde{\omega}_{\min}}^{\tilde{\omega}_{\max}} \frac{\tilde{\omega}^2}{(1-\tilde{\omega}^2)^2 + \tilde{\omega}^2\zeta^2} d\tilde{\omega}, \tag{33}$$

Completing the integration using limits $\tilde{\omega}_{\min} = 0$ and $\tilde{\omega}_{\max} = N\Delta\omega/\Omega$ produces

$$\begin{aligned} J_q &= \frac{2}{\pi\gamma}\bigg[\alpha \arctan\left(\frac{\sqrt{2}}{\alpha}\frac{M}{m}\right) + \beta \arctan\left(\frac{\sqrt{2}}{\beta}\frac{M}{m}\right) \\ &\quad - \alpha \arctan\left(\frac{\sqrt{2}}{\alpha}\frac{M}{Nm}\right) - \beta \arctan\left(\frac{\sqrt{2}}{\beta}\frac{M}{Nm}\right)\bigg], \\ \alpha &= \sqrt{1 - 2\left(\frac{\pi}{2\zeta}\right)^2 + \frac{\gamma}{\sqrt{2}}}, \\ \beta &= \sqrt{1 - 2\left(\frac{\pi}{2\zeta}\right)^2 - \frac{\gamma}{\sqrt{2}}}, \\ \frac{\gamma}{\sqrt{2}} &= \sqrt{1 - \left(\frac{\pi}{\zeta}\right)^2}. \end{aligned} \tag{34}$$

The extended bandwidth case implies $\tilde{\omega}_{\max} \gg 1$ (this condition can also be reached with $\Omega \to 0$) approaching infinity, resulting in the expression for the test particle energy:

$$J_q \approx \frac{2}{\pi}\arctan(M/m). \tag{35}$$

The absorbed energy by the test particle shows it to be an asymptotically increasing function of M/m; the heavier the test particle, the more energy it asymptotically absorbs. Moreover, energy absorption shows a saturation as M increases, approaching its maximum value $J_q = 1$ (i.e., $E_q = E_b/N$).

B. Finite bandwidth, $\omega_i \in (\omega_{\min},\omega_{\max})$

For a finite N, the integral has a finite upper bound and a nonvanishing lower bound, producing a reactive term χ in the denominator of the integrand in Eq. (32). Substitution for χ from (31) in Eq. (32) yields

$$\begin{aligned} J_q = 2\frac{\zeta}{\pi}\int_{\tilde{\omega}_{\min}}^{\tilde{\omega}_{\max}} &\bigg|(1-\tilde{\omega}^2) + \frac{\tilde{\omega}\zeta}{\pi}\bigg[\ln\left|\frac{\tilde{\omega}-\tilde{\omega}_{\min}}{\tilde{\omega}-\tilde{\omega}_{\max}}\right| \\ &- \ln\left|\frac{\tilde{\omega}+\tilde{\omega}_{\min}}{\tilde{\omega}+\tilde{\omega}_{\max}}\right|\bigg] + j\zeta\tilde{\omega}\bigg|^{-2}\tilde{\omega}^2 d\tilde{\omega}. \end{aligned} \tag{36}$$

In its present form, integral (36) does not admit closed-form solutions. However, as shown below, useful approximate expressions can help explain the salient features of the energy sharing process between the particle and bath for several cases of interest.

C. Resonant region, $\Omega \in [\omega_{\min},\omega_{\max}]$

For small values of ζ, the integrand in Eq. (36) develops a sharp peak, which permits approximation of the integral as a product of the integrand amplitude at $\omega = \Omega$ and its effective bandwidth, $\pi\zeta/2$:

$$J_q = \left(1 + \frac{1}{\pi^2}\left[\ln\left|\frac{1-\tilde{\omega}_{\min}}{1-\tilde{\omega}_{\max}}\right| - \ln\left|\frac{1+\tilde{\omega}_{\min}}{1+\tilde{\omega}_{\max}}\right|\right]^2\right)^{-1}. \tag{37}$$

The infinite bandwidth case follows as $J_q = 1$ by substituting for $\tilde{\omega}_{\min} \to 0$ and $\tilde{\omega}_{\min} \to \infty$. As an illustration of the effect of a finite bandwidth, consider the test particle with frequency $\Omega = (\omega_{\max} + \omega_{\min})/2$ and half-bandwidth $BW = (\omega_{\max} - \omega_{\min})/2$:

$$J_q = \left(1 + \frac{1}{\pi^2}\left[\ln\left(\frac{2+\tilde{BW}}{2-\tilde{BW}}\right)\right]^2\right)^{-1} \to 0.88 < J_q < 1, \tag{38}$$

revealing a reduction of energy absorbed for larger values of $\tilde{BW} = (\omega_{\max} - \omega_{\min})/2\Omega$. However, the results suggest that for $\zeta \ll 1$, in the resonant region the test particle is in thermal equilibrium with the bath, with some overcooling effect observed when the value of the bandwidth BW becomes comparable with that of Ω.

In the case of large values of ζ,

$$J_q \approx \frac{1}{\pi}\frac{\tilde{\omega}_{\max} - \tilde{\omega}_{\min}}{\zeta} = \frac{2}{\pi^2}\frac{M}{m_b}\frac{(\rho-1)^2}{\rho}, \tag{39}$$

which shows that the retained energy by the test particle, even in the resonant region, is very small due to the mass ratio, $(M/m_b) \ll 1$.

D. Nonresonant region

Integral expression for energy absorbed by the test particle J_q can be approximated by retaining the first-order term in the Taylor expansion of the ln terms in $\chi(\tilde{\omega})$:

$$\chi(\tilde{\omega}) \approx \chi_C + \chi_C'(\tilde{\omega} - \tilde{\omega}_C), \quad \tilde{\omega}_C = \tfrac{1}{2}(\tilde{\omega}_{\min} + \tilde{\omega}_{\max}), \tag{40}$$

where

$$\chi_C = \frac{\zeta}{\pi}\ln\left(\frac{\tilde{\omega}_C + \tilde{\omega}_{\min}}{\tilde{\omega}_C + \tilde{\omega}_{\max}}\right) = \frac{\zeta}{\pi}\phi,$$

$$\chi_C' = \frac{\zeta}{\pi}\frac{2(\tilde{\omega}_{\max} - \tilde{\omega}_{\min})(\tilde{\omega}_C^2 + \tilde{\omega}_{\min}\tilde{\omega}_{\max})}{(\tilde{\omega}_C^2 - \tilde{\omega}_{\min}^2)(\tilde{\omega}_C^2 - \tilde{\omega}_{\max}^2)} = \frac{\zeta}{\pi}\psi$$

and, in terms of ρ,

$$\phi = \ln\left(\frac{3+\rho}{3\rho+1}\right), \quad \psi = \frac{8(\rho-1)(\rho^2+6\rho+1)}{\tilde{\omega}_{\min}(\rho^2+2\rho-3)(-3\rho^2+2\rho+1)},$$

$$\rho = \frac{\tilde{\omega}_{\max}}{\tilde{\omega}_{\min}},$$

which, when substituted in Eq. (32), produce a more tractable form of the integral:

$$J_q = \frac{2\zeta}{\pi}\int_{\tilde{\omega}_{\min}}^{\tilde{\omega}_{\max}} \frac{\tilde{\omega}^2 d\tilde{\omega}}{(1 - A\,\tilde{\omega}^2 - B\,\tilde{\omega})^2 + \zeta^2\tilde{\omega}^2}, \tag{41}$$

with $A = 1 + \zeta\psi/\pi$ and $B = (\zeta/\pi)[\phi - \psi\tilde{\omega}_{\min}(1+\rho)/2]$.

When evaluating the expression J_q for energy absorption by a test particle, say a MEMS or a NEMS resonator immersed in a bath, the nonresonant cases can be described using the electromagnetic wave analogy: radio waves or infrared (IR) when $\Omega \ll \omega_{\min}$ and ultraviolet (UV) when $\Omega \gg \omega_{\max}$, respectively.

1. Radio-waves or IR approximation, $\tilde{\omega}_{\min} \gg 1$

In this case, the test particle frequency falls below the bath frequencies, simplifying the denominator of the integrand in Eq. (32) with $(1-\tilde{\omega}^2) \approx -\tilde{\omega}^2$, which then can be evaluated in closed form using Eq. (41) to yield the nondimensional energy ratio $J_q = E_q/(E_b/N)$:

$$J_q = \frac{2\zeta}{\pi}\int_{\tilde{\omega}_{\min}}^{\tilde{\omega}_{\max}} \frac{\tilde{\omega}^2 d\tilde{\omega}}{(A\,\tilde{\omega}^2 + B\,\tilde{\omega})^2 + \zeta^2\tilde{\omega}^2}$$
$$= \frac{2}{\pi}\frac{1}{A}\left[\arctan\left(\frac{A\tilde{\omega}_{\max}+B}{\zeta}\right) - \arctan\left(\frac{A\tilde{\omega}_{\min}+B}{\zeta}\right)\right]. \tag{42}$$

For $\rho \gg 1$, we can approximate $A \approx 1$, $B \approx \zeta/3\pi$, and $\zeta \approx (\pi\, m_b\, \tilde{\omega}_{\min})/2M$, to obtain

$$J_q \approx \frac{2}{\pi}\left[\arctan\left(\frac{2}{\pi}\frac{M}{m_b}\rho + \frac{1}{3\pi}\right) - \arctan\left(\frac{2}{\pi}\frac{M}{m_b} + \frac{1}{3\pi}\right)\right]. \tag{43}$$

The energy J_q has a simple dependence on the mass ratio m_b/M that displays an increase up to a maximum after which the energy decreases with increasing m_b/M. The value $(m_b/M)_{IR}$ at which J_q attains its maximum value can be found as

$$\left(\frac{m_b}{M}\right)_{IR} \approx \frac{2}{\pi}\sqrt{\frac{\rho}{1+a^2}}, \quad a = \frac{1}{3\pi}. \tag{44}$$

The corresponding peak value of the nondimensional energy takes the form

$$J_q \approx \frac{2}{\pi}\left[\arctan(\sqrt{\rho(1+a^2)} + a) - \arctan\left(\sqrt{\frac{1+a^2}{\rho}} + a\right)\right]. \tag{45}$$

With these approximations, the peak value of J_q shows that when the bandwidth of the bath is such that for $\rho = \tilde{\omega}_{\max}/\tilde{\omega}_{\min} > 50$, then the peak value of the nondimensional energy falls between $0.88 < J_q < 1$. This result suggests that for cases $\rho \gg 1$, the test particle reaches a temperature close to that of the bath and that energy exchange takes place between the particle and the bath even outside the bandwidth due to their coupling that appears in $Z(\omega)$, as long as the mass ratio is near $(m_b/M)_{IR}$. For vastly different mass ratios, the particle overcools and its temperature becomes much lower than that of the bath. We also note that for a given mass ratio and bandwidth, the particle energy is not affected by the exact value of the particle natural frequency Ω within the infrared region.

2. Ultraviolet approximation, $\tilde{\omega}_{\max} \ll 1$

For low frequencies, such that $\tilde{\omega} \ll 1$, the integral for J_q in Eq. (41) can be approximated for the case when $\zeta \ll 1$ as

$$J_q \approx \frac{2\zeta}{3\pi}\left(\tilde{\omega}_{\max}^3 - \tilde{\omega}_{\min}^3\right) = \frac{m_b}{3M}\frac{\rho^3-1}{\rho-1}\rho\tilde{\omega}_{\min}^4 \quad (\zeta \ll 1). \tag{46}$$

For cases where $\zeta \gg 1$, products of small and large quantities appear in the denominator and to approximate the integral (41), we first define orders of magnitude of the terms. For $\tilde{\omega} = O(\epsilon)$ and $\zeta = O(1/\epsilon^p)$ with $p > 0$ and integer, order of the terms in the denominator of (41) are

$$A\tilde{\omega}^2 = O(\epsilon^{2-p}), \quad B\tilde{\omega} = O(\epsilon^{1-p}), \quad \zeta^2\tilde{\omega}^2 = O(\epsilon^{2-2p}).$$

With such orders of magnitudes, we can approximate the denominator as follows:

$$(1 - A\tilde{\omega}^2 - B\tilde{\omega})^2 + \zeta^2\tilde{\omega}^2$$
$$\approx \begin{cases} (1-B\tilde{\omega})^2 + \zeta^2\tilde{\omega}^2 & \text{for } p = 1, \\ \zeta^2\tilde{\omega}^2 & \text{for } p \geqslant 2. \end{cases} \tag{47}$$

Approximation of the integral (41) for $p = 1$ produces

$$J_q \approx \frac{2}{\pi}\frac{1}{(B^2+\zeta^2)^2}\bigg\{B\,\ln(B^2\tilde{\omega}^2 - 2B\tilde{\omega} + \tilde{\omega}^2\zeta^2 + 1)$$
$$+ \tilde{\omega}(B^2+\zeta^2) + (B^2/\zeta - \zeta)$$
$$\times \arctan\left(\frac{B^2\tilde{\omega} - B + \tilde{\omega}\zeta^2}{\zeta}\right)\bigg\}\bigg|_{\tilde{\omega}_{\min}}^{\tilde{\omega}_{\max}} \quad (\zeta \gg 1). \tag{48}$$

Since for values $\rho \gg 1$, $B \approx \zeta/3\pi = a\zeta$:

$$J_q \approx \frac{2}{\pi}\frac{1}{\zeta^4(a^2+1)^2}\{-a\zeta\,\ln[\zeta^2\tilde{\omega}^2(a^2+1) + 2a\zeta\tilde{\omega} + 1]$$
$$+ \tilde{\omega}\zeta^2(a^2+1) + \zeta(a^2-1)$$
$$\times \arctan[\zeta\tilde{\omega}(a^2+1) + a]\}\big|_{\tilde{\omega}_{\min}}^{\tilde{\omega}_{\max}} \quad (\zeta \gg 1), \tag{49}$$

which is a monotonically decreasing function of ζ ($\propto m_b/M$).

A simpler expression can be obtained for only the $p \geqslant 2$ case for which

$$J_q \approx \frac{2}{\pi}\frac{\tilde{\omega}_{\max} - \tilde{\omega}_{\min}}{\zeta}$$
$$= \left(\frac{2}{\pi}\right)^2\frac{M}{m_b}\left(1 - \frac{1}{\rho}\right)(\rho - 1) \quad (\zeta \gg 1). \tag{50}$$

Equations (46), (49), and (50) show opposing trends in terms of ζ for values $\zeta \ll 1$ and $\zeta \gg 1$, respectively. These trends suggest that a maximum for J_q should exist during the transition from $\zeta \ll 1$ to $\zeta \gg 1$, which can be roughly determined for $p \geqslant 2$ as the intersection of the equations (46) and (50):

$$\left(\frac{m_b}{M}\right)_{UV} = \frac{\sqrt{3}}{\tilde{\omega}_{\min}^2}\frac{2}{\pi}\sqrt{\frac{(\rho-1)^3}{\rho^2(\rho^3-1)}}, \tag{51}$$

which again demonstrates energy exchange outside the bandwidth for specific values of the mass ratio.

TABLE I. Summary of normalized energy absorption J_q scenarios between a test particle and a bath that has Caldeira-Leggett frequency distribution (with the approximations specified in Sec. IV). $\rho = \tilde{\omega}_{\max}/\tilde{\omega}_{\min};\ a = 1/3\pi$.

Energy scenarios	$\zeta \ll 1$	$\zeta \gg 1$	Critical mass ratio
RW/IR $1 \ll \tilde{\omega}_{\min} < \tilde{\omega}_{\max}$	$J_q \approx \frac{2}{\pi}\arctan\left(\frac{2}{\pi}\frac{M}{m_b}\rho + a\right) - \frac{2}{\pi}\arctan\left(\frac{2}{\pi}\frac{M}{m_b} + a\right)$		$\left(\frac{m_b}{M}\right)_{IR} \approx \frac{2}{\pi}\sqrt{\frac{\rho}{1+a^2}}$
Resonant $\tilde{\omega}_{\min} < 1 < \tilde{\omega}_{\max}$	$J_q \approx \left(1 + \frac{1}{\pi^2}\left[\ln\left\lvert\frac{(1-\tilde{\omega}_{\min})(1+\tilde{\omega}_{\max})}{(1-\tilde{\omega}_{\max})(1+\tilde{\omega}_{\min})}\right\rvert\right]^2\right)^{-1}$	$J_q \approx \frac{2}{\pi^2}\frac{M}{m_b}\frac{(\rho-1)^2}{\rho}$	None
UV $\tilde{\omega}_{\min} < \tilde{\omega}_{\max} \ll 1$	$J_q \approx \tilde{\omega}_{\min}^4 \frac{\rho}{3}\frac{m_b}{M}\frac{(\rho^3-1)}{(\rho-1)}$	$J_q \approx \left(\frac{2}{\pi}\right)^2 \frac{M}{m_b}\frac{(\rho-1)^2}{\rho}$	$\left(\frac{m_b}{M}\right)_{UV} \approx \frac{2\sqrt{3}}{\pi\tilde{\omega}_{\min}^2}\sqrt{\frac{(\rho-1)^3}{\rho^2(\rho^3-1)}}$

A summary of scenarios for energy exchange between the test particle and bath are given in Table I.

V. ENERGY DISTRIBUTION IN THE BATH

The zeroth-order equation (9) describing a bath particle response yields a nonzero solution even in the long-time range when $q_0(t)$ vanishes asymptotically, and x_{i0} can be expressed as

$$x_{i0}(t) = \omega_i \sin\omega_i t \int_0^t q_0(\tau)\cos\omega_i\tau\, d\tau - \omega_i\cos\omega_i t\int_0^t q_0(\tau) \times \sin\omega_i\tau\, d\tau + x_i(0)\cos\omega_i t + \frac{\dot{x}_i(0)}{\omega_i}\sin\omega_i t, \quad (52)$$

where $x_i(0)$ and $\dot{x}_i(0)$ denote the initial conditions that represent the fluctuations in the bath. For long times, where q_0 vanishes, the upper limit t of the integrals can be replaced by infinity, leading to real, $Q_0^R(\omega_i)$, and imaginary, $Q_0^I(\omega_i)$, parts of the Fourier transform of q_0 evaluated at $\omega = \omega_i$; $Q_0(\omega_i) = Q_0^R(\omega_i) + jQ_0^I(\omega_i)$. The bath particle displacement (52) becomes

$$\lim_{t\to\infty} x_{i0}(t) = \sqrt{\left[\frac{\dot{x}_i(0)}{\omega_i} + \omega_i Q_0^R(\omega_i)\right]^2 + \left[x_i(0) + \omega_i Q_0^I(\omega_i)\right]^2} \times \sin(\omega_i t + \phi_i). \quad (53)$$

The corresponding energy within the bath can also be expressed for long times as

$$\lim_{t\to\infty} e_i(t) = \frac{1}{2}\frac{dm}{d\omega}\bigg|_{\omega=\omega_i} \dot{x}_{i0}^2(t) + \frac{1}{2}\frac{dm}{d\omega}\bigg|_{\omega=\omega_i} \omega_i^2[x_{i0}(t) - q_0(t)]^2 = \frac{1}{2}\frac{dm}{d\omega}\bigg|_{\omega=\omega_i}\left[\dot{x}_{i0}^2(t) + \omega_i^2 x_{i0}^2(t)\right]. \quad (54)$$

Substituting for $x_{i0}(t)$ from above gives the energy spectrum in the bath as

$$\lim_{t\to\infty} e_i(t) = \frac{1}{2}\frac{dm}{d\omega}\bigg|_{\omega=\omega_i}\omega_i^2\left[\left(\frac{\dot{x}_i(0)}{\omega_i}\right)^2 + x_i^2(0) + \omega_i^2|Q_0(\omega_i)|^2 + 2\dot{x}_i(0)Q_0^R(\omega_i) + 2\omega_i x_i(0)Q_0^I(\omega_i)\right].$$

The first two terms in the bracket represent the initial energy density $e_i(0)$

$$\lim_{t\to\infty} e_i(t) = e_i(0) + \frac{2}{\pi}D(\omega_i)\left[\frac{1}{2}\omega_i^2|Q_0(\omega_i)|^2 + Q_0^R(\omega_i)\dot{x}_i(0) + \omega_i Q_0^I(\omega_i)x_i(0)\right]. \quad (55)$$

The term $Q_0(\omega_i)$ can be calculated from Fourier transform $\mathcal{F}$ of Eq. (8)

$$Q_0(\omega_i) = \frac{\mathcal{F}\{\Pi_0\}_{\omega=\omega_i}}{M\left(\Omega^2 - \omega_i^2\right) + j\omega_i D(\omega_i) - \omega_i R(\omega_i)}. \quad (56)$$

Substitution of Eq. (56) into (55) yields the bath energy for long times:

$$\lim_{t\to\infty} e_i = e_i(0) + \frac{2}{\pi}\frac{D(\omega_i)}{|Z(\omega_i)|^2}\bigg\{\frac{1}{2}\left(\left[\mathbf{Re}\mathcal{F}\{\Pi_0\}_{\omega=\omega_i}\right]^2 + \mathbf{Im}\left[\mathcal{F}\{\Pi_0\}_{\omega=\omega_i}\right]^2\right) - x_i(0)\left(\left[\mathbf{Re}\mathcal{F}\{\Pi_0\}_{\omega=\omega_i}\right]\mathbf{Re}Z^*(\omega_i) + \mathbf{Im}\mathcal{F}\{\Pi_0\}_{\omega=\omega_i}\mathbf{Im}Z^*(\omega_i)\right) + \frac{\dot{x}_i(0)}{\omega_i}\left(\mathbf{Re}Z^*(\omega_i)\mathbf{Im}\left[\mathcal{F}\{\Pi_0\}_{\omega=\omega_i}\right] + \left[\mathbf{Re}\mathcal{F}\{\Pi_0\}_{\omega=\omega_i}\right]\mathbf{Im}Z^*(\omega_i)\right)\bigg\}, \quad (57)$$

where $Z(\omega)$ is the coupled particle impedance, given by Eq. (29).

A simplified form of Eq. (57) can be obtained with certain assumptions. Since virial theorem suggests that the average values of kinetic and potential energies are equal, we assume the initial energy to be entirely potential and thus set $\dot{x}_i(0) = 0$; Eq. (57) simplifies to

$$\lim_{t\to\infty} e_i = e_i(0) + \frac{2}{\pi}\frac{D(\omega_i)}{|Z(\omega_i)|^2}\bigg\{\frac{1}{2}\left(\left[\mathbf{Re}\mathcal{F}\{\Pi_0\}_{\omega=\omega_i}\right]^2 + \left[\mathbf{Im}\mathcal{F}\{\Pi_0\}_{\omega=\omega_i}\right]^2\right) - x_i(0)\left(\left[\mathbf{Re}\mathcal{F}\{\Pi_0\}_{\omega=\omega_i}\right]\mathbf{Re}Z^*(\omega_i) + \left[\mathbf{Im}\mathcal{F}\{\Pi_0\}_{\omega=\omega_i}\right]\mathbf{Im}Z^*(\omega_i)\right)\bigg\}, \quad (58)$$

where $\mathcal{F}\{\Pi_0\}$ follows from Eqs. (5) analogous to Eqs. (16) and (17):

$$\mathbf{Re}[\mathcal{F}\{\Pi_0\}] = D(\omega)x_\omega(0),$$
$$\mathbf{Im}[\mathcal{F}\{\Pi_0\}] = -\mathcal{H}\{D(\omega)x_\omega(0)\}.$$

Equation (58) is further simplified by invoking the relationship $\mathcal{H}\{D(\omega)x_\omega(0)\} = x_\omega(0)\mathcal{H}\{D(\omega)\} = -x_\omega(0)R(\omega)$ if the conditions for the use of the modulation theorem hold [27]. Under this condition, using $x_\omega(0) = \sqrt{\pi\, e_\omega(0)/D(\omega)}$ from Eq. (54) with $\dot{x} = 0$, we can express

$$\mathbf{Im}[\mathcal{F}\{\Pi_0\}] = \sqrt{\frac{\pi\, e_\omega(0)}{D(\omega)}}R(\omega), \quad \mathbf{Re}[\mathcal{F}\{\Pi_0\}] = \sqrt{\pi\, e_\omega(0)D(\omega)}$$

and rewrite Eq. (58) as

$$\lim_{t\to\infty} J_i = 1 - \frac{\tilde{\omega}_i^2[\zeta^2(\omega_i)+\chi^2(\omega_i)] - 2\left(1-\tilde{\omega}_i^2\right)\tilde{\omega}_i\chi(\tilde{\omega}_i)}{\tilde{\omega}_i^2[\zeta^2(\omega_i)+\chi^2(\omega_i)] - 2\left(1-\tilde{\omega}_i^2\right)\tilde{\omega}_i\chi(\tilde{\omega}_i) + \left(1-\tilde{\omega}_i^2\right)^2}, \tag{59}$$

where $J_i = e_i\Delta\omega/(E_B/N)$. The above equation can be simplified to isolate the terms that represent deviation from equipartitioning:

$$\lim_{t\to\infty} J_i = \left[\frac{1}{1+\Lambda_i}\right], \tag{60}$$

where

$$\Lambda_i = \frac{\tilde{\omega}_i^2\zeta^2(\tilde{\omega})[1+\chi^2(\tilde{\omega}_i)/\zeta^2(\tilde{\omega}_i)] - 2\left(1-\tilde{\omega}_i^2\right)\tilde{\omega}_i\chi(\tilde{\omega}_i)}{\left(1-\tilde{\omega}_i^2\right)^2} \tag{61}$$

represents the deviation.

A. Infinite bandwidth

Expressing Eq. (61) for infinite bandwidth with a Caldeira-Leggett distribution results in

$$\Lambda_i = \frac{\tilde{\omega}_i^2\zeta^2}{\left(1-\tilde{\omega}_i^2\right)^2}.$$

Inspection of Λ_i and (60) shows equipartitioning at all frequencies except in the neighborhood of $\tilde{\omega}_i = 1$, where the bath has zero energy since all the energy at that frequency is absorbed by the test particle. These results show that, in an infinite bath, equipartitioning is reached for small values of ζ, except near the resonance where there is localized energy absorption by the test particle from the bath.

B. Finite bandwidth

To investigate how a finite-bandwidth influences the equipartitioning process in a bath, we continue with the Caldeira-Leggett distribution within the range of bath frequencies $\omega_i \in (\omega_{\min},\omega_{\max})$ that yields a constant dissipation $D(\omega) = C_D$.

In this case, the reactive part given in Eq. (31) is nonzero ($\chi \neq 0$) and is retained in Eq. (61):

$$\Lambda_i = \frac{\tilde{\omega}_i^2\zeta^2}{\left(1-\tilde{\omega}_i^2\right)^2}\left[1+\frac{1}{\pi^2}\left(\ln\left|\frac{\tilde{\omega}_i-\tilde{\omega}_{\min}}{\tilde{\omega}_i-\tilde{\omega}_{\max}}\right| - \ln\left|\frac{\tilde{\omega}_i+\tilde{\omega}_{\min}}{\tilde{\omega}_i+\tilde{\omega}_{\max}}\right|\right)^2 + \frac{2\left(1-\tilde{\omega}_i^2\right)}{\pi\,\tilde{\omega}_i\zeta}\left(\ln\left|\frac{\tilde{\omega}_i-\tilde{\omega}_{\min}}{\tilde{\omega}_i-\tilde{\omega}_{\max}}\right| - \ln\left|\frac{\tilde{\omega}_i+\tilde{\omega}_{\min}}{\tilde{\omega}_i+\tilde{\omega}_{\max}}\right|\right)\right]. \tag{62}$$

For a finite-bandwidth bath, the location of the resonator frequency matters. For instance, in the infrared regime, where $1 \ll \omega_{\min} < \omega_i < \omega_{\max}$, the denominator is very large, making Λ_i vanishingly small:

$$\Lambda_i = -2\frac{\zeta}{\pi\,\tilde{\omega}_i}\left(\ln\left|\frac{\tilde{\omega}_i-\tilde{\omega}_{\min}}{\tilde{\omega}_i-\tilde{\omega}_{\max}}\right| - \ln\left|\frac{\tilde{\omega}_i+\tilde{\omega}_{\min}}{\tilde{\omega}_i+\tilde{\omega}_{\max}}\right|\right) \approx 0. \tag{63}$$

Such small values of Λ_i suggest the presence of a small distortion to equipartition, implying that a test particle in the IR region does not otherwise alter equipartition significantly.

A further effect of having a finite bandwidth is the development of sharp peaks at the bandwidth boundaries. These peaks stem from the singularities in the arguments of the ln expressions in Λ_i and are expected to distort the equipartitioned energy distribution at the bandwidth boundaries. We refer to these singularities as "edge effects" that occur in conjunction with the reactive component of the impedance and can supply energy to the test particle even when outside of the frequency band.

When the resonator frequencies are in the ultraviolet region such that $\omega_{\min} < \omega_i < \tilde{\omega}_{\max} \ll 1$, Λ_i again nearly vanishes, except at the "edges", as in the IR case:

$$\Lambda_i = -2\frac{\zeta\,\tilde{\omega}_i}{\pi}\left(\ln\left|\frac{\tilde{\omega}_i-\tilde{\omega}_{\min}}{\tilde{\omega}_i-\tilde{\omega}_{\max}}\right| - \ln\left|\frac{\tilde{\omega}_i+\tilde{\omega}_{\min}}{\tilde{\omega}_i+\tilde{\omega}_{\max}}\right|\right) \approx 0. \tag{64}$$

Finally, in the case of the so-called resonant region, $\tilde{\omega}_{\min} < 1 < \tilde{\omega}_{\max}$, energy distribution in the bath is no longer expected to be equipartitioned, because the test particle now interacts locally with the resonating oscillators in the bath, altering their energy distribution. These results suggest existence of an energy absorption path for the test particle outside the resonant bandwidth that depends on certain mass ratios. Considering Eq. (61), in the IR range, while small, Λ_i, decreases with frequency, see Eq. (63), and thus the bath energy increases. Conversely, in the UV regime, Λ_i, shown in Eq. (64), increases with frequency and the bath energy decreases.

VI. DISCUSSION

We developed a cascade of Langevin-like equations with a perturbation approach. The resulting equations allow separation of the mix of scales in the classical Langevin equation where the dissipation and forcing terms represent macroscopic and microscopic quantities, respectively.

The equations derived here are also used to examine the transient and asymptotic, or steady-state, conditions. Two pairs of equations are obtained by retaining the zeroth-order and first-order terms in the equations of motion for a test particle and for the finite heat bath it is immersed in. The zeroth-order term for the particle response vanishes in long times but it is the source of excitation of the first- and higher-order terms in the bath and thus is still necessary to investigate the transient cases.

The present formulation is also amenable to closed-form solutions in certain cases to show how energy is distributed among the degrees of freedom of the bath and the energy sharing between the test particle and the bath.

Energy sharing between the test particle and the bath are examined according to the relative value of the resonant frequency of the test particle with respect to the bandwidth of the bath: (i) the resonant behavior describes the cases when the particle frequency falls within bath bandwidth, (ii) the radio-waves or infrared response refers to test particle frequency much lower than the frequencies in the bath, and (iii) the ultraviolet case when the test particle frequency is much higher than the bath frequencies.

As discussed above, all three cases of energy sharing processes can be completely described by three nondimensional parameters $\tilde{\omega}_{\max} = \omega_{\max}/\Omega$, $\tilde{\omega}_{\min} = \omega_{\min}/\Omega$, $\zeta = (\pi\tilde{\omega}_{\max})(m_b/M)$, which represent the test particle natural frequency, bandwidth limits, and the mass ratio, respectively. For each scenario we provide simple closed-form expressions that describe the energy retained by the test particle in terms of these parameters.

The resonant regime displays two different behaviors depending on whether ζ, indicating the mass ratio, is very small or very large. For small values of ζ, energy absorbed by the test particle is close to that expected at equilibrium under equipartition conditions. Also, as long as $\zeta \ll 1$, the energy absorption does not depend on the specific value of the ratio m_b/M.

For $\zeta \gg 1$ or large m_b/M, the energy retained asymptotically by the bath is small and much lower than that predicted by equipartition, and inversely depends on the ratio m_b/M. Equilibrium is not reached in this case.

The out-of-bandwidth cases are characterized by a nonresonant response of the test particle, which interacts with the bath by a different mechanism. A test particle with frequency Ω, outside of the bandwidth, still extracts energy from the bath, primarily at the edge frequencies, as a result of the coupling of bath and particle at certain m_b/M values, while a test particle with a frequency that falls within the bandwidth has both resonance effects as well as edge effects.

APPENDIX

In order to estimate the magnitude of the additional fluctuation represented by the last term in Eq. (13), we rewrite it explicitly:

$$\begin{aligned}\int_0^t &\Gamma(t-\tau)\dot{q}_0(\tau)\,d\tau\\ &\approx \sum_i^N m_i\omega_i^2 \int_{-\infty}^{+\infty} \cos\omega_i(t-\tau)\dot{q}_0(\tau)d\tau\\ &= -\sum_i^N m_i\omega_i^2 \mathbf{Re}[j\omega_i Q_0^*(\omega_i)e^{j\omega_i t}]\\ &= \sum_i^N m_i\omega_i^3|Q_0(\omega_i)|\cos(\omega_i t + \phi_{0_i}),\end{aligned} \tag{A1}$$

and the power spectral density of the convolution

$$S_{\Gamma*\dot{q}_0} = \left(m_i\omega_i^3\right)^2 \frac{\overline{|Q_0(\omega_i)|^2}}{\Delta\omega}$$

where the upper integral limit represents the long-time approximation since $\dot{q}_0(t)$ vanishes as $t\to\infty$ and the lower limit is allowed since $\dot{q}_0(t) = 0$ for all values of $t < 0$. As a result, $S_{Q_0}(\omega_i) = \overline{|Q_0(\omega_i)|^2}\Delta\omega$, and using Eq. (56) $S_{Q_0}(\omega_i) = S_{\Pi_0}(\omega_i)/\omega_i^2|Z(\omega_i)|^2$, we have

$$S_{\Gamma*\dot{q}_0} = \left[\frac{2}{\pi}\frac{D(\omega_i)}{|Z(\omega_i)|}\right]^2 S_{\Pi_0}(\omega_i).$$

Since from Eq. (23), we have $S_{\Pi_0}(\omega_i) = \lim_{\Delta\omega\to 0} S_{\Pi}(\omega_i) = 0$, $S_{\Gamma*q_0}$ is small compared to S_Π and, therefore, the last term in Eq. (13) can be neglected.

[1] M. Campisi, P. Talkner, and P. Hänggi, Phys. Rev. E **80**, 031145 (2009).
[2] A. Rohrbach, C. Tischer, D. Neumayer, E.-L. Florin, and E. H. Stelzer, Rev. Sci. Instrum. **75**, 2197 (2004).
[3] M. R. Vanner, Phys. Rev. X **1**, 021011 (2011).
[4] J. M. Dobrindt and T. J. Kippenberg, Phys. Rev. Lett. **104**, 033901 (2010).
[5] I. Favero and K. Karrai, Nat. Photon. **3**, 201 (2009).
[6] P. DelHaye, A. Schliesser, O. Arcizet, T. Wilken, R. Holzwarth, and T. Kippenberg, Nature (London) **450**, 1214 (2007).
[7] S. Rips, I. Wilson-Rae, and M. J. Hartmann, Phys. Rev. A **89**, 013854 (2014).
[8] X. Chew, G. Zhou, F. S. Chau, and J. Deng, J. Nanophoton. **5**, 059503 (2011).
[9] Z. Lindenfeld and R. Lifshitz, Phys. Rev. B **87**, 085448 (2013).
[10] H. Hamann and H. Wörn, Swarm Intelligence **2**, 209 (2008).
[11] J. F. Rhoads, S. W. Shaw, and K. L. Turner, J. Dyn. Syst., Meas., Control **132**, 034001 (2010).
[12] A. Cleland and M. Roukes, Sens. Actuators, A **72**, 256 (1999).
[13] K. Ekinci and M. Roukes, Rev. Sci. Instrum. **76**, 061101 (2005).
[14] R. Kubo, M. Toda, and N. Hatsune, *Statistical Physics II: Nonequilibrium Statistical Mechanics* (Springer-Verlag, Berlin, 1985).
[15] R. P. Feynman and F. Vernon, Jr., Ann. Phys. (NY) **24**, 118 (1963).
[16] A. O. Caldeira and A. J. Leggett, Physica A **121**, 587 (1983).
[17] A. Caldeira and A. Leggett, Ann. Phys. (NY) **149**, 374 (1983).
[18] G. Ford, M. Kac, and P. Mazur, J. Math. Phys. **6**, 504 (1965).
[19] G. Ford and M. Kac, J. Stat. Phys. **46**, 803 (1987).

[20] C. Presilla, R. Onofrio, and M. Patriarca, J. Phys. A: Math. Gen. **30**, 7385 (1997).

[21] Q. Wei, S. T. Smith, and R. Onofrio, Phys. Rev. E **79**, 031128 (2009).

[22] M. Patriarca, Phys. E (Amsterdam, Neth.) **29**, 243 (2005).

[23] A. Carcaterra and A. Akay, J. Acoust. Soc. Am. **121**, 1971 (2007).

[24] A. Carcaterra and A. Akay, Phys. Rev. E **84**, 011121 (2011).

[25] A. Carcaterra and A. Akay, J. Acoust. Soc. Am. **115**, 683 (2004).

[26] A. Carcaterra, A. Akay, and C. Bernardini, Mech. Syst. Signal Process. **26**, 1 (2012).

[27] R. N. Bracewell and R. Bracewell, *The Fourier Transform and its Applications* (McGraw-Hill, New York, 1986), Vol. 31999.