PROCEEDINGS A

rspa.royalsocietypublishing.org

Research

Cite this article: Temizer İ, Stupkiewicz S. 2016 Formulation of the Reynolds equation on a time-dependent lubrication surface. *Proc. R. Soc. A* **472**: 20160032.
http://dx.doi.org/10.1098/rspa.2016.0032

Received: 13 January 2016
Accepted: 25 February 2016

Subject Areas:
mechanical engineering

Keywords:
Reynolds equation, elastohydrodynamic lubrication, soft interfaces, time-dependent curvilinear coordinates, finite deformations, objectivity

Author for correspondence:
İ. Temizer
e-mail: temizer@bilkent.edu.tr

Formulation of the Reynolds equation on a time-dependent lubrication surface

İ. Temizer[1] and S. Stupkiewicz[2]

[1]Department of Mechanical Engineering, Bilkent University, 06800 Ankara, Turkey
[2]Institute of Fundamental Technological Research (IPPT), Polish Academy of Sciences, Pawińskiego 5B, 02-106 Warsaw, Poland

İT, 0000-0003-3043-7521

The Reynolds equation, which describes the lubrication effect arising through the interaction of two physical surfaces that are separated by a thin fluid film, is formulated with respect to a continuously evolving third surface that is described by a time-dependent curvilinear coordinate system. The proposed formulation essentially addresses lubrication mechanics at interfaces undergoing large deformations and *a priori* satisfies all objectivity requirements, neither of which are features of the classical Reynolds equation. As such, this formulation may be particularly suitable for non-stationary elastohydrodynamic lubrication problems associated with soft interfaces. The ability of the formulation to capture finite-deformation effects and the influence of the choice of the third surface are illustrated through analytical examples.

1. Introduction

Since the publication of Reynolds' seminal work [1], the Reynolds equation has been the central, driving formulation behind theoretical and computational lubrication analysis. The mechanics of a vanishingly thin fluid film at the interface between two moving surfaces has been described by the Reynolds equation in a broad range of lubrication problems with indisputable success [2,3]. Owing to the underlying assumption of an infinitesimal film thickness in the theoretical basis for its derivation from the Navier–Stokes equations, the Reynolds equation is a surface formulation. In practice,

THE ROYAL SOCIETY PUBLISHING

the interacting *physical surfaces* are always separated by a finite film thickness and, therefore, it is not always immediately clear which of these surfaces should be chosen in order to formulate the Reynolds equation, whether this choice has any impact on the results or whether choosing one of the physical surfaces is necessary at all. Indeed, to maintain generality, it is of interest to be able to carry out this formulation on an independent third surface, to be referred to as the *lubrication surface*, that does not necessarily coincide with either of the physical surfaces—see [4] for an early example. Moreover, the physical surfaces can significantly deform in soft interfaces [5–9] and also dynamically evolve in non-stationary problems of impact and sliding [10], which also necessitates the lubrication surface to evolve. The goal of this work is to formulate the Reynolds equation with respect to such a continuously evolving surface. Overall, the proposed formulation essentially addresses lubrication interfaces undergoing large deformations and *a priori* satisfies all objectivity requirements, neither of which are features of the classical Reynolds equation.

When the problem is stationary and the physical surfaces are curved, it is convenient to choose the lubrication surface coincident with the geometry of one of the physical surfaces. An example of this setting is the journal bearing. Here, either the non-moving (bearing) surface geometry may be chosen or the geometry of the moving (shaft) surface may be employed. In either case, the lubrication formulation is Eulerian with respect to the fluid flow at the interface. Recently, it has been demonstrated [9] that there is a small influence of the choice of the third surface for this class of problems in the context of soft elastohydrodynamic lubrication owing to the finiteness of the film thickness. When the physical surfaces have a complex topography, for instance in the case of nominally flat rough surfaces [11], it may be possible to employ an intermediate stationary flat plane, which again leads to an Eulerian-type formulation. In this context, an independently moving lubrication surface would lead to an arbitrary Lagrangian–Eulerian formulation [12] with respect to an underlying *time-dependent* curvilinear coordinate system. When the problem is non-stationary and involves significant changes of the interface geometry, finite-deformation effects must additionally be taken into account, irrespective of the impact of the particular definition of the lubrication surface. These effects are naturally addressed once the lubrication surface is assigned independent kinematics.

Although fluid dynamics problems are routinely formulated with respect to a curvilinear coordinate system, this system is in most cases admitted to be *time independent* [13]. Formulation with respect to a time-dependent curvilinear coordinate system has been a source of discussion until recently [14]. A similar situation exists for the theory of lubrication, except that an explicit discussion regarding a time-dependent system does not appear to exist, despite extensive early [15] and recent [16] relevant work on surface flows. One approach to accomplishing the stated goal of this work is to start from the formulation of the Navier–Stokes equations with respect to a time-dependent curvilinear coordinate system and subsequently invoke the thin-film assumption to arrive at the desired lubrication formulation. However, this is more easily achieved by starting from the formulation of the Reynolds problem with respect to a stationary but generally curved lubrication surface and subsequently state the generalization to the time-dependent case, which is the approach that will be pursued presently.

2. Interface geometry

The interface geometry is described by the two physical surfaces $\mathcal{P}^{(\mathrm{I})}$, $\mathrm{I}=1$ or 2, which are associated with the interacting solids, and the lubrication surface $\mathcal{L}$ on which the mechanics of the thin fluid film will be described. This set-up is summarized in figure 1. It is important to note that $\mathcal{L}$ is depicted intermediate to $\mathcal{P}^{(\mathrm{I})}$ and will be enforced to satisfy this constraint, which is to be discussed in §9. The physical surfaces are assigned position vectors $\boldsymbol{x}^{(\mathrm{I})}$ and convected curvilinear coordinates $\xi^{(\mathrm{I}),\alpha}$, where $\alpha=1$ or 2. The position vector and the convected curvilinear coordinates for the lubrication surface will be denoted by $\boldsymbol{y}$ and η^{α}, respectively. The curvilinear coordinate systems are depicted globally although they may also be locally constructed, for instance within

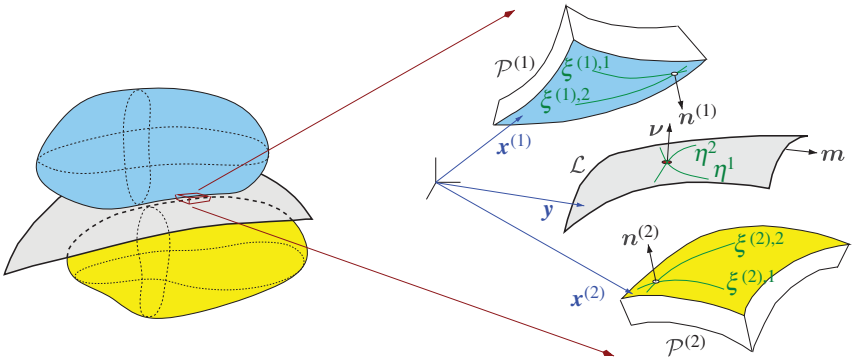

Figure 1. The interface geometry for the three-surface set-up. (Online version in colour.)

individual finite elements. Overall, the following representations hold:

$$\boldsymbol{x}^{(\mathrm{I})} = \boldsymbol{x}^{(\mathrm{I})}(\xi^{(\mathrm{I}),1}, \xi^{(\mathrm{I}),2}, t) \quad \text{and} \quad \boldsymbol{y} = \boldsymbol{y}(\eta^1, \eta^2, t). \tag{2.1}$$

The outward unit normals to $\mathcal{P}^{(\mathrm{I})}$ are denoted by $\boldsymbol{n}^{(\mathrm{I})}$. The lubrication surface $\mathcal{L}$ is assigned a unit normal $\boldsymbol{\nu}$, pointing in the direction from $\mathcal{P}^{(2)}$ to $\mathcal{P}^{(1)}$, as well as a tangential vector $\boldsymbol{m}$ that is simultaneously an outward unit normal to $\partial\mathcal{L}$ which indicates the boundary of the domain within which the Reynolds equation is solved.

Following standard differential geometry [13,17], the curvilinear coordinate systems are employed to construct covariant basis vectors $\boldsymbol{a}_\alpha^{(\mathrm{I})} = \partial \boldsymbol{x}^{(\mathrm{I})}/\partial \xi^{(\mathrm{I}),\alpha}$ on $\mathcal{P}^{(\mathrm{I})}$ and $\boldsymbol{g}_\alpha = \partial \boldsymbol{y}/\partial \eta^\alpha$ on $\mathcal{L}$. The metric components on $\mathcal{L}$ will be denoted by $g_{\alpha\beta} = \boldsymbol{g}_\alpha \cdot \boldsymbol{g}_\beta$, the inverse of which has components $g^{\alpha\beta}$ that induce the contravariant basis vectors $\boldsymbol{g}^\alpha = g^{\alpha\beta}\boldsymbol{g}_\beta = \partial \eta^\alpha/\partial \boldsymbol{y}$. Note that $\boldsymbol{g}_\alpha \cdot \boldsymbol{\nu} = 0$ and $\boldsymbol{a}_\alpha^{(\mathrm{I})} \cdot \boldsymbol{n}^{(\mathrm{I})} = 0$ by construction but, for instance, $\boldsymbol{a}_\alpha^{(\mathrm{I})} \cdot \boldsymbol{\nu} \neq 0$ in general. With respect to $\mathcal{L}$, any vector $\boldsymbol{z}$ may be decomposed into its normal

$$z_{\mathrm{N}} = \boldsymbol{z} \cdot \boldsymbol{\nu} \quad \longrightarrow \quad \boldsymbol{z}_{\mathrm{N}} = z_{\mathrm{N}}\boldsymbol{\nu} \tag{2.2}$$

and tangential parts

$$\boldsymbol{z}_{\mathrm{T}} = \boldsymbol{z} - \boldsymbol{z}_{\mathrm{N}} = z_{\mathrm{T}}^\alpha \boldsymbol{g}_\alpha. \tag{2.3}$$

The *surface-gradient* operator will be defined with respect to the geometry of $\mathcal{L}$

$$\nabla = \boldsymbol{g}^\alpha \frac{\partial}{\partial \eta^\alpha} \tag{2.4}$$

so that the *surface divergence* of a vector may be expressed as

$$\nabla \cdot \boldsymbol{z} = \frac{\partial \boldsymbol{z}}{\partial \eta^\alpha} \cdot \boldsymbol{g}^\alpha. \tag{2.5}$$

For a tangential vector $\boldsymbol{z}_{\mathrm{T}}$, it takes the specific form

$$\nabla \cdot \boldsymbol{z}_{\mathrm{T}} = \frac{\partial \boldsymbol{z}_{\mathrm{T}}}{\partial \eta^\alpha} \cdot \boldsymbol{g}^\alpha = \frac{\partial z_{\mathrm{T}}^\alpha}{\partial \eta^\alpha} + z_{\mathrm{T}}^\beta \Gamma_{\beta\alpha}^\alpha, \tag{2.6}$$

where $\Gamma_{\beta\gamma}^\alpha = \partial \boldsymbol{g}_\beta/\partial \eta^\gamma \cdot \boldsymbol{g}^\alpha$ are the Christoffel symbols of the second kind. For a non-tangential vector field, (2.5) may now be evaluated as

$$\nabla \cdot \boldsymbol{z} = \nabla \cdot \boldsymbol{z}_{\mathrm{T}} + \nabla \cdot \boldsymbol{z}_{\mathrm{N}}, \tag{2.7}$$

where the normal contribution is

$$\nabla \cdot \boldsymbol{z}_{\mathrm{N}} = z_{\mathrm{N}} \frac{\partial \boldsymbol{\nu}}{\partial \eta^\alpha} \cdot \boldsymbol{g}^\alpha = \kappa z_{\mathrm{N}}. \tag{2.8}$$

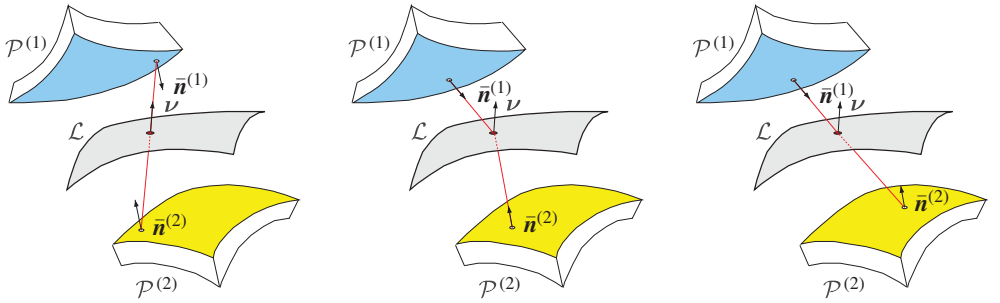

Figure 2. Inverse–orthogonal, orthogonal (closest point) and slave/master projection approaches for the definition of the gap between the surfaces. (Online version in colour.)

Here, $\kappa = -2H$ with $H = \frac{1}{2}g^{\alpha\beta}b_{\alpha\beta}$ as the mean curvature and $b_{\alpha\beta} = \partial \boldsymbol{g}_\alpha / \partial \eta^\beta \cdot \boldsymbol{\nu}$. It is also useful to recall the *surface-divergence theorem* for a continuously differentiable tangential vector field,

$$\int_{\mathcal{L}} \nabla \cdot \boldsymbol{z}_{\mathrm{T}} \mathrm{d}a = \int_{\partial \mathcal{L}} \boldsymbol{z}_{\mathrm{T}} \cdot \boldsymbol{m} \, \mathrm{d}\ell. \tag{2.9}$$

3. Definition of the film thickness

An incompressible fluid is assumed in this work such that the density can be assigned a unit value without loss of generality. Hence, no distinction will be made between mass and volume. The fluid film thickness h is a measure of the gap between the surfaces and delivers the volume $\int_{\mathcal{L}} h \, \mathrm{d}a$ of the fluid at the interface. In a relatively general setting, h may be decomposed into measures of gap $h^{(\mathrm{I})}$ between each $\mathcal{P}^{(\mathrm{I})}$ and $\mathcal{L}$,

$$h = h^{(1)} - h^{(2)}. \tag{3.1}$$

As a particular choice, $h^{(\mathrm{I})}$ may be defined via *inverse–orthogonal* projections $\bar{\boldsymbol{x}}^{(\mathrm{I})} = \boldsymbol{x}^{(\mathrm{I})}(\bar{\xi}^{(\mathrm{I}),1}, \bar{\xi}^{(\mathrm{I}),2})$ onto $\mathcal{P}^{(\mathrm{I})}$ according to the normal $\boldsymbol{\nu}$ at $\boldsymbol{y} \in \mathcal{L}$,

$$h^{(\mathrm{I})} = (\bar{\boldsymbol{x}}^{(\mathrm{I})} - \boldsymbol{y}) \cdot \boldsymbol{\nu} \quad \longrightarrow \quad \bar{\boldsymbol{x}}^{(\mathrm{I})} - \boldsymbol{y} = h^{(\mathrm{I})} \boldsymbol{\nu}. \tag{3.2}$$

Here and in the following discussion, the notation $\bar{(\bullet)}$ denotes the evaluation of a quantity at the point of projection.

The film thickness definition based on (3.2) is depicted in figure 2 along with alternative ones. The present definition fits together with similar continuum theories such as shells within which a transition is made from a three-dimensional formulation to a two-dimensional one and where the thickness of the medium is defined with respect to the normal to the surface [18]. On the other hand, within a general computational tribology framework where contact and lubrication problems must be handled simultaneously, notably in the mixed lubrication regime, a definition of the gap between the interacting surfaces that is common to both aspects of the problem may be more advantageous. In this respect, it should be noted that the definition of the gap in computational contact mechanics is based predominantly [10,19] (but not exclusively [20]) on an *orthogonal* (or *closest point*) projection. This alternative definition has been employed as a basis for the numerical implementation of lubrication formulations via techniques that were originally introduced for contact problems [21]. Following standard contact mechanics terminology where one of the physical surfaces is referred to as the *slave* and the other as the *master*, yet another choice is to carry out a projection directly from the slave onto the master via an orthogonal or inverse–orthogonal projection method—see [9] for an implementation based on this definition. The first two definitions are symmetric, in the sense that the gap is independent from the labelling of the physical surfaces, whereas the third definition is non-symmetric. If $\mathcal{L}$ is chosen to coincide

with one of the physical surfaces, then the third definition is equivalent to one of the first two, depending on the choice of projection method.

From a mathematical point of view, the gap between microscopically flat surfaces is ideally zero during contact, and the lubrication theory also assumes a vanishingly small gap, so that the difference between the alternative definitions should disappear in this limit. From a numerical point of view, on the other hand, the particular choice will make a difference, particularly at coarse discretizations, similar to the choice of the kinematics for the lubrication surface. The core theoretical developments of upcoming sections are independent of the gap definition. A specific definition of the gap will be necessary only in §7, where the inverse–orthogonal projection (3.2) will be invoked. Irrespective of the specific definition employed and whether or not the particular choice of the lubrication surface impacts the results at all, the major issue of the lubrication interface undergoing finite deformations requires special treatment that is delineated in upcoming sections.

4. Kinematics at the interface

The physical surfaces $\mathcal{P}^{(\mathrm{I})}$ are described in a finite-element setting through their respective meshes. These meshes can significantly deform, particularly in the soft elastohydrodynamic lubrication regime. However, they do not necessarily constitute *material surfaces*, because at least one of the associated solids may, for instance, rotate at a high velocity, so that it is numerically more effective to carry out the solid formulation of the problem within an arbitrary Lagrangian–Eulerian setting where the meshes are assigned simplified kinematics. Presently, on the other hand, $\mathcal{P}^{(\mathrm{I})}$ are assumed to be material surfaces.

In upcoming discussions, the standard notation $\partial/\partial t$ will be retained for partial differentiation with respect to time, but $\mathrm{d}/\mathrm{d}t$ will be employed when the sole argument of the function is time. The total (material) time derivative on the physical surfaces $\mathcal{P}^{(\mathrm{I})}$ tracks their material points and when applied to $\boldsymbol{x}^{(\mathrm{I})}$ delivers their velocity distributions $\boldsymbol{v}^{(\mathrm{I})}$ that may also be expressed as

$$\boldsymbol{v}^{(\mathrm{I})} = \frac{\partial}{\partial t}\boldsymbol{x}^{(\mathrm{I})}(\xi^{(\mathrm{I}),1}, \xi^{(\mathrm{I}),2}, t). \tag{4.1}$$

Because $\mathcal{L}$ is not a material surface, the specific notation $\mathrm{D}/\mathrm{D}t$ will be employed to highlight the total time derivative that tracks points of this surface and delivers, for instance, its velocity distribution $\boldsymbol{w}$,

$$\boldsymbol{w} = \frac{\mathrm{D}\boldsymbol{y}}{\mathrm{D}t} = \frac{\partial}{\partial t}\boldsymbol{y}(\eta^1, \eta^2, t). \tag{4.2}$$

Additionally, the *transformation derivative* [22,23]

$$\frac{\delta(\bullet)}{\delta t} = \frac{\mathrm{D}(\bullet)}{\mathrm{D}t} - \frac{\partial(\bullet)}{\partial \eta^\alpha} w_{\mathrm{T}}^\alpha \tag{4.3}$$

is introduced which corresponds to the time derivative along the normal trajectory of the evolving lubrication surface. Indeed, note that $\delta\boldsymbol{y}/\delta t = \boldsymbol{w}_{\mathrm{N}}$.

Finally, recalling the notation $(\bar{\bullet})$ for evaluation at the projection point, one can define the relative velocities

$$\bar{\boldsymbol{u}}^{(\mathrm{I})} = \bar{\boldsymbol{v}}^{(\mathrm{I})} - \bar{\boldsymbol{w}}^{(\mathrm{I})}, \tag{4.4}$$

where the *projection velocities* $\bar{\boldsymbol{w}}^{(\mathrm{I})}$ are defined as

$$\bar{\boldsymbol{w}}^{(\mathrm{I})} = \frac{\mathrm{D}\bar{\boldsymbol{x}}^{(\mathrm{I})}}{\mathrm{D}t} \neq \boldsymbol{w}. \tag{4.5}$$

Here, $\bar{\boldsymbol{v}}^{(\mathrm{I})}$ represents the material velocity of $\mathcal{P}^{(\mathrm{I})}$ evaluated at the projection point $\bar{\boldsymbol{x}}^{(\mathrm{I})}$, whereas $\bar{\boldsymbol{w}}^{(\mathrm{I})}$ essentially corresponds to the velocity of the projection image $\bar{\boldsymbol{x}}^{(\mathrm{I})} \in \mathcal{P}^{(\mathrm{I})}$ of $\boldsymbol{y} \in \mathcal{L}$ which itself

moves with velocity $\boldsymbol{w}$. Following an established analysis in contact mechanics [10,19], going back to the early works of [24,25], the projection velocities may be expressed as

$$\bar{\boldsymbol{w}}^{(\mathrm{I})} = \bar{\boldsymbol{v}}^{(\mathrm{I})} + \frac{\mathrm{D}\bar{\xi}^{(\mathrm{I}),\alpha}}{\mathrm{D}t}\,\bar{\boldsymbol{a}}_{\alpha}^{(\mathrm{I})}, \tag{4.6}$$

leading to

$$\bar{\boldsymbol{u}}^{(\mathrm{I})} = -\frac{\mathrm{D}\bar{\xi}^{(\mathrm{I}),\alpha}}{\mathrm{D}t}\,\bar{\boldsymbol{a}}_{\alpha}^{(\mathrm{I})}. \tag{4.7}$$

The evolution $\mathrm{D}\bar{\xi}^{(\mathrm{I}),\alpha}/\mathrm{D}t$ of the projection coordinates will be addressed in §7.

5. Time-independent lubrication surface

The Reynolds equation formulated with respect to a curved stationary lubrication surface is relatively standard. For instance, among other possibilities, this case would be encountered if one of $\mathcal{P}^{(\mathrm{I})}$ is stationary and $\mathcal{L}$ is chosen to coincide with it. The corresponding formulation is [3,6]

$$\boxed{\frac{\partial h}{\partial t} = -\nabla\cdot\boldsymbol{q}_{\mathrm{T}}.} \tag{5.1}$$

Here, the *physical fluid flux* $\boldsymbol{q}_{\mathrm{T}}$ is a purely tangential vector and hence the expression (2.6) applies directly to its surface divergence. It has the constitutive form

$$\boxed{\boldsymbol{q}_{\mathrm{T}} = -\frac{h^3}{12\mu}\nabla p + \frac{h}{2}\left(\bar{\boldsymbol{v}}_{\mathrm{T}}^{(1)} + \bar{\boldsymbol{v}}_{\mathrm{T}}^{(2)}\right),} \tag{5.2}$$

where μ is the viscosity of the fluid, p is the pressure that is generated at the interface and the surface gradient is also with respect to $\mathcal{L}$, delivering the tangential vector

$$\nabla p = \frac{\partial p}{\partial \eta^{\alpha}}\,\boldsymbol{g}^{\alpha}. \tag{5.3}$$

Upon integrating (5.1) on $\mathcal{L}$ and making use of the surface-divergence theorem (2.9) on the tangential vector $\boldsymbol{q}_{\mathrm{T}}$, the Reynolds equation with respect to a time-independent lubrication surface may be expressed as a balance equation in integral form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{L}} h\,\mathrm{d}a = \int_{\partial\mathcal{L}} f\,\mathrm{d}\ell, \tag{5.4}$$

where $f = -\boldsymbol{q}_{\mathrm{T}}\cdot\boldsymbol{m}$ is the boundary flux on $\partial\mathcal{L}$. Note that $\mathrm{d}/\mathrm{d}t$ can here be transferred into the integral simply as $\partial/\partial t$, because $\mathcal{L}$ is stationary.

6. Time-dependent lubrication surface

The Reynolds equation for a time-dependent lubrication surface which has a velocity distribution $\boldsymbol{w}$ will now be derived from a balance equation that retains the integral form in (5.4):

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{L}} h\,\mathrm{d}a}_{1} = \underbrace{\int_{\partial\mathcal{L}} f'\,\mathrm{d}\ell}_{2}. \tag{6.1}$$

The individual terms in this expression are as follows:

1. *Rate of change of volume*: let $\mathcal{L}_o$ be a time-independent reference surface that is convected to $\mathcal{L}$, where the area mapping $\mathrm{d}a = J\,\mathrm{d}a_o$ applies. The rate of the Jacobian J is associated

with the surface divergence (2.7) of the non-tangential vector field $\boldsymbol{w}$ [16]:

$$\frac{1}{J}\frac{\mathrm{D}J}{\mathrm{D}t} = \nabla\cdot\boldsymbol{w} = \nabla\cdot\boldsymbol{w}_{\mathrm{T}} + \kappa w_{\mathrm{N}}. \tag{6.2}$$

Making use of the area mapping, the transport theorem associated with the left-hand side may now be stated as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{L}} h\,\mathrm{d}a = \int_{\mathcal{L}}\left(\frac{\mathrm{D}h}{\mathrm{D}t} + h\nabla\cdot\boldsymbol{w}\right)\mathrm{d}a = \int_{\mathcal{L}}\left(\frac{\mathrm{D}h}{\mathrm{D}t} + h(\nabla\cdot\boldsymbol{w}_{\mathrm{T}} + \kappa w_{\mathrm{N}})\right)\mathrm{d}a. \tag{6.3}$$

The last terms associated with $\nabla\cdot\boldsymbol{w}$ take into account the fact that, even if h is a constant, the volume of the fluid at the interface will increase (decrease) owing to variable tangential velocity or when a curved surface moves normal to itself, because the area expands (or contracts) in both processes. Note that the derivation of the Reynolds equation essentially requires negligible curvature on the physical surfaces $\mathcal{P}^{(\mathrm{I})}$ with respect to the thickness of the fluid film [3]. In practice, however, their ratio is small yet finite. Moreover, together with $\mathcal{P}^{(\mathrm{I})}$, $\mathcal{L}$ can undergo significant deformations such that irrespective of the magnitude of the curvature the finite area changes associated with normal motion need to be accounted for, which is automatically accomplished by the curvature term.

2. *Flux across the boundary*: f' is the flux on the boundary $\partial\mathcal{L}$ which is associated with the *relative tangential flux* $\boldsymbol{q}'_{\mathrm{T}}$ that is observed with respect to the moving surface $\mathcal{L}$,

$$\boldsymbol{q}'_{\mathrm{T}} = \boldsymbol{q}_{\mathrm{T}} - \frac{h}{2}\left(\bar{\boldsymbol{w}}^{(1)}_{\mathrm{T}} + \bar{\boldsymbol{w}}^{(2)}_{\mathrm{T}}\right) \quad\longrightarrow\quad f' = -\boldsymbol{q}'_{\mathrm{T}}\cdot\boldsymbol{m}. \tag{6.4}$$

$\boldsymbol{q}'_{\mathrm{T}}$ may be given a form identical to the form (5.2) for $\boldsymbol{q}_{\mathrm{T}}$ if the $\bar{\boldsymbol{v}}^{(\mathrm{I})}_{\mathrm{T}}$ therein are replaced, making use of (4.4), by the relative tangential velocities to obtain

$$\boxed{\boldsymbol{q}'_{\mathrm{T}} = -\frac{h^3}{12\mu}\nabla p + \frac{h}{2}\left(\bar{\boldsymbol{u}}^{(1)}_{\mathrm{T}} + \bar{\boldsymbol{u}}^{(2)}_{\mathrm{T}}\right).} \tag{6.5}$$

It is also useful to define a *total tangential flux*

$$\boldsymbol{q}''_{\mathrm{T}} = \boldsymbol{q}'_{\mathrm{T}} + h\boldsymbol{w}_{\mathrm{T}} \quad\longrightarrow\quad f'' = -\boldsymbol{q}''_{\mathrm{T}}\cdot\boldsymbol{m}. \tag{6.6}$$

Non-tangential flux vectors may also be defined by augmentation through a normal contribution,

$$\boldsymbol{q} = \boldsymbol{q}_{\mathrm{T}} + h\boldsymbol{w}_{\mathrm{N}}, \quad \boldsymbol{q}' = \boldsymbol{q}'_{\mathrm{T}} + h\boldsymbol{w}_{\mathrm{N}} \quad\text{and}\quad \boldsymbol{q}'' = \boldsymbol{q}''_{\mathrm{T}} + h\boldsymbol{w}_{\mathrm{N}}. \tag{6.7}$$

Note that $\boldsymbol{q}''$ may alternatively be expressed via $(6.6)_1$ as $\boldsymbol{q}'' = \boldsymbol{q}'_{\mathrm{T}} + h\boldsymbol{w}$, which therefore represents the total flux in the sense that it combines the tangential flux relative to $\mathcal{L}$ with the flux that emanates from its motion. Because $\boldsymbol{m}\cdot\boldsymbol{v} = 0$, the relations $f = -\boldsymbol{q}\cdot\boldsymbol{m}$, $f' = -\boldsymbol{q}'\cdot\boldsymbol{m}$ and $f'' = -\boldsymbol{q}''\cdot\boldsymbol{m}$ also hold although the surface-divergence theorem (2.9) applies only to the tangential parts of the flux vectors.

The local counterpart of (6.1) which represents the Reynolds equation on a time-dependent lubrication surface now follows from the surface-divergence theorem and significantly differs from its time-independent counterpart (5.1):

$$\boxed{\frac{\mathrm{D}h}{\mathrm{D}t} + h\nabla\cdot\boldsymbol{w} = -\nabla\cdot\boldsymbol{q}'_{\mathrm{T}}.} \tag{6.8}$$

Two additional equivalent forms follow from the expansion of $\nabla\cdot\boldsymbol{w}$,

$$\frac{\mathrm{D}h}{\mathrm{D}t} + h(\nabla\cdot\boldsymbol{w}_{\mathrm{T}} + \kappa w_{\mathrm{N}}) = -\nabla\cdot\boldsymbol{q}'_{\mathrm{T}} \quad\longrightarrow\quad \frac{\mathrm{D}h}{\mathrm{D}t} + h\nabla\cdot\boldsymbol{w}_{\mathrm{T}} = -\nabla\cdot\boldsymbol{q}'. \tag{6.9}$$

With small modifications, the following alternatives are also valid:

$$\frac{\mathrm{D}h}{\mathrm{D}t} - \nabla h \cdot \boldsymbol{w}_{\mathrm{T}} + h\kappa w_{\mathrm{N}} = -\nabla \cdot \boldsymbol{q}''_{\mathrm{T}} \quad \longrightarrow \quad \frac{\mathrm{D}h}{\mathrm{D}t} - \nabla h \cdot \boldsymbol{w}_{\mathrm{T}} = -\nabla \cdot \boldsymbol{q}''. \tag{6.10}$$

This time-dependent formulation may also be expressed with respect to a surface which momentarily coincides with the evolving lubrication surface and follows its instantaneous normal trajectory. Applying (4.3) to the film thickness, one obtains

$$\frac{\delta h}{\delta t} = \frac{\mathrm{D}h}{\mathrm{D}t} - \nabla h \cdot \boldsymbol{w}_{\mathrm{T}}, \tag{6.11}$$

which, upon substitution in (6.10), delivers

$$\frac{\delta h}{\delta t} + h\kappa w_{\mathrm{N}} = -\nabla \cdot \boldsymbol{q}''_{\mathrm{T}} \quad \longrightarrow \quad \frac{\delta h}{\delta t} = -\nabla \cdot \boldsymbol{q}''. \tag{6.12}$$

In particular, the second expression is a compact Eulerian-type form which resembles the classical Reynolds equation (5.1) for a time-independent lubrication surface, but here the transformation derivative (6.11) appears as the appropriate time derivative on the left-hand side and the total flux $(6.7)_3$ appears as the appropriate flow rate on the right-hand side. Consequently, it is clear that one does not obtain the form of the classical Reynolds equation even if $\mathcal{L}$ is chosen to coincide with one of $\mathcal{P}^{(\mathrm{I})}$, precisely owing to the inability of the classical formulation to properly account for finite-deformation effects.

The numerical implementation of the time-dependent formulation (6.9) or (6.10) within a finite-element setting may be carried out in a number of ways. Although the numerical aspects of the problem are outside the scope of this work, the relevant weak form is shortly commented upon. Suppose that the pressure is controlled on the Dirichlet portion of the boundary $\partial\mathcal{L}^{\mathrm{D}} \subset \partial\mathcal{L}$. Let π be a test function that represents the variation of the pressure and therefore is non-zero only on the Neumann boundary $\partial\mathcal{L}^{\mathrm{N}} = \partial\mathcal{L} \backslash \partial\mathcal{L}^{\mathrm{D}}$ where either the relative boundary flux f' or the total one f'' may be prescribed to a value $\hat{f}'$ or $\hat{f}''$, respectively. Multiplying the strong form $(6.9)_1$ by π, integrating over $\mathcal{L}$ and manipulating the right-hand side using the surface-divergence theorem (2.9) delivers the weak form

$$\int_{\mathcal{L}} \pi \left(\frac{\mathrm{D}h}{\mathrm{D}t} + h(\nabla \cdot \boldsymbol{w}_{\mathrm{T}} + \kappa w_{\mathrm{N}}) \right) \mathrm{d}a = \int_{\mathcal{L}} \nabla\pi \cdot \boldsymbol{q}'_{\mathrm{T}} \, \mathrm{d}a + \int_{\partial\mathcal{L}^{\mathrm{N}}} \pi \hat{f}' \, \mathrm{d}\ell. \tag{6.13}$$

Alternatively, starting from the strong form $(6.10)_1$, one obtains

$$\int_{\mathcal{L}} \pi \left(\frac{\mathrm{D}h}{\mathrm{D}t} - \nabla h \cdot \boldsymbol{w}_{\mathrm{T}} + h\kappa w_{\mathrm{N}} \right) \mathrm{d}a = \int_{\mathcal{L}} \nabla\pi \cdot \boldsymbol{q}''_{\mathrm{T}} \, \mathrm{d}a + \int_{\partial\mathcal{L}^{\mathrm{N}}} \pi \hat{f}'' \, \mathrm{d}\ell. \tag{6.14}$$

The curvature contribution in either form may alternatively be evaluated implicitly within the isoparametric setting of finite elements by replacing $\pi h\kappa w_{\mathrm{N}}$ with one of the following equivalent expressions:

$$\pi h\kappa w_{\mathrm{N}} = \pi h \nabla \cdot \boldsymbol{w}_{\mathrm{N}} = \pi \nabla \cdot (h\boldsymbol{w}_{\mathrm{N}}) = \nabla \cdot (\pi h \boldsymbol{w}_{\mathrm{N}}). \tag{6.15}$$

7. Total time derivative of film thickness

The total time derivative term $\mathrm{D}h/\mathrm{D}t$ is evaluated at a fixed point (η^1, η^2) of $\mathcal{L}$. In order to calculate this term, a specific measure of the gap between the surfaces is necessary. Any one of the three choices depicted in figure 2 involves similar calculation steps. Invoking the definition (3.2) based on an inverse–orthogonal projection as a particular choice and observing that the derivative of $\boldsymbol{v} \cdot \boldsymbol{v} = 1$ vanishes, one may write

$$\frac{\mathrm{D}h^{(\mathrm{I})}}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\{(\bar{\boldsymbol{x}}^{(\mathrm{I})} - \boldsymbol{y}) \cdot \boldsymbol{v}\} = (\bar{\boldsymbol{w}}^{(\mathrm{I})} - \boldsymbol{w}) \cdot \boldsymbol{v}. \tag{7.1}$$

It follows via (3.1) and (4.6) that

$$\frac{\mathrm{D}h}{\mathrm{D}t} = \left(\bar{v}_{\mathrm{N}}^{(1)} - \bar{v}_{\mathrm{N}}^{(2)}\right) + \left(\frac{\mathrm{D}\bar{\xi}^{(1),\alpha}}{\mathrm{D}t}\bar{\boldsymbol{a}}_{\alpha}^{(1)} - \frac{\mathrm{D}\bar{\xi}^{(2),\alpha}}{\mathrm{D}t}\bar{\boldsymbol{a}}_{\alpha}^{(2)}\right)\cdot\boldsymbol{v}, \tag{7.2}$$

which also delivers $\delta h/\delta t$ in (6.12) via (6.11). The evolution of the projection coordinates follows from the fact that $(\bar{\boldsymbol{x}}^{(\mathrm{I})} - \boldsymbol{y})\cdot\boldsymbol{g}_{\alpha} = 0$ and hence the derivative of this equality is also zero. Explicitly, again using (3.2) and noting $\mathrm{D}\boldsymbol{g}_{\alpha}/\mathrm{D}t = \partial\boldsymbol{w}/\partial\eta^{\alpha}$, one may write

$$0 = \frac{\mathrm{D}}{\mathrm{D}t}\{(\bar{\boldsymbol{x}}^{(\mathrm{I})} - \boldsymbol{y})\cdot\boldsymbol{g}_{\alpha}\} = \left(\bar{\boldsymbol{v}}^{(\mathrm{I})} + \frac{\mathrm{D}\bar{\xi}^{(\mathrm{I}),\beta}}{\mathrm{D}t}\bar{\boldsymbol{a}}_{\beta}^{(\mathrm{I})} - \boldsymbol{w}\right)\cdot\boldsymbol{g}_{\alpha} + h^{(\mathrm{I})}\boldsymbol{v}\cdot\frac{\partial\boldsymbol{w}}{\partial\eta^{\alpha}} \tag{7.3}$$

or, in the form of a linear system of equations based on the definition $c_{\alpha\beta}^{(\mathrm{I})} = \boldsymbol{g}_{\alpha}\cdot\bar{\boldsymbol{a}}_{\beta}^{(\mathrm{I})}$,

$$-c_{\alpha\beta}^{(\mathrm{I})}\frac{\mathrm{D}\bar{\xi}^{(\mathrm{I}),\beta}}{\mathrm{D}t} = (\bar{\boldsymbol{v}}^{(\mathrm{I})} - \boldsymbol{w})\cdot\boldsymbol{g}_{\alpha} + h^{(\mathrm{I})}\boldsymbol{v}\cdot\frac{\partial\boldsymbol{w}}{\partial\eta^{\alpha}}, \tag{7.4}$$

which completes the calculation of $\mathrm{D}h/\mathrm{D}t$. The left-hand side also corresponds to $\bar{u}_{\mathrm{T}}^{(\mathrm{I}),\alpha}$ in view of (4.7) and, therefore, (7.4) can be additionally employed in the evaluation of the relative tangential flux (6.5). Note that the last term of (7.4) has the explicit form

$$\boldsymbol{v}\cdot\frac{\partial\boldsymbol{w}}{\partial\eta^{\alpha}} = \frac{\partial w_{\mathrm{N}}}{\partial\eta^{\alpha}} + b_{\alpha\beta}w_{\mathrm{T}}^{\beta}. \tag{7.5}$$

In the preceding analysis, the projection velocities have been expressed in the form (4.6). For future reference, it is instructive to express them in an alternative form by first observing from the particular gap definition (3.2) that

$$\bar{\boldsymbol{x}}^{(\mathrm{I})} = \boldsymbol{y} + h^{(\mathrm{I})}\boldsymbol{v} \tag{7.6}$$

and subsequently applying the definition (4.5) to obtain

$$\bar{\boldsymbol{w}}^{(\mathrm{I})} = \boldsymbol{w} + \frac{\mathrm{D}h^{(\mathrm{I})}}{\mathrm{D}t}\boldsymbol{v} + h^{(\mathrm{I})}\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}, \tag{7.7}$$

where the last term is a tangential vector with components

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}\cdot\boldsymbol{g}_{\alpha} = -\boldsymbol{v}\cdot\frac{\partial\boldsymbol{w}}{\partial\eta^{\alpha}}. \tag{7.8}$$

Hence, the normal part of $\bar{\boldsymbol{w}}^{(\mathrm{I})}$ is

$$\bar{\boldsymbol{w}}_{\mathrm{N}}^{(\mathrm{I})} = \left(w_{\mathrm{N}} + \frac{\mathrm{D}h^{(\mathrm{I})}}{\mathrm{D}t}\right)\boldsymbol{v} \tag{7.9}$$

while the tangential part is, making use of (7.5),

$$\bar{\boldsymbol{w}}_{\mathrm{T}}^{(\mathrm{I})} = \left(w_{\mathrm{T}}^{\alpha} - h^{(\mathrm{I})}\left\{b_{\alpha\beta}w_{\mathrm{T}}^{\beta} + \frac{\partial w_{\mathrm{N}}}{\partial\eta^{\alpha}}\right\}\right)\boldsymbol{g}_{\alpha}. \tag{7.10}$$

Clearly, $\bar{\boldsymbol{w}}_{\mathrm{T}}^{(\mathrm{I})} \neq \boldsymbol{w}_{\mathrm{T}}$ in general. However, when the curvature or the tangential velocity of $\mathcal{L}$ vanishes and its normal velocity is a constant, $\bar{\boldsymbol{w}}_{\mathrm{T}}^{(\mathrm{I})} = \boldsymbol{w}_{\mathrm{T}}$ holds. Sample scenarios which satisfy these assumptions will be analysed in §10. In such cases, one observes from (6.4) that the relative tangential flux takes the simplified form $\boldsymbol{q}_{\mathrm{T}}' = \boldsymbol{q}_{\mathrm{T}} - h\boldsymbol{w}_{\mathrm{T}}$ and, as a consequence, the difference between the total and the physical flux vectors vanishes, i.e. $\boldsymbol{q}'' = \boldsymbol{q}$ and $f'' = f$.

8. Objectivity

The proposed time-dependent formulation satisfies objectivity requirements. To demonstrate this, an observer is denoted by $\mathcal{O}$, and the observer transformation $\mathcal{O} \to \mathcal{O}^+$ is invoked that is characterized by the position vector transformations

$$\boldsymbol{x}^{(\mathrm{I}),+} = \boldsymbol{Q}(t)\boldsymbol{x}^{(\mathrm{I})} + \boldsymbol{c}(t) \quad \text{and} \quad \boldsymbol{y}^+ = \boldsymbol{Q}(t)\boldsymbol{y} + \boldsymbol{c}(t), \tag{8.1}$$

where $\boldsymbol{Q}$ is a proper orthogonal tensor. Additionally, without any loss of generality, the invariance $t^+ = t$ as well as $\xi^{(\mathrm{I}),\alpha,+} = \xi^{(\mathrm{I}),\alpha}$ and $\eta^{\alpha,+} = \eta^\alpha$ will be invoked. Clearly, this observer transformation leads to a rotation of all basis and normal vectors through $\boldsymbol{Q}$. In particular, it is noted that

$$\boldsymbol{g}_\alpha^+ = \boldsymbol{Q}\boldsymbol{g}_\alpha \quad \text{and} \quad \boldsymbol{\nu}^+ = \boldsymbol{Q}\boldsymbol{\nu}. \tag{8.2}$$

The normal and tangential parts of a vector $\boldsymbol{z}^+$ with respect to $\mathcal{O}^+$ are therefore

$$z_{\mathrm{N}}^+ = \boldsymbol{z}^+ \cdot \boldsymbol{\nu}^+ \quad \longrightarrow \quad \boldsymbol{z}_{\mathrm{N}}^+ = z_{\mathrm{N}}^+ \boldsymbol{\nu}^+ \tag{8.3}$$

and

$$\boldsymbol{z}_{\mathrm{T}}^+ = \boldsymbol{z}^+ - \boldsymbol{z}_{\mathrm{N}}^+ = z_{\mathrm{T}}^{\alpha,+} \boldsymbol{g}_\alpha^+. \tag{8.4}$$

Moreover, it follows that

$$\nabla^+ = \boldsymbol{Q}\nabla, \quad h^+ = h \quad \text{and} \quad \kappa^+ = \kappa. \tag{8.5}$$

For compactness, the notation

$$\dot{\boldsymbol{Q}} = \frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}t} \quad \text{and} \quad \dot{\boldsymbol{c}} = \frac{\mathrm{d}\boldsymbol{c}}{\mathrm{d}t} \tag{8.6}$$

will additionally be employed.

Now, objectivity requires that the time-dependent formulation of the Reynolds equation for $\mathcal{O}$ preserves its form with respect to $\mathcal{O}^+$. Making use of (6.8), this requirement translates into satisfying

$$\underbrace{\frac{\mathrm{D}h^+}{\mathrm{D}t}}_{1} + \underbrace{h^+\nabla^+ \cdot \boldsymbol{w}^+}_{2} = \underbrace{-\nabla^+ \cdot \boldsymbol{q}_{\mathrm{T}}'^+}_{3}. \tag{8.7}$$

The individual terms in this expression are analysed next:

1. Projection coordinate evolutions $\mathrm{D}\bar{\xi}^{(\mathrm{I}),\alpha}/\mathrm{D}t$ remain invariant under observer transformations which may be verified, for the particular gap definition (3.2), by revisiting the discussion of §7. One therefore concludes from (4.6) that

$$\bar{\boldsymbol{w}}^{(\mathrm{I}),+} = \boldsymbol{Q}\bar{\boldsymbol{w}}^{(\mathrm{I})} + \dot{\boldsymbol{Q}}\bar{\boldsymbol{x}}^{(\mathrm{I})} + \dot{\boldsymbol{c}}. \tag{8.8}$$

   It follows that $\mathrm{D}h/\mathrm{D}t$ is invariant under an observer transformation:

$$\frac{\mathrm{D}h^+}{\mathrm{D}t} = \frac{\mathrm{D}h}{\mathrm{D}t}. \tag{8.9}$$

   Note, however, that $\delta h/\delta t$ in (6.11) is not invariant.

2. One observes that

$$\nabla^+ \cdot \boldsymbol{w}^+ = \boldsymbol{Q}\nabla \cdot (\boldsymbol{Q}\boldsymbol{w} + \dot{\boldsymbol{Q}}\boldsymbol{y} + \dot{\boldsymbol{c}}) = \nabla \cdot \boldsymbol{w} + \nabla \cdot (\boldsymbol{\Omega}\boldsymbol{y}), \tag{8.10}$$

   where $\boldsymbol{\Omega} = \boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{Q}}$ is a skew-symmetric tensor. Recalling the expression $\boldsymbol{g}^\alpha = \partial\eta^\alpha/\partial\boldsymbol{y}$ and indicating the Kronecker delta with δ_{ij}, the last term may be expressed via (2.5) as

$$\nabla \cdot (\boldsymbol{\Omega}\boldsymbol{y}) = \boldsymbol{\Omega}\frac{\partial \boldsymbol{y}}{\partial \eta^\alpha} \cdot \frac{\partial \eta^\alpha}{\partial \boldsymbol{y}} = \Omega_{ij}\frac{\partial y_j}{\partial \eta^\alpha}\frac{\partial \eta^\alpha}{\partial y_i} = \Omega_{ij}\delta_{ji} = \Omega_{ii} = 0. \tag{8.11}$$

   Because $h^+ = h$, the invariance of the second term in (8.7) follows. Note that the normal and tangential contributions to $\nabla \cdot \boldsymbol{w}$ are not individually invariant.

3. Material frame-indifference is invoked which, based on (6.5), delivers

$$\boldsymbol{q}_{\mathrm{T}}^{\prime,+}=-\frac{(h^{+})^{3}}{12\mu^{+}}\nabla^{+}p^{+}+\frac{h^{+}}{2}\left(\bar{\boldsymbol{u}}_{\mathrm{T}}^{(1),+}+\bar{\boldsymbol{u}}_{\mathrm{T}}^{(2),+}\right). \tag{8.12}$$

Invoking the invariance statements $\mu^{+}=\mu$ and $p^{+}=p$, one may employ earlier transformation rules to first observe from (4.7) that the relative velocities transform objectively, i.e. $\bar{\boldsymbol{u}}_{\mathrm{T}}^{(\mathrm{I}),+}=\boldsymbol{Q}\bar{\boldsymbol{u}}_{\mathrm{T}}^{(\mathrm{I})}$, and subsequently conclude the objectivity of the relative tangential flux,

$$\boldsymbol{q}_{\mathrm{T}}^{\prime,+}=-\frac{h^{3}}{12\mu}\boldsymbol{Q}\nabla p+\frac{h}{2}\boldsymbol{Q}\left(\bar{\boldsymbol{u}}_{\mathrm{T}}^{(1)}+\bar{\boldsymbol{u}}_{\mathrm{T}}^{(2)}\right)=\boldsymbol{Q}\boldsymbol{q}_{\mathrm{T}}^{\prime}. \tag{8.13}$$

Therefore, the last term in (8.7) remains invariant:

$$-\nabla^{+}\cdot\boldsymbol{q}_{\mathrm{T}}^{\prime,+}=-\boldsymbol{Q}\nabla\cdot\boldsymbol{Q}\boldsymbol{q}_{\mathrm{T}}^{\prime}=-\nabla\cdot\boldsymbol{q}_{\mathrm{T}}^{\prime}. \tag{8.14}$$

This completes the proof of objectivity of the time-dependent formulation (6.8). Note that neither $\boldsymbol{q}$ nor $\boldsymbol{q}''$ transforms objectively. As a particular implication, the classical time-independent formulation (5.1) does not satisfy objectivity requirements because, by construction, the lubrication surface is assumed to be stationary or deform at most infinitesimally, which precludes large rotations as well—see also §10. In this sense, the difference between the two formulations (5.1) and (6.8) is reminiscent of the difference between small and large strain elasticity formulations.

9. Construction of the lubrication surface

So far, an explicit description of the lubrication surface $\mathcal{L}$ has not been necessary. Clearly, however, the construction of $\mathcal{L}$ cannot be arbitrary because it will significantly influence the solution of the lubrication problem. In fact, even in a time-independent setting with rigid physical surfaces, the choice of $\mathcal{L}$ may be non-trivial. Various scenarios with the additional simplification of stationary surfaces ($\boldsymbol{v}^{(\mathrm{I})}=\boldsymbol{0}$) have been reviewed in [26,27], and the importance of the choice of $\mathcal{L}$ has been pointed out, with certain constructions leading to a significant overestimation of experimental measurements or numerical results based on Stokes flow. Presently, no attempt is made to examine the effects of various possible choices or improve the predictive capability of the Reynolds equation beyond its intended domain of applicability. Instead, a particular choice will be made that will help to highlight the importance of finite deformations in §10.

Based on the general expression (3.1), the lubrication surface $\mathcal{L}$ is constructed so as to ensure that the geometry of $\mathcal{L}$ is representative of the geometries of $\mathcal{P}^{(\mathrm{I})}$ *at all times*. A simple construction of $\mathcal{L}$ that attains this goal is to restrict $\mathcal{L}$ to lie intermediate to $\mathcal{P}^{(\mathrm{I})}$ and additionally ensure that the ratio

$$\lambda=\frac{h^{(1)}}{h}\in[0,1] \tag{9.1}$$

is constant on $\mathcal{L}$. For instance, if (3.2) is invoked as a particular definition of $h^{(\mathrm{I})}$, then the geometry of $\mathcal{L}$ is described via

$$\boldsymbol{y}=(1-\lambda)\bar{\boldsymbol{x}}^{(1)}+\lambda\bar{\boldsymbol{x}}^{(2)}. \tag{9.2}$$

Consequently, the local slope and curvature of $\mathcal{L}$ are directly acquired from $\mathcal{P}^{(\mathrm{I})}$ and cannot be arbitrary. This guarantees that the formulation of the lubrication problem on $\mathcal{L}$ will respect the series of assumptions underlying the derivation of the Reynolds equation, provided that the formulation of the problem on either of the physical surfaces satisfies them.

When $\lambda=\frac{1}{2}$, the geometry of $\mathcal{L}$ is reminiscent of a shell geometry [18], with $\mathcal{L}$ representing the mid-plane. When $\lambda=0$ or 1, $\mathcal{L}$ coincides with one of the physical surfaces. For varying λ, therefore, $\mathcal{L}$ will span the interface between $\mathcal{P}^{(\mathrm{I})}$, as proposed in [9]. Moreover, with this construction, $\mathcal{L}$ has a non-zero normal velocity w_{N} only if $\mathcal{P}^{(\mathrm{I})}$ have relative normal velocities. In this sense, $\mathcal{L}$ is completely locked on to the kinematics of $\mathcal{P}^{(\mathrm{I})}$ in the normal direction. Note, however, that

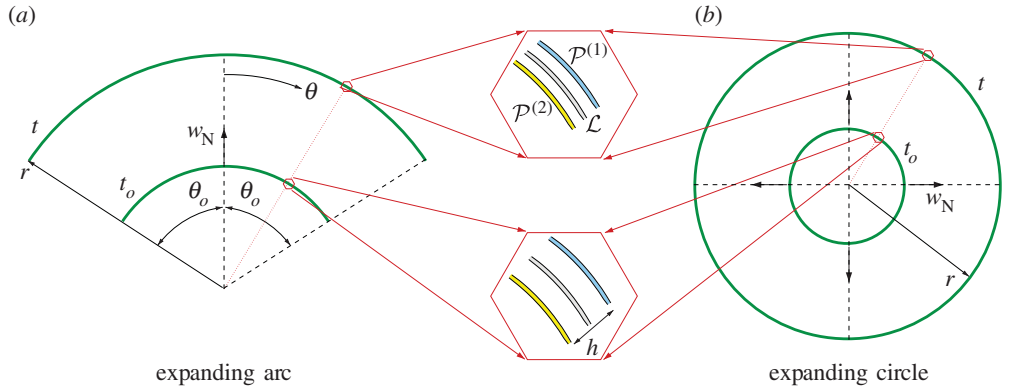

Figure 3. (*a*,*b*) Expanding interfaces with pure normal velocities. (Online version in colour.)

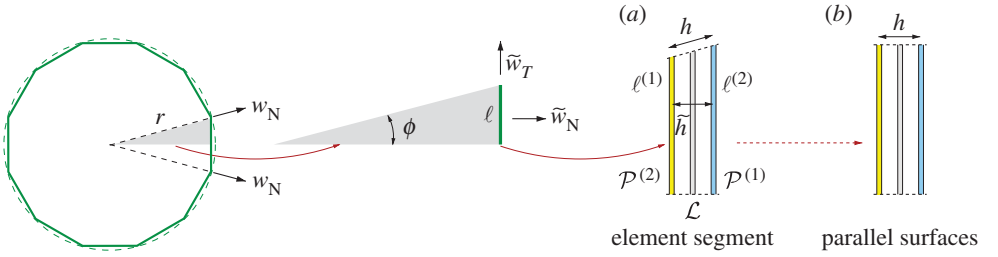

Figure 4. (*a*,*b*) Approximation of the expanding circle with linear finite elements. (Online version in colour.)

the tangential kinematics of $\mathcal{L}$ is still independent, because $\boldsymbol{w}_T$ is only weakly related to the kinematics of $\mathcal{P}^{(1)}$, to ensure that $\mathcal{L}$ continuously encompasses the potential lubrication domain.

The explicit construction of $\mathcal{L}$ for values of λ between 0 and 1 through the calculation of its points $\boldsymbol{y}$ according to the constraint (9.1) may not be straightforward. This is observed in relation (9.2) based on an inverse–orthogonal projection, because the calculation of the right-hand side implicitly depends on the geometry of $\mathcal{L}$. A similar situation holds for the orthogonal projection method of figure 2. In this sense, the slave–master projection is a facilitating approach [9]. Alternatively, choosing $\mathcal{L}$ to coincide with one of the physical surfaces entirely circumvents the need to construct a third surface. The optimal specification of $\mathcal{L}$ depends on the particular problem and will not be elaborated upon further.

10. Illustrative examples

Here, a series of analytical examples will be discussed in order to illustrate the effect of finite deformations, the choice of the lubrication surface and the influence of numerical discretization. For compactness, the notation $(\bar{\bullet})$ for the evaluation of a quantity at the point of projection is dropped. It is recalled from the discussion in §7 that $\boldsymbol{q}''_T = \boldsymbol{q}_T$ for the examples in figures 3 and 4.

Figure 3 depicts two simple settings that highlight the importance of curvature at finite deformations. Here, the physical surfaces expand from their initial states at time t_o and the lubrication surface with radial position r follows them, because it is constrained to lie in-between the two by construction. In both cases, the film thickness is assumed to be constant along the interface, the physical surfaces are assigned normal velocities only, and the curvature is a constant $\kappa = 1/r$. Owing to the simple geometry of the surfaces, the film thickness condenses to the same expression for alternative definitions and can be expressed as $h = r^{(1)} - r^{(2)}$ in terms of the radial

positions $r^{(\mathrm{I})}$ of the two surfaces. Likewise, its rate of change can be expressed in terms of the normal velocities of the physical surfaces: $\mathrm{d}h/\mathrm{d}t = v_{\mathrm{N}}^{(1)} - v_{\mathrm{N}}^{(2)}$.

In the case of the expanding arc (figure 3*a*), the film thickness $h(t)$ is constrained to remain constant in time. Consequently, fluid will be sucked into the interface at the boundaries, leading to a non-zero boundary flux $f' = f''$ and hence to non-zero flux vectors $\boldsymbol{q}'_{\mathrm{T}} = \boldsymbol{q}''_{\mathrm{T}}$ that will generate a pressure distribution. Here, the film thickness satisfies a simplified version of $(6.9)_1$ or $(6.10)_1$,

$$\frac{hw_{\mathrm{N}}}{r} = \nabla\cdot\frac{h^3}{12\mu}\nabla p. \tag{10.1}$$

Prescribing $p = p_o$ at the two ends $\theta = \pm\theta_o$, one obtains the quadratic distribution

$$p(\theta) = p_o + \frac{6\mu}{h^2} r w_{\mathrm{N}}(\theta^2 - \theta_o^2). \tag{10.2}$$

Clearly, unless the curvature term is present, one cannot account for the continuous influx of fluid through the boundaries that will lead to a significant change in the volume of the interface at large deformations and subsequently to a pressure generation. Because h is enforced to remain constant, $v_{\mathrm{N}}^{(1)} = v_{\mathrm{N}}^{(2)} = w_{\mathrm{N}}$ is required. Hence, the influence of the choice of $\mathcal{L}$ only reflects through a linear dependence on the ratio λ from (9.1), because $r = r^{(2)} + (1-\lambda)h$ in (10.2).

In the case of the expanding circle (figure 3*b*), where $\boldsymbol{q}'_{\mathrm{T}} = \boldsymbol{q}''_{\mathrm{T}} = \boldsymbol{0}$, the balance equation simplifies to

$$\frac{\mathrm{d}h}{\mathrm{d}t} + \frac{hw_{\mathrm{N}}}{r} = 0 \tag{10.3}$$

and hence the film thickness must continuously decrease to preserve the volume. Indeed, because $w_{\mathrm{N}} = \mathrm{d}r/\mathrm{d}t$, this expression is equivalent to $\mathrm{d}(hr)/\mathrm{d}t = 0$. As a further specialized scenario, suppose both physical surfaces are initially stationary and $\mathcal{P}^{(2)}$ is subsequently mobilized with a velocity $v_{\mathrm{N}}^{(2)}$. For the sake of simplicity, $\mathcal{L}$ can be chosen to coincide with this surface so that $w_{\mathrm{N}} = v_{\mathrm{N}}^{(2)}$, i.e. $\lambda = 1$ in (9.1). Defining

$$\tau = \frac{h}{r^{(2)}} \ll 1 \tag{10.4}$$

the balance equation (10.3) then delivers

$$v_{\mathrm{N}}^{(1)} = (1-\tau)v_{\mathrm{N}}^{(2)} < v_{\mathrm{N}}^{(2)}. \tag{10.5}$$

This constraint will also automatically initiate the motion of $\mathcal{P}^{(1)}$. For an arbitrary choice of λ, observing $w_{\mathrm{N}} = (1-\lambda)v_{\mathrm{N}}^{(1)} + \lambda v_{\mathrm{N}}^{(2)}$, one obtains

$$v_{\mathrm{N}}^{(1)} = \left(\frac{1+\tau(1-2\lambda)}{1+2\tau(1-\lambda)}\right) v_{\mathrm{N}}^{(2)}. \tag{10.6}$$

In the other extreme case of $\lambda = 0$, from (10.6) one obtains $v_{\mathrm{N}}^{(1)} = (1+\tau)/(1+2\tau)v_{\mathrm{N}}^{(2)}$. The estimation of $v_{\mathrm{N}}^{(1)}$ based on this expression differs from (10.5) already by less than 2% for $\tau = 0.1$. Hence, the observed influence of the choice of $\mathcal{L}$ in this case is a finite film thickness effect and it will vanish if the thin-film assumption $\tau \ll 1$ is strictly invoked, which can easily be verified by linearizing the expression for $\lambda = 0$ with respect to τ to obtain (10.5).

The influence of the tangential motion of $\mathcal{L}$ at finite deformations can be examined in the context of a finite-element approximation of the expanding circle based on the discretization of the surfaces $\{\mathcal{P}^{(\mathrm{I})}, \mathcal{L}\}$ with linear elements (figure 4). Owing to the discretization, the surfaces will pick velocities $\{\tilde{\boldsymbol{v}}^{(\mathrm{I})}, \tilde{\boldsymbol{w}}\}$ which differ from $\{\boldsymbol{v}^{(\mathrm{I})}, \boldsymbol{w}\}$ of the preceding analysis and the film thickness between the elements $\tilde{h} = h\cos\phi$ will also differ from h. By symmetry, it is sufficient to consider a segment of length $\ell = r\sin\phi$ that corresponds to half of an element of $\mathcal{L}$, which has a constant pressure over its length (figure 4*a*). The segment lengths for the physical surface elements will be denoted by $\ell^{(\mathrm{I})} = r^{(\mathrm{I})}\sin\phi$. The pure radial expansion of the circle is prescribed on the nodes of the surface elements through normal velocities $\{v_{\mathrm{N}}^{(\mathrm{I})}, w_{\mathrm{N}}\}$. This translates into a velocity field $\tilde{\boldsymbol{w}}$ on the segment with a constant normal component $\tilde{w}_{\mathrm{N}} = w_{\mathrm{N}}\cos\phi$, which does not have an effect owing

to zero curvature, and a tangential part $\tilde{w}_\mathrm{T}$ with magnitude $\tilde{w}_\mathrm{T} = w_\mathrm{N} \sin\phi$ on the node. Similarly, $\tilde{v}_\mathrm{N}^{(\mathrm{I})} = v_\mathrm{N}^{(\mathrm{I})} \cos\phi$ and $\tilde{v}_\mathrm{T}^{(\mathrm{I})} = v_\mathrm{N}^{(\mathrm{I})} \sin\phi$. Note that, in general, neither the relative boundary flux f' nor the total one f'' vanishes on the edge of an element in this case. Now, invoking $(6.10)_1$, one obtains the balance statement

$$\frac{\mathrm{d}\tilde{h}}{\mathrm{d}t} = -\frac{\tilde{h}}{2}\nabla\cdot\left(\tilde{\boldsymbol{v}}_\mathrm{T}^{(1)} + \tilde{\boldsymbol{v}}_\mathrm{T}^{(2)}\right), \tag{10.7}$$

where $\mathrm{d}\tilde{h}/\mathrm{d}t = \tilde{v}_\mathrm{N}^{(1)} - \tilde{v}_\mathrm{N}^{(2)}$ again holds and, in view of the linear variation of $\tilde{v}_\mathrm{T}^{(\mathrm{I})}$ on $\mathcal{P}^{(\mathrm{I})}$, $\nabla\cdot\boldsymbol{v}_\mathrm{T}^{(\mathrm{I})} = \tilde{v}_\mathrm{T}^{(\mathrm{I})}/\ell^{(\mathrm{I})}$. Again, as a specialized scenario, suppose that the nodes on $\mathcal{P}^{(2)}$ are mobilized with a velocity $v_\mathrm{N}^{(2)}$. The balance statement then delivers normal velocities for the nodes on $\mathcal{P}^{(1)}$

$$v_\mathrm{N}^{(1)} = \left(\frac{1 - h/2r^{(2)}}{1 + h/2r^{(1)}}\right) v_\mathrm{N}^{(2)} = \left(\frac{(2-\tau)(1+\tau)}{(2+3\tau)}\right) v_\mathrm{N}^{(2)} < v_\mathrm{N}^{(2)}, \tag{10.8}$$

where the definition (10.4) has been employed. This result is independent of the choice of $\mathcal{L}$ and also independent from the number of elements, i.e. from ϕ. The estimations of $v_\mathrm{N}^{(1)}$ based on (10.8) and (10.5) differ by less than 10% for $\tau = 0.3$ and already by less than 1% for $\tau = 0.1$. Hence, the observed difference is again a finite film thickness effect. Indeed, the linearization of (10.8) with respect to τ delivers (10.5). Overall, this example also demonstrates that the improper resolution of curvature in standard finite-element discretizations will be accounted for by the initiation of tangential motion.

A simpler analysis for demonstrating the influence of surface expansion and the related tangential motion at finite deformations may be carried out by considering the modified set-up in figure 4*b* where parallel segments of equal length $\ell^{(\mathrm{I})} = \ell$ elongate with a pure tangential velocity of magnitude $v_\mathrm{T}^{(\mathrm{I})} = w_\mathrm{T}$ at the top and zero at the bottom. Additionally, invoking zero relative flux boundary conditions $f' = 0$, one concludes that the pressure must remain constant at the interface. Consequently, $\boldsymbol{q}_\mathrm{T}'$ vanishes in this case, so that it is convenient to employ $(6.9)_1$, which leads to

$$\frac{\mathrm{d}h}{\mathrm{d}t} + h\nabla\cdot\boldsymbol{w}_\mathrm{T} = 0. \tag{10.9}$$

Observing that $\nabla\cdot\boldsymbol{w}_\mathrm{T} = (1/\ell)(\mathrm{d}\ell/\mathrm{d}t)$, this result is equivalent to $\mathrm{d}(h\ell)/\mathrm{d}t = 0$, which is the statement of volume conservation. This constraint will force the physical surfaces to approach each other in the normal direction. Note that if $\mathcal{L}$ expanded as above but the physical surfaces did not have tangential velocities and zero total flux boundary conditions $f'' = 0$ were applied then the pressure should trivially remain constant. In this case, $\boldsymbol{q}_\mathrm{T}''$ would vanish, so that it is more convenient to invoke $(6.10)_1$, which would directly imply $\mathrm{d}h/\mathrm{d}t = 0$ as required, irrespective of the tangential motion of $\mathcal{L}$.

In all of these examples, rigid body translations or rotations of the interfaces as a whole should not lead to non-physical outcomes. This is guaranteed by the objectivity of the formulation. As a specific scenario, consider the circular interface geometry of figure 3*b*, further simplified by omitting expansion. Subsequently, the circle as a whole ($\mathcal{P}^{(\mathrm{I})}$ and $\mathcal{L}$) is subjected to a rigid body rotation with a prescribed angular velocity Ω, which induces a physical tangential flux $\boldsymbol{q}_\mathrm{T}$ owing to the resulting physical tangential velocities $\boldsymbol{v}_\mathrm{T}^{(\mathrm{I})}$ with magnitude $\Omega r^{(\mathrm{I})}$. Consequently, the time-independent formulation (5.1) would make the incorrect prediction of a time-varying gap. On the other hand, because the projection coordinates are not altered by rotation, the relative tangential flux $\boldsymbol{q}_\mathrm{T}'$ vanishes and the time-dependent formulation (6.8) correctly predicts the trivial solution of a stationary gap, the second contribution vanishing automatically in view of (8.11).

11. Conclusion

Elastohydrodynamic lubrication at dynamically evolving soft interfaces requires the ability to formulate and solve the Reynolds equation with respect to a time-dependent curvilinear coordinate system. Such a formulation which satisfies objectivity requirements has been presented and discussed in this work. Several equivalent forms of the Reynolds equation with

respect to a time-dependent lubrication surface have been derived and various finite-deformation effects have been illustrated through analytical examples. The numerical implementation of this time-dependent formulation within a computational tribology framework that can simultaneously address contact and lubrication, which has been carried out in a time-independent setting for both hard [4,21] and soft [6,9] interfaces, constitutes a subject of future interest.

Authors' contributions. The authors have contributed equally.

Competing interests. The authors have no competing interests.

Funding. İ.T. acknowledges support by the Scientific and Technological Research Council of Turkey (TÜBİTAK) under the 1001 Programme (grant no. 114M406). S.S. acknowledges support by the National Science Centre (NCN) in Poland under grant no. 2011/01/B/ST8/07434.

References

1. Reynolds O. 1886 On the theory of lubrication and its application to Mr. Beauchamp Tower's experiments, including an experimental determination of the viscosity of olive oil. *Phil. Trans. R. Soc. Lond. A* **177**, 157–234. (doi:10.1098/rstl.1886.0005)
2. Hamrock BJ, Schmid SR, Jacobson BO. 2004 *Fundamentals of fluid film lubrication*, 2nd edn. New York, NY: Marcel Dekker.
3. Szeri AZ. 2011 *Fluid film lubrication*, 2nd edn. Cambridge, UK: Cambridge University Press.
4. Curnier AR, Taylor RL. 1982 A thermomechanical formulation and solution of lubricated contacts between deformable solids. *J. Lubr. Technol.* **104**, 109–117. (doi:10.1115/1.3253155)
5. Skotheim JM, Mahadevan L. 2004 Soft lubrication. *Phys. Rev. Lett.* **92**, 245509. (doi:10.1103/PhysRevLett.92.245509)
6. Stupkiewicz S. 2009 Finite element treatment of soft elastohydrodynamic lubrication problems in the finite deformation regime. *Comput. Mech.* **44**, 605–619. (doi:10.1007/s00466-009-0394-3)
7. Schmidt T, André M, Poll G. 2010 A transient 2D-finite-element approach for the simulation of mixed lubrication effects of reciprocating hydraulic rod seals. *Tribol. Int.* **43**, 1775–1785. (doi:10.1016/j.triboint.2009.11.012)
8. Scaraggi M, Carbone G. 2012 A two-scale approach for lubricated soft-contact modeling: an application to lip-seal geometry. *Adv. Tribol.* **2012**, 412190. (doi:10.1155/2012/412190)
9. Stupkiewicz S, Lengiewicz J, Sadowski P, Kucharski S. 2016 Finite deformation effects in soft elastohydrodynamic lubrication problems. *Tribol. Int.* **93**, 511–522. (doi:10.1016/j.triboint.2015.03.016)
10. Laursen TA. 2003 *Computational contact and impact mechanics*, 1st edn. Berlin, Germany: Springer. (Corr. 2nd printing.)
11. Kabacaoğlu G, Temizer İ. 2015 Homogenization of soft interfaces in time-dependent hydrodynamic lubrication. *Comput. Mech.* **56**, 421–441. (doi:10.1007/s00466-015-1179-5)
12. Belytschko T, Liu WK, Moran B. 2001 *Nonlinear finite elements for continua and structures*. London, UK: Wiley.
13. Aris R. 1989 *Vectors, tensors and the basic equations of fluid mechanics*. New York, NY: Dover.
14. Luo H, Bewley TR. 2004 On the contravariant form of the Navier-Stokes equations in time-dependent curvilinear coordinate systems. *J. Comput. Phys.* **199**, 355–375. (doi:10.1016/j.jcp.2004.02.012)
15. Scriven LE. 1960 Dynamics of a fluid interface. *Chem. Eng. Sci.* **12**, 98–108. (doi:10.1016/0009-2509(60)87003-0)
16. Rangamani P, Agrawal A, Mandadapu KK, Oster G, Steigmann DJ. 2013 Interaction between surface shape and intra-surface viscous flow on lipid membranes. *Biomech. Model. Mechanobiol.* **12**, 833–845. (doi:10.1007/s10237-012-0447-y)
17. Kreyszig E. 1991 *Differential geometry*. New York, NY: Dover.
18. Naghdi PM. 1972 The theory of shells and plates. In *Handbuch der Physik*, vol. VIa/2 (ed. S Flügge), pp. 425–640. Berlin, Germany: Springer.
19. Wriggers. P. 2006 *Computational contact mechanics*, 2nd edn. Berlin, Germany: Springer.
20. Tur M, Fuenmayor FJ, Wriggers P. 2009 A mortar-based frictional contact formulation for large deformations using Lagrange multipliers. *Comput. Methods Appl. Mech. Eng.* **198**, 2860–2873. (doi:10.1016/j.cma.2009.04.007)
21. Yang B, Laursen TA. 2009 A mortar-finite element approach to lubricated contact problems. *Comput. Methods Appl. Mech. Eng.* **198**, 3656–3669. (doi:10.1016/j.cma.2009.07.008)

22. Thomas TY. 1961 *Plastic flow and fracture in solids*. San Diego, CA: Academic Press.
23. Petryk H, Mróz Z. 1986 Time derivatives of integrals and functionals defined on varying volume and surface domains. *Arch. Mech.* **38**, 697–724.
24. He Q-C, Curnier A. 1993 Anisotropic dry friction between two orthotropic surfaces undergoing large displacements. *Eur. J. Mech. A Solids* **12**, 631–666.
25. Klarbring A. 1995 Large displacement frictional contact: a continuum framework for finite element discretization. *Eur. J. Mech. A Solids* **14**, 237–253.
26. Mallikamas W. 2008 A new improved depth-integrated flow equation for single rough-walled fractures, incorporating the roles of fracture mid-plane inclination. PhD thesis, University of Colorado, Boulder, CO, USA.
27. Mallikamas W, Rajaram H. 2010 An improved two-dimensional depth-integrated flow equation for rough-walled fractures. *Water Resour. Res.* **46**, W08506. (doi:10.1029/2009WR008779)