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Chebyshev Polynomials on Generalized Julia Sets

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Abstract Let $(f_n)_{n=1}^{\infty}$ be a sequence of non-linear polynomials satisfying some mild conditions. Furthermore, let $F_m(z) := (f_m \circ f_{m-1} \cdots \circ f_1)(z)$ and ρ_m be the leading coefficient of F_m. It is shown that on the Julia set $J_{(f_n)}$, the Chebyshev polynomial of degree $\deg F_m$ is of the form $F_m(z)/\rho_m - \tau_m$ for all $m \in \mathbb{N}$ where $\tau_m \in \mathbb{C}$. This generalizes the result obtained for autonomous Julia sets in Kamo and Borodin (Mosc. Univ. Math. Bull. 49:44–45, 1994).

Keywords Chebyshev polynomials · Extremal polynomials · Julia sets · Widom factors

Mathematics Subject Classification 37F10 · 41A50

1 Introduction

Let $(f_n)_{n=1}^{\infty}$ be a sequence of rational functions in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let us define the associated compositions by $F_m(z) := (f_m \circ \cdots f_1)(z)$ for each $m \in \mathbb{N}$. Then the set of points in $\overline{\mathbb{C}}$ for which $(F_n)_{n=1}^{\infty}$ is normal in the sense of Montel is called the *Fatou set* for $(f_n)_{n=1}^{\infty}$. The complement of the Fatou set is called the *Julia set* for $(f_n)_{n=1}^{\infty}$ and is denoted by $J_{(f_n)}$. The metric considered here is the chordal metric. Julia sets

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corresponding to a sequence of rational functions, to our knowledge, were considered first in [9]. Several papers that have appeared in the literature (see e.g. [3,6,8,18]) show the possibility of adapting the results on autonomous Julia sets to this more general setting with some minor changes. By an autonomous Julia set, we mean the set $J_{(f_n)}$ with $f_n(z) = f(z)$ for all $n \in \mathbb{N}$ where f is a rational function.

The Julia set $J_{(f_n)}$ is never empty provided that $\deg f_n \geq 2$ for all n. If, in addition, we assume that $f_n = f$ for all n then $f(J(f)) = f^{-1}(J(f)) = J(f)$ where $J(f) := J_{(f_n)}$. But without the last assumption, we only have $F_k^{-1}(F_k(J_{(f_n)})) = J_{(f_n)}$ and $J_{(f_n)} = F_k^{-1}(J_{(f_{k+n})})$ for all $k \in \mathbb{N}$ in general, where $(f_{k+n}) = (f_{k+1}, f_{k+2}, f_{k+3}, \ldots)$. That is the main reason why further techniques are needed in this framework.

Let $K \subset \mathbb{C}$ be a compact set with Card $K \geq m$ for some $m \in \mathbb{N}$. Recall that, for every $n \in \mathbb{N}$ with $n \leq m$, the unique monic polynomial P_n of degree n satisfying

$$\|P_n\|_K = \min\{\|Q_n\|_K : Q_n \text{ monic of degree } n\}$$

is called the nth *Chebyshev polynomial* on K where $\|\cdot\|_K$ is the sup-norm on K.

If f is a non-linear complex polynomial then $J(f) = \partial\{z \in \mathbb{C} : f^{(n)}(z) \to \infty\}$ and $J(f)$ is an infinite compact subset of $\mathbb{C}$ where $f^{(n)}$ is the nth iteration of f. The next result is due to Kamo and Borodin [12]:

Theorem 1 *Let $f(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0$ be a non-linear complex polynomial and $T_k(z)$ be a Chebyshev polynomial on $J(f)$. Then $(T_k \circ f^{(n)})(z)$ is also a Chebyshev polynomial on $J(f)$ for each $n \in \mathbb{N}$. In particular, this implies that there exists a complex number τ such that $f^{(n)}(z) - \tau$ is a Chebyshev polynomial on $J(f)$ for all $n \in \mathbb{N}$.*

In Sect. 2, we review some facts about generalized Julia sets and Chebyshev polynomials. In the last section, we present a result which can be seen as a generalization of Theorem 1. Polynomials considered in these sections are always non-linear complex polynomials unless stated otherwise. For a deeper discussion of Chebyshev polynomials, we refer the reader to [15,16,19]. For different aspects of the theory of Julia sets, see [2,4,13] among others.

2 Preliminaries

Autonomous polynomial Julia sets enjoy plenty of nice properties. These sets are non-polar compact sets which are regular with respect to the Dirichlet problem. Moreover, there are a couple of equivalent ways to describe these sets. For further details, see [13]. In order to have similar features for the generalized case, we need to put some restrictions on the given polynomials. The conditions used in the following definition are from [4, Sec. 4].

Definition 1 Let $f_n(z) = \sum_{j=0}^{d_n} a_{n,j} \cdot z^j$ where $d_n \geq 2$ and $a_{n,d_n} \neq 0$ for all $n \in \mathbb{N}$. We say that (f_n) is a *regular polynomial sequence* if the following properties are satisfied:

- There exists a real number $A_1 > 0$ such that $|a_{n,d_n}| \geq A_1$, for all $n \in \mathbb{N}$.
- There exists a real number $A_2 \geq 0$ such that $|a_{n,j}| \leq A_2|a_{n,d_n}|$ for $j = 0, 1, \ldots, d_n - 1$ and $n \in \mathbb{N}$.
- There exists a real number A_3 such that

$$\log|a_{n,d_n}| \leq A_3 \cdot d_n,$$

for all $n \in \mathbb{N}$.

If (f_n) is a regular polynomial sequence then we use the notation $(f_n) \in \mathcal{R}$. Here and in the rest of this paper, $F_l(z) := (f_l \circ \cdots \circ f_1)(z)$ and ρ_l is the leading coefficient of F_l. Let $\mathcal{A}_{(f_n)}(\infty) := \{z \in \overline{\mathbb{C}} : (F_n(z))_{n=1}^{\infty}$ goes locally uniformly to $\infty\}$ and $\mathcal{K}_{(f_n)} := \{z \in \mathbb{C}\colon (F_n(z))_{n=1}^{\infty}$ is bounded$\}$. In the next theorem, we list some facts that will be necessary for the subsequent results.

Theorem 2 [4,6] *Let $(f_n) \in \mathcal{R}$. Then the following hold:*

(a) *$J_{(f_n)}$ is a compact set in $\mathbb{C}$ with positive logarithmic capacity.*
(b) *For each $R > 1$ satisfying*

$$A_1 R\left(1 - \frac{A_2}{R-1}\right) > 2, \tag{1}$$

we have $\mathcal{A}_{(f_n)}(\infty) = \cup_{k=1}^{\infty} F_k{}^{-1}(\triangle_R)$ and $f_n(\overline{\triangle_R}) \subset \triangle_R$ where

$$\triangle_R = \{z \in \overline{\mathbb{C}}\colon |z| > R\}$$

Furthermore, $\mathcal{A}_{(f_n)}(\infty)$ is a domain in $\overline{\mathbb{C}}$ containing $\triangle_R$.
(c) *$\triangle_R \subset \overline{F_k^{-1}(\triangle_R)} \subset F_{k+1}^{-1}(\triangle_R) \subset \mathcal{A}_{(f_n)}(\infty)$ for all $k \in \mathbb{N}$ and each $R > 1$ satisfying* (1).
(d) *$\partial\mathcal{A}_{(f_n)}(\infty) = J_{(f_n)} = \partial\mathcal{K}_{(f_n)}$ and $\mathcal{K}_{(f_n)} = \overline{\mathbb{C}} \setminus \mathcal{A}_{(f_n)}(\infty)$. Thus, $\mathcal{K}_{(f_n)}$ is a compact subset of $\mathbb{C}$ and $J_{(f_n)}$ has no interior points.*

The next result is an immediate consequence of Theorem 2.

Proposition 1 *Let $(f_n) \in \mathcal{R}$. Then*

$$\lim_{k\to\infty}\left(\sup_{a\in\overline{\mathbb{C}}\setminus F_k^{-1}(\triangle_R)} \operatorname{dist}(a, \mathcal{K}_{(f_n)})\right) = 0,$$

where R be a real number satisfying (1).

Proof Using the part (c) of Theorem 2, we have $\overline{\mathbb{C}} \setminus F_{k+1}^{-1}(\triangle_R) \subset \overline{\mathbb{C}} \setminus F_k^{-1}(\triangle_R)$ which implies that

$$(a_k) := \left(\sup_{a\in\overline{\mathbb{C}}\setminus F_k^{-1}(\triangle_R)} \operatorname{dist}(a, \mathcal{K}_{(f_n)})\right)$$

is a decreasing sequence.

Suppose that $a_k \to \epsilon$ as $k \to \infty$ for some $\epsilon > 0$. Then, by compactness of the set $\overline{\mathbb{C}} \backslash F_k^{-1}(\triangle_R)$, there exists a number $b_k \in \overline{\mathbb{C}} \backslash F_k^{-1}(\triangle_R)$ for each k such that $\operatorname{dist}(b_k, \mathcal{K}_{(f_n)}) \geq \epsilon$. But since $\cap_{k=1}^{\infty} \overline{\mathbb{C}} \backslash F_k^{-1}(\triangle_R) = \mathcal{K}_{(f_n)}$ by parts (*b*) and (*d*) of Theorem 2, (b_k) should have an accumulation point b in $\mathcal{K}_{(f_n)}$ with $\operatorname{dist}(b, \mathcal{K}_{(f_n)}) > \epsilon/2$ which is clearly impossible. This completes the proof. □

For a compact set $K \subset \mathbb{C}$, the smallest closed disk $\overline{D(a, r)}$ containing K is called the *Chebyshev disk* for K. The center a of this disk is called the *Chebyshev center* of K. These concepts were crucial and widely used in the paper [14]. The next result which is vital for the proof of Lemma 1 is from [14]:

Theorem 3 *Let $L \subset \mathbb{C}$ be a compact set with* $\operatorname{card} L \geq 2$ *having the origin as its Chebyshev center. Let $L_p = p^{-1}(L)$ for some monic complex polynomial p with* $\deg p = n$. *Then p is the unique Chebyshev polynomial of degree n on L_p.*

3 Results

First, we begin with a lemma which is also interesting in its own right.

Lemma 1 *Let f and g be two non-constant complex polynomials and K be a compact subset of $\mathbb{C}$ with* $\operatorname{card} K \geq 2$. *Furthermore, let α be the leading coefficient of f. Then the following propositions hold.*

(a) *The Chebyshev polynomial of degree* $\deg f$ *on the set $(g \circ f)^{-1}(K)$ is of the form $f(z)/\alpha - \tau$ where $\tau \in \mathbb{C}$.*
(b) *If g is given as a linear combination of monomials of even degree and $K = \overline{D(0, R)}$ for some $R > 0$ then the* $\deg f$*th Chebyshev polynomial on $(g \circ f)^{-1}(K)$ is $f(z)/\alpha$.*

Proof Let $K_1 := g^{-1}(K)$. Then $(g \circ f)^{-1}(K) = f^{-1}(K_1) = (f/\alpha)^{-1}(K_1/\alpha)$ where $K_1/\alpha - \tau = \{z : z = z_1/\alpha - \tau$ for some $z_1 \in K_1\}$. By the fundamental theorem of algebra, $\operatorname{card}(K_1/\alpha) = \operatorname{card} K_1 \geq \operatorname{card} K$ and K_1 is compact by the continuity of $g(z)$. The set K_1/α is also compact since the compactness of a set is preserved under a linear transformation. Let τ be the Chebyshev center for K_1/α. Then $K_1/\alpha - \tau$ is a compact set with the Chebyshev center as the origin. Note that, $\operatorname{card}(K_1/\alpha - \tau) = \operatorname{card}(K_1/\alpha)$ and $(f/\alpha)^{-1}(K_1/\alpha) = (f/\alpha - \tau)^{-1}(K_1/\alpha - \tau)$. Using Theorem 3, for $p(z) = f(z)/\alpha - \tau$ and $L = K_1/\alpha - \tau$, we see that $p(z)$ is the $\deg f$th Chebyshev polynomial on $L_p = (g \circ f)^{-1}(K)$. This proves the first part of the lemma.

Suppose further that $g(z) = \sum_{j=0}^{n} a_j \cdot z^{2j}$ for some $n \geq 1$ and $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$ with $a_n \neq 0$. Let $K = \overline{D(0, R)}$ for some $R > 0$. Then the Chebyshev center for $K_1/\alpha = g^{-1}(K)/\alpha = g^{-1}(\overline{D(0, R)})/\alpha$ is the origin since $g(z)/\alpha = g(-z)/\alpha$ for all $z \in \mathbb{C}$. Thus, $f(z)/\alpha$ is the $\deg f$th Chebyshev polynomial for $(g \circ f)^{-1}(K)$ under these extra assumptions. □

The next theorem shows that it is possible to obtain similar results to Theorem 1 in a richer setting.

Theorem 4 *Let $(f_n) \in \mathcal{R}$. Then the following hold:*

(a) *For each* $m \in \mathbb{N}$*, the* $\deg F_m$*th Chebyshev polynomial on* $J_{(f_n)}$ *is of the form* $F_m(z)/\rho_m - \tau_m$ *where* $\tau_m \in \mathbb{C}$.

(b) *If, in addition, each* f_n *is given as a linear combination of monomials of even degree then* $F_m(z)/\rho_m$ *is the* $\deg F_m$*th Chebyshev polynomial on* $J_{(f_n)}$ *for all* m.

Proof Let $m \in \mathbb{N}$ be given and $R > 1$ satisfy (1). For each natural number $l > m$, define $g_l := f_l \circ \cdots \circ f_{m+1}$. Then $F_l = g_l \circ F_m$ for each such l. Using part (a) of Lemma 1 for $g = g_l$, $f = F_m$ and $K = \overline{D(0, R)}$, we see that the $(d_1 \cdots d_m)$th Chebyshev polynomial on $(g_l \circ F_m)^{-1}(\overline{D(0, R)})$ is of the form $F_m(z)/\rho_m - \tau_l$ where $\tau_l \in \mathbb{C}$. Let $C_l := \|F_m/\rho_m - \tau_l\|_{(g_l \circ F_m)^{-1}(K)}$. Note that, by part (c) of Theorem 2,

$$F_t^{-1}(\overline{D(0, R)}) \subset F_s^{-1}(\overline{D(0, R)}) \subset \overline{D(0, R)} \tag{2}$$

provided that $s < t$. This implies that $(C_j)_{j=m+1}^{\infty}$ is a decreasing sequence of positive numbers and hence has a limit C. The last follows from the observation that the norms of the Chebyshev polynomials of same degree on a decreasing sequence of compact sets constitute a decreasing sequence on $\mathbb{R}$.

Let $P_{d_1 \cdots d_m}(z) = \sum_{j=0}^{d_1 \cdots d_m} a_j z^j$ be the $(d_1 \cdots d_m)$th Chebyshev polynomial on $\mathcal{K}_{(f_n)}$. Since $\mathcal{K}_{(f_n)} \subset (g_l \circ F_m)^{-1}(\overline{D(0, R)})$ for each l, we have $C_0 := \|P_{d_1 \cdots d_m}\|_{\mathcal{K}_{(f_n)}} \leq C$. Suppose that $C_0 < C$.

Let $\epsilon = \min\{C - C_0, 1\}$. Using the compactness of $\overline{D(0, R)}$ let us choose a $\delta > 0$ such that for all $|z_1 - z_2| < \delta$ and $z_1, z_2 \in \overline{D(0, R)}$ we have

$$|P_{d_1 \cdots d_m}(z_1) - P_{d_1 \cdots d_m}(z_2)| < \frac{\epsilon}{2}$$

By Proposition 1, there exists a real number $N_0 > m$ such that $N > N_0$ with $N \in \mathbb{N}$ implies that

$$\sup_{z \in \overline{\mathbb{C}} \setminus F_N^{-1}(\triangle_R)} \operatorname{dist}(z, \mathcal{K}_{(f_n)}) < \delta.$$

Therefore, for any $z \in F_{N_0+1}^{-1}(\overline{D(0, R)})$, there exists a $z' \in \mathcal{K}_{(f_n)}$ with $|z - z'| < \delta$. Hence, for each $z \in F_{N_0+1}^{-1}(\overline{D(0, R)})$, we have

$$|P_{d_1 \cdots d_m}(z)| < |P_{d_1 \cdots d_m}(z')| + \frac{\epsilon}{2} < C \leq \left\| \frac{F_m}{\rho_m} - \tau_{N_0+1} \right\|_{F_{N_0+1}^{-1}(\overline{D(0,R)})},$$

where in the first inequality, we use $z, z' \in \overline{D(0, R)}$. This contradicts with the fact that $F_m(z)/\rho_m + \tau_{N_0+1}$ is the $(d_1 \cdots d_m)$th Chebyshev polynomial on $F_{N_0+1}^{-1}(\overline{D(0, R)})$. Thus, $C_0 = C$.

Using the triangle inequality in (4) and (5), the monotonicity of $(C_l)_{l=m+1}^{\infty}$ in (6) and (2) in (7), we have

$$|\tau_l| = \left\| -\frac{F_m}{\rho_m} + \frac{F_m}{\rho_m} - \tau_l \right\|_{F_l^{-1}(\overline{D(0,R)})} \tag{3}$$

$$\leq \left\| \frac{F_m}{\rho_m} - \tau_l \right\|_{F_l^{-1}(\overline{D(0,R)})} + \left\| \frac{F_m}{\rho_m} \right\|_{F_l^{-1}(\overline{D(0,R)})} \tag{4}$$

$$\leq C_l + |\tau_{m+1}| + \left\| \frac{F_m}{\rho_m} - \tau_{m+1} \right\|_{F_l^{-1}(\overline{D(0,R)})} \tag{5}$$

$$\leq C_{m+1} + |\tau_{m+1}| + \left\| \frac{F_m}{\rho_m} - \tau_{m+1} \right\|_{F_l^{-1}(\overline{D(0,R)})} \tag{6}$$

$$\leq 2C_{m+1} + |\tau_{m+1}|. \tag{7}$$

for $l \geq m+1$. This shows that $(\tau_l)_{l=m+1}^{\infty}$ is a bounded sequence. Thus, $(\tau_l)_{l=m+1}^{\infty}$ has at least one convergent subsequence $(\tau_{l_k})_{k=1}^{\infty}$ with a limit τ_m. Therefore,

$$C \leq \lim_{k\to\infty} \left\| \frac{F_m}{\rho_m} - \tau_m \right\|_{F_{l_k}^{-1}(\overline{D(0,R)})} \leq \lim_{k\to\infty} (C_{l_k} + |\tau_{l_k} - \tau_m|) = C. \tag{8}$$

By the uniqueness of Chebyshev polynomials and (8), $F_m(z)/\rho_m - \tau_m$ is the $(d_1 \cdots d_m)$th Chebyshev polynomial on $\mathcal{K}_{(f_n)}$. By the maximum principle, for any polynomial Q, we have

$$\|Q\|_{\mathcal{K}_{(f_n)}} = \|Q\|_{\partial\mathcal{K}_{(f_n)}} = \|Q\|_{J_{(f_n)}}.$$

Hence, the Chebyshev polynomials on $\mathcal{K}_{(f_n)}$ and $J_{(f_n)}$ should coincide. This proves the first assertion.

Suppose that the assumption given in part (b) is satisfied. Then by the part (b) of Lemma 1, for $g = g_l$, $f = F_m$ and $K = \overline{D(0, R)}$, the $(d_1 \cdots d_m)$th Chebyshev polynomial on $(g_l \circ F_m)^{-1}(\overline{D(0, R)})$ is of the form $F_m(z)/\rho_m - \tau_l$ where $\tau_l = 0$ for $l > m$. Thus, arguing as above, we can reach the conclusion that $F_m(z)/\rho_m$ is the $(d_1 \cdots d_m)$th Chebyshev polynomial for $J_{(f_n)}$ provided that the assumption in the part (b) holds. This completes the proof. □

This theorem gives the total description of 2^n degree Chebyshev polynomials for the most studied case, i.e., $f_n(z) = z^2 + c_n$ with $c_n \in \mathbb{C}$ for all n. If $(c_n)_{n=1}^{\infty}$ is bounded then the logarithmic capacity of $J_{(f_n)}$ is 1. Moreover, by [5], we know that if $|c_n| \leq 1/4$ for all n then $J_{(f_n)}$ is connected. If $|c_n| < c < 1/4$, then $J_{(f_n)}$ is a quasicircle and hence a Jordan curve. See [3], for the definition of a quasicircle and proof of the above fact.

For a non-polar compact set $K \subset \mathbb{C}$, let us define the sequence $(W_n(K))_{n=1}^{\infty}$ by $W_n(K) = \|P_n\|/(\mathrm{Cap}(K))^n$ for all $n \in \mathbb{N}$. There are recent studies on the asymptotic behavior of these sequences on several occasions. See e.g. [1,10,20].

In [1,20], sufficent conditions are given for $(W_n(K))_{n=1}^{\infty}$ to be bounded in terms of the smoothness of the outer boundary of K. There is also an old and open question (we consider this as an open problem since we could not find any concrete examples in the literature although in [17], Pommerenke says that "D. Wrase in Karlsruhe has shown that an example constructed by J. Clunie [Ann. of Math., 69 (1959), 511–519] for a different purpose has the required property.") proposed by Pommerenke [17] which is in the inverse direction: Find (if possible) a continuum K with $\mathrm{Cap}(K) = 1$ such that $(W_n(K))_{n=1}^{\infty}$ is unbounded. To answer this question positively, it is very natural to consider a continuum with a non-rectifiable outer boundary. Thus, we make the following conjecture:

Conjecture 1 *Let $f(z) = z^2 + 1/4$. Then, $(W_n(J(f))_{n=1}^{\infty}$ is unbounded.*

By [11, Thm. 1], for $f(z) = z^2 + 1/4$, $J(f)$ has Hausdorff dimension greater than 1 and in this case (see e.g. [7, p. 130]) $J(f)$ is not a quasicircle. Hence, [1, Thm. 2] is not applicable for $J(f)$ since it requires even stronger assumptions on the outer boundary.

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