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ABSTRACT

We study a war scenario in which the winner occupies the loser's territory. Attacking a territory increases the chance of winning, but also causes harm, which in turn decreases the territory's value (i.e. the reward of winning). This paper highlights the effects of this trade-off on the equilibrium strategies of the warring states in a contest game with endogenous rewards. Providing both static and dynamic models, our analysis captures insights regarding strategic behavior in asymmetric contests with such conflict.

Introduction

Armed confrontations or wars typically involve conflicts over the control or possession of particular pieces of territory between two states. The contending parties expend means of destruction to increase their probability of capturing the rival's territory. However, attacking a territory causes harm, thereby decreasing the value of the contested territory. This creates a trade-off since an increase in one's attack level increases one's winning probability, but decreases one's reward. This trade-off is clearly emphasized in the 'Art of War' by Sun Tzu as follows: 'In the practical art of war, the best thing of all is to take the enemy's country whole and intact; to shatter and destroy it is not so good'. This paper highlights the effects of this trade-off on the equilibrium strategies of the warring states in static and dynamic contest games with endogenous rewards.

Contests have been extensively utilized under the assumption of an exogenously fixed reward to model conflicts in various contexts such as sports, rent-seeking, patent races, economics of advertising, and political campaigns (for an excellent book on contests, see Konrad 2009). However, the endogenous nature of rewards depending on the efforts of contestants appears not to have been rigorously analyzed in the theoretical conflict literature. Among few exceptions, Alexeev and Leitzel (1996) considered static rent-seeking contests in which the reward of winning decreases in the aggregate effort of the other players; Chung (1996) studied positive externalities in rent-seeking contests by assuming that the reward of winning increases in the aggregate effort of all players; Shaffer (2006) analyzed the equilibrium behavior in a contest with two symmetric players, considering positive and negative externalities imposed by both players' efforts; and Hirai and Szidarovszky (2013) investigated the existence and uniqueness of Nash equilibrium after defining reward as any function of the aggregate effort of all players. Finally, Chowdhury and Sheremeta (2011) generalized such models by constructing a two-player Tullock contest where the payoff of each player is a linear function of rewards, own effort, and the rival's effort.

In this paper, we assume that each player is endowed with a territory and study a two-player Tullock contest in which the winner occupies the loser's territory. A novel feature of our analysis is the assumption that the value of the territory player i aims to occupy decreases in player i's own effort. In that regard, we contribute to the literature on contests with endogenous rewards. Another novelty lies

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in the incorporation of the possibility of a draw. To the best of our knowledge, although the first referral to draw in contest models was by Nalebuff and Stiglitz (1983), this possibility has been disregarded in the existing literature on contests with endogenous rewards. We fill this gap by assuming that if the contest results in a draw, both players keep their own territories, which implies that although an increase in a player's effort decreases his/her own payoff from winning, it also decreases the rival's payoff in case of a draw. This makes a remarkable impact on the equilibrium efforts.[1] Furthermore, we utilize the possibility of a draw in extending our analysis to a dynamic contest.

First, we analyze the static version of this armed conflict in an asymmetric contest game. This allows us to investigate what implications the differences in the initial territory values and the vulnerability to threat may have on a player's choice of contest effort. We find the unique Nash equilibrium of the model and provide comparative statics analyses. Next, we extend our analysis to a dynamic contest with two periods. If there is a winner in the first period, then the contest ends with the winner occupying the loser's territory. If the first period results in a draw, however, then the contest proceeds to the second period. Assuming that the losses in territory values accumulate, we analyze subgame perfect Nash equilibrium of this model in several numerical examples and illustrate how the prospect of proceeding to the second period might affect the equilibrium behavior.

Our paper is related to the game-theoretic analysis of territorial disputes. Chang, Potter, and Sanders (2007) characterized the outcome of a territorial dispute between two rival parties under the assumption that engaging in conflict destructs a fixed ratio of the resources. Later, Chang and Luo (2013) endogenized the destruction assuming that the resources decrease in the amount of guns used in warfare. Both of these papers analyzed the conditions under which war or settlement arises between the contending parties. In this paper, although we only concentrate on the case of conflict, our analysis allows us to derive conditions characterizing when players prefer to avoid war.[2]

The paper is organized as follows. In the second section, we present the static contest game, characterize its unique Nash equilibrium, and provide comparative statics analyses for the equilibrium strategies. We also extend our analysis to a dynamic contest game. The last section concludes.

The Model

There are two players $i = 1, 2$. Each player i is endowed with a territory which has a certain value, denoted by $V_i > 0$. They are in a war, which is modeled as a two-player contest game. Each player i has the strategy set $[0, B_i]$, such that the upper bound corresponds to the highest destruction power possible with the weapons/bombs/etc. player i already owns (for which the cost is fixed). The players simultaneously choose how much destruction power to use in this war. The winner is determined by a Tullock contest success function. In particular, player i wins the contest with probability

$$p_i(b_1, b_2) = \frac{b_i}{b_i + b_j + d}$$

where b_i is the destruction power used by player i and $d > 0$ represents the case of a draw. Accordingly, the probability of a draw is[3,4]

$$p_d(b_1, b_2) = \frac{d}{b_i + b_j + d}.$$

The winner occupies the opposition's territory, so that the winner has two territories and the loser has nothing at the end of the war. Finally, if the contest results in a draw, both parties keep their own territories.

In this paper, we assume that both territories are vulnerable to incoming attacks, meaning that their values decrease in the incoming attack level. Therefore, the value of player i's territory is actually a *decreasing* function $\mathcal{V}_i : [0, B_j] \rightarrow (0, V_i]$ satisfying $\mathcal{V}_i(0) = V_i$. This assumption creates a trade-off since an increase in one's attack level increases one's winning probability, but decreases one's reward.

In this model, the expected utility function to be maximized by player i is

$$\frac{b_i}{b_i + b_j + d}(\mathcal{V}_i(b_j) + \mathcal{V}_j(b_i)) + \frac{d}{b_i + b_j + d}\mathcal{V}_i(b_j)$$

given that player j used b_j amount of destruction power. Then, the first-order condition[5] with respect to b_i can be written as

$$\frac{b_j + d}{(b_i + b_j + d)^2}(\mathcal{V}_i(b_j) + \mathcal{V}_j(b_i)) + \frac{b_i}{b_i + b_j + d}\frac{\partial \mathcal{V}_j(b_i)}{\partial b_i} - \frac{d}{b_i + b_j + d}\mathcal{V}_i(b_j) = 0.$$

From this point onward, in order to have more concrete ideas about the outcome of this contest, we consider a specific functional form for the value of each territory. For every $i = 1, 2$, assume that

$$\mathcal{V}_i(b_j) = V_i - \alpha_i b_j \tag{1}$$

such that $V_i > \alpha_i B_j > 0$ where α_i is the territory's vulnerability to incoming attacks.

Given this territory value function, we find the following best response function for player 1[6]:

$$\mathcal{BR}_1(b_2) = -(b_2 + d) + \sqrt{(b_2 + d)^2 + \frac{(b_2 + d)V_2 + b_2 V_1 - \alpha_1 b_2^2}{\alpha_2}}.$$

Player 2's best response function is symmetric:

$$\mathcal{BR}_2(b_1) = -(b_1 + d) + \sqrt{(b_1 + d)^2 + \frac{(b_1 + d)V_1 + b_1 V_2 - \alpha_2 b_1^2}{\alpha_1}}.$$

We can see that $\mathcal{BR}_i(b_j) > 0$ for every $i = 1, 2$ and every $b_j \in [0, B_j]$. This means that each player always prefers to *actively* participate in this war. The intersection of these best response functions $(\bar{b}_1, \bar{b}_2)$ is unique on the positive domain:

$$\bar{b}_1 = \frac{\frac{(V_1 + V_2)^2}{2} + 2\alpha_1 d V_2}{\alpha_2(V_1 + V_2) + 2\alpha_1\alpha_2 d + \sqrt{\alpha_1\alpha_2[(V_1 + V_2)^2 + 4d(\alpha_1 V_2 + \alpha_2 V_1) + 4\alpha_1\alpha_2 d^2]}}$$

$$\bar{b}_2 = \frac{\frac{(V_1 + V_2)^2}{2} + 2\alpha_2 d V_1}{\alpha_1(V_1 + V_2) + 2\alpha_1\alpha_2 d + \sqrt{\alpha_1\alpha_2[(V_1 + V_2)^2 + 4d(\alpha_1 V_2 + \alpha_2 V_1) + 4\alpha_1\alpha_2 d^2]}} \tag{2}$$

It is also worth emphasizing that $\mathcal{BR}_i(b_j) > B_i$ is possible for some $i = 1, 2$ and some $b_j \in [0, B_j]$. In such cases, it might turn out that the intersection of these best response functions lies outside the set of strategy profiles. This would push players to the boundaries of their strategy sets in the equilibrium. Using this information, the following proposition characterizes the unique Nash equilibrium of the model.

Proposition 1: *In the above-described two-player territory occupation game, the unique Nash equilibrium can be characterized by:*

$$\begin{aligned}(b_1^*, b_2^*) &= (\bar{b}_1, \bar{b}_2) && \text{if } \bar{b}_i \leq B_i \text{ for every } i \in \{1, 2\} \\ &= (B_1, \mathcal{BR}_2(B_1)) && \text{if } \bar{b}_1 > B_1 \text{ and } \mathcal{BR}_2(B_1) \leq B_2 \\ &= (\mathcal{BR}_1(B_2), B_2) && \text{if } \bar{b}_2 > B_2 \text{ and } \mathcal{BR}_1(B_2) \leq B_1 \\ &= (B_1, B_2) && \text{if otherwise.}\end{aligned}$$

Now that we have characterized the unique Nash equilibrium of this game, we turn to comparative statics analyses. First, we focus on the best response function: $\mathcal{BR}_i$. This best response function

increases in V_1 and V_2, but decreases in α_1 and α_2. Thus, both players become more aggressive when either of the territories' starting value increases since the reward of winning would increase, meaning that they would have more room to destruct.[7] Moreover, both players become less aggressive when either of the territories becomes more vulnerable to attack since using the same amount of destruction power would lead to more destruction.[8] Finally, the effect of an increase in d on best responses is mixed. In particular, $\mathcal{BR}_i$ increases in d when $V_j > 2\sqrt{\alpha_j b_j(V_i - \alpha_i b_j)}$, which is possible for sufficiently high values of α_i and V_j or sufficiently low values of α_j and V_i.[9] The interpretation is as follows. If d increases, then the effect of b_i on player i's winning probability diminishes. This might discourage player i, leading to a decrease in his/her best response to a given b_j. On the other hand, such a diminishing effect in the probability of winning also decreases the attacking cost for player i since the cost is incurred only when player i wins the contest. From this perspective, player i's best response to a given b_j might increase in d. Hence, the overall effect is mixed.

From this point onward, we concentrate on the interior equilibrium: $(\bar{b}_1, \bar{b}_2)$. First, the equilibrium strategy $\bar{b}_i$ decreases in α_j, so that trying to occupy a more vulnerable territory indicates a lower equilibrium effort. Unfortunately, comparative statics with respect to the other model parameters are not straightforward. Yet, for most cases, we numerically see that $\bar{b}_i$ decreases in α_i, but increases in V_1 and V_2.[10]

As an advantage of studying an asymmetric model, we can additionally write:

(i) When $\alpha_1 = \alpha_2$, if player i starts with a more valuable territory than player j does (i.e. $V_i > V_j$), then player i uses less destruction power in the equilibrium than player j does.
(ii) When $V_1 = V_2$, if player i has a more vulnerable territory than player j has (i.e. $\alpha_i > \alpha_j$), then player i uses more destruction power in the equilibrium than player j does.

Below we present two special cases of our model. These special cases have better looking interior equilibria, thereby allowing us to provide a full characterization of comparative statics.

Remark 1: If players are symmetric, i.e. if $V_1 = V_2 = V$ and $\alpha_1 = \alpha_2 = \alpha$, then the interior equilibrium would reduce to:

$$\left(\frac{V}{2\alpha}, \frac{V}{2\alpha} \right).$$

In this case, the equilibrium strategies are both increasing in V and decreasing in α. Moreover, the equilibrium strategies do not depend on d.

Remark 2: When there is no possibility of a draw, i.e. when $d = 0$, the interior equilibrium would reduce to:

$$\left(\frac{V_1 + V_2}{2(\alpha_2 + \sqrt{\alpha_1 \alpha_2})}, \frac{V_1 + V_2}{2(\alpha_1 + \sqrt{\alpha_1 \alpha_2})} \right).$$

In this case, the equilibrium strategies are both increasing in V_1 and V_2, whereas they are both decreasing in α_1 and α_2. Interestingly, the difference between starting territory values has no effect in the equilibrium strategies.

Finally, we can relate our findings to the literature on the cost of conflict analyzing the conditions under which the contending parties choose to avoid war. In our model, the expected utilities at the unique Nash equilibrium can be compared with the pre-war utility V_i, so that players do not engage in warfare only if the latter is no less than the former for both players. Along similar lines, if a binding side payment agreement is possible, a war can be avoided if the total expected utility at the unique Nash equilibrium is less than $V_1 + V_2$.

An Extension: The Dynamic Version

Here, we consider a dynamic contest with two periods. If there is a winner in the first period, then the contest ends with the winner occupying the loser's territory. If the first period results in a draw, then players do not receive any payoff and the contest proceeds to the second period. The subgame in this period is similar to the static version with two important differences: (i) losses in territory values

accumulate and (ii) the strategy set of player i in the second period is restricted to $[0, B_i - b_i^1]$ where b_i^1 denotes the destruction power used by player i in the first period.

We aim to analyze the subgame perfect Nash equilibrium of this model. Assume that (b_1^1, b_2^1) is played in the first period. Observing this outcome, in the second period, each player i tries to maximize the following expected utility function:

$$\frac{b_i^2}{b_i^2 + b_j^2 + d}[\mathcal{V}_i(b_j^1 + b_j^2) + \mathcal{V}_j(b_i^1 + b_i^2)] + \frac{d}{b_i^2 + b_j^2 + d}\mathcal{V}_i(b_j^1 + b_j^2).$$

where b_i^t denotes the destruction power used by player i in period $t \in \{1, 2\}$. Taking the first-order condition and solving the corresponding system of equations yield the solution $(\bar{b}_1^2, \bar{b}_2^2)$ which is already given in Equation (2) with an important difference: instead of V_i in Equation (2), we now have $V_i - \alpha_i b_i^1$ since this is the territory's value at the beginning of period 2. Obviously, if one of the players does not possess such a destruction power in the second period, i.e. if $\bar{b}_i^2 > B_i - b_i^1$ for some $i = 1, 2$, then the boundary conditions should be utilized as described in the equilibrium analysis of the static version. The unique Nash equilibrium (b_1^{2*}, b_2^{2*}) of this subgame can accordingly be formulated.

This equilibrium yields the following expected utility function for player i:

$$EU_i^*(\cdot) = \frac{b_i^{2*}}{b_i^{2*} + b_j^{2*} + d}[\mathcal{V}_i(b_j^1 + b_j^{2*}) + \mathcal{V}_j(b_i^1 + b_i^{2*})] + \frac{d}{b_i^{2*} + b_j^{2*} + d}\mathcal{V}_i(b_j^1 + b_j^{2*}).$$

Anticipating this outcome, each player i maximizes the following expected utility function in period 1:

$$\frac{b_i^1}{b_i^1 + b_j^1 + d}(\mathcal{V}_i(b_j^1) + \mathcal{V}_j(b_i^1)) + \frac{d}{b_i^1 + b_j^1 + d}EU_i^*(\cdot).$$

As it turns out, this dynamic model is too complicated to be solved analytically. Accordingly, we provide a detailed numerical example through which we can understand how proceeding to the second period after a draw might affect equilibrium behavior (see Case 3 in Table 1). We further analyze four variants of this numerical example and summarize all these results in Table 1.

Consider that $V_1 = 10$, $V_2 = 16$, $\alpha_1 = 2$, $\alpha_2 = 4$, and $d = 1$. First, in the static model, the unique Nash equilibrium is $(1.921, 2.658)$. The corresponding winning probabilities are 0.344 and 0.476, respectively. As for the dynamic model, in the second period, the unique equilibrium indicates the play of

$$b_1^{2*} = \frac{1}{4}\left(30 - 4b_1^1 - 2b_2^1 - \sqrt{2}\sqrt{249 - 60b_1^1 - 34b_2^1 + (2b_1^1 + b_2^1)^2}\right)$$

$$b_2^{2*} = \frac{1}{2}\left(2b_1^1 + b_2^1 - 17 + \sqrt{2}\sqrt{249 - 60b_1^1 - 34b_2^1 + (2b_1^1 + b_2^1)^2}\right).$$

Anticipating these strategies, in the first period, players choose $(1.815, 2.668)$. This reduces the territory values to 4.664 and 8.74 at the beginning of the second period, respectively. Moreover, it yields the following equilibrium strategies in the second period: $(1.015, 1.322)$. Note that the corresponding winning probabilities are 0.331 and 0.487 in the first period, whereas 0.304 and 0.396 in the second period.

In the following, an *underdog/top dog* is defined to be the player who uses less/more destruction power in the interior equilibrium of the static version in comparison to the other player. In the numerical example above, we see that the underdog's destruction power used in the first period might be less than the destruction power he/she would use in the static version, while the top dog is using a destruction power which is more than he/she would use in the static version. This favors the top dog in

Table 1. Equilibrium Predictions when $d = 1$.

		Static		Dynamic			
		b_1^*	b_2^*	b_1^{1*}	b_2^{1*}	b_1^{2*}	b_2^{2*}
Case 1	$V_1 = 10, V_2 = 10$ $\alpha_1 = 2, \alpha_2 = 2$	2.500	2.500	2.429	2.429	1.285	1.285
Case 2	$V_1 = 10, V_2 = 16$ $\alpha_1 = 2, \alpha_2 = 2$	3.350	3.150	3.162	3.180	1.839	1.490
Case 3	$V_1 = 10, V_2 = 16$ $\alpha_1 = 2, \alpha_2 = 4$	1.921	2.658	1.815	2.668	1.015	1.322
Case 4	$V_1 = 16, V_2 = 16$ $\alpha_1 = 2, \alpha_2 = 4$	2.292	3.416	2.260	3.279	1.117	1.866
Case 5	$V_1 = 16, V_2 = 16$ $\alpha_1 = 4, \alpha_2 = 4$	2.000	2.000	1.937	1.937	1.031	1.031

the first period in terms of the ratio of winning probabilities; however, if the second period is reached, then the underdog becomes more advantaged.

Interestingly, we see the converse of this situation in Case 2. In comparison to the respective equilibrium strategies in the static version, the top dog uses less destruction power and the underdog uses more destruction power in the first period. As a matter of fact, these changes are such that the top dog starts to use less destruction power than the underdog does. However, if the second period is reached, then the top dog makes a remarkable return. To put it differently, the top dog saves his/her power to become very dominant in the second period.

Case 4 is an example in which both players' equilibrium destruction powers used in the first period are lower than they would use in the static model. Afterward, having a more vulnerable territory becomes an important advantage for player 2 in the second period, as the ratio of winning probabilities significantly changes in favor of player 2.

In the remaining cases, we consider symmetric values of parameters. It seems that, as in Case 4, the prospect of proceeding to the second period after a draw leads both players to use less destruction power in the first period. Since the symmetric nature of the game is carried over to the next period, players equally decrease their destruction powers in the second period. As a consequence, in contrast to Case 4, no player has an advantage in the second period as the ratio of winning probabilities remains to be $1/2$.

Conclusion

In this paper, we have assumed that each player is endowed with a territory and studied a war scenario in which the winner occupies the loser's territory. The novel feature of our analysis lies in the assumptions that (i) the value of the territory player i aims to occupy decreases in player i's own effort and (ii) the contest may result in a draw, meaning that each player gets to keep his/her own territory. We have considered both static and dynamic versions of the model. In the static version, we have characterized the unique Nash equilibrium. Then assuming that the game reaches the second period only when the first period results in a draw, we have numerically analyzed several cases in the dynamic version. Our analysis captures insights regarding strategic behavior in asymmetric contests with endogenous rewards.

Notes

1. Our model captures the case of no draw as a special case. As it is shown in a remark, the equilibrium strategies are much simpler if the war cannot result in a draw.
2. In that regard, this paper is also related to the literature on the cost of conflict. Noting that engaging in conflict often imposes costs (even when the destruction costs are not considered), this literature investigates the escalation

of conflict between the contending parties. The interested reader is referred to Kimbrough and Sheremeta (2013) for a model without destruction and to Smith et al. (2014) for a model with destruction.

3. For an axiomatization of this contest success function, see Blavatskyy (2010).
4. We choose not to impose any upper bound for d. However, it is worth noting that if d is greater than B_i for every $i = 1, 2$, then the winning probabilities p_1 and p_2 can never be greater than the probability of a draw. Accordingly, one can argue that when d is too high, the contest is not so competitive.
5. Under our assumptions, the second derivative of the expected utility function is always negative. Therefore, the first-order condition is sufficient for the best response analysis.
6. The reader is referred to the Appendix for a detailed analysis.
7. Keeping everything else constant, if V_i increases, the reward of winning would not increase for player i, but losing the contest would be more costly.
8. Keeping everything else constant, if α_i increases, the reward of winning would not decrease for player i, but losing the contest would be less costly.
9. If $\alpha_i V_j^2 > \alpha_j V_i^2$, then $\mathcal{BR}_i$ always increases in d. On top of that, if this inequality holds, $\mathcal{BR}_j$ decreases in d for some intermediate values of b_i.
10. When α_i increases, knowing that the rival's equilibrium effort decreases, player i's equilibrium effort also decreases. On the other hand, when V_1 or V_2 increases, since the reward of winning and/or the cost of losing would increase, we observe an increase in destruction powers used at the equilibrium.

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Disclosure Statement

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Appendix

This appendix explains how to find the best response functions reported above. Recall that if player j uses b_j amount of destruction power, player i maximizes

$$\frac{b_i}{b_i + b_j + d}(\mathcal{V}_i(b_j) + \mathcal{V}_j(b_i)) + \frac{d}{b_i + b_j + d}\mathcal{V}_i(b_j).$$

Given the value function $\mathcal{V}_i(b_j) = V_i - \alpha_i b_j$, the first-order condition with respect to b_i is

$$\frac{b_j + d}{(b_i + b_j + d)^2}(V_i - \alpha_i b_j + V_j - \alpha_j b_i) - \frac{\alpha_j b_i}{b_i + b_j + d} - \frac{d}{(b_i + b_j + d)^2}(V_i - \alpha_i b_j) = 0.$$

For player 1, this first-order condition becomes

$$b_2(V_1 + V_2 - \alpha_1 b_2 - \alpha_2 b_1) - \alpha_2 b_1(b_1 + b_2 + d) + d(V_2 - \alpha_2 b_1) = 0$$

which can also be written as

$$\alpha_2 \cdot b_1^2 + 2\alpha_2(b_2 + d) \cdot b_1 + \alpha_1 b_2^2 - (b_2 + d)V_2 - b_2 V_1 = 0.$$

Solving for b_1 yields the following best response function for player 1:

$$\mathcal{BR}_1(b_2) = -(b_2 + d) + \sqrt{(b_2 + d)^2 + \frac{(b_2 + d)V_2 + b_2 V_1 - \alpha_1 b_2^2}{\alpha_2}}.$$

Considering the symmetric first-order condition with respect to b_2 derived from player 2's problem, the best response function for player 2 is:

$$\mathcal{BR}_2(b_1) = -(b_1 + d) + \sqrt{(b_1 + d)^2 + \frac{(b_1 + d)V_1 + b_1 V_2 - \alpha_2 b_1^2}{\alpha_1}}.$$