J Comb Optim (2018) 36:845–860
https://doi.org/10.1007/s10878-017-0177-2

Robust trading mechanisms over 0/1 polytopes

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Published online: 22 September 2017

Abstract The problem of designing a trade mechanism (for an indivisible good) between a seller and a buyer is studied in the setting of discrete valuations of both parties using tools of finite-dimensional optimization. A robust trade design is defined as one which allows both traders a dominant strategy implementation independent of other traders' valuations with participation incentive and no intermediary (i.e., under budget balance). The design problem which is initially formulated as a mixed-integer non-linear non-convex feasibility problem is transformed into a linear integer feasibility problem by duality arguments, and its explicit solution corresponding to posted price optimal mechanisms is derived along with full characterization of the convex hull of integer solutions. A further robustness concept is then introduced for a central planner unsure about the buyer or seller valuation distribution, a corresponding worst-case design problem over a set of possible distributions is formulated as an integer linear programming problem, and a polynomial solution procedure is given. When budget balance requirement is relaxed to feasibility only, i.e., when one allows an intermediary maximizing the expected surplus from trade, a characterization of the optimal robust trade as the solution of a simple linear program is given. A modified VCG mechanism turns out to be optimal.

Keywords Mechanism design · Bilateral trade · Robustness · Integer programming · Duality · VCG mechanism

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Thanks are due to two anonymous reviewers for their comments that helped improve the paper.

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1 Introduction and background

The trade of a good between a seller and a buyer is a ubiquitous situation in several economic contexts. The way in which this trade is carried out is a fundamental object of study in microeconomic theory. However, it is a question which received little attention from the optimization community. Departing from the assumption of continuous type space in Hagerty and Rogerson (1987) we consider the problem of designing a robust trading mechanism between a single buyer and a single seller where valuations (or types) of both parties are private information (we have a discrete type space) and take values in finite sets according to some given probability mass functions (referred to as a "prior") using tools of finite-dimensional optimization. Robustness in this context is understood as a dominant strategy implementation where each trader has an optimal strategy independent of others'. Besides dominant strategy implementation, the robust mechanism should fulfil ex-post individual rationality (no trader is worse off by participating) and budget balance (i.e., the seller gets exactly what the buyer pays) for both parties. The problem of robust trade design is initially formulated here as a mixed-integer non-linear and non-convex feasibility problem. While the constraints causing non-linearity can be ignored initially to be addressed later (see proof of Theorem 2), a more interesting avenue is opened via linear programming/network optimization duality. The initial formulation is shown to be equivalent to an integer linear programming formulation of which the linear relaxation polytope has all extreme points integral, and constructed explicitly. The number of extreme points is equal to the cardinality m of the discrete type space (assumed to be common for simplicity) plus one, which implies that the polytope is an m-simplex (m is the number of valuations as well as the largest valuation in our context). Hence, we obtain an explicit representation of the convex hull. This result immediately leads to the fact that all extreme point solutions of the linear relaxation correspond to posted price mechanisms. A posted price mechanism is one in which a price is announced in advance, and trade occurs if the valuation of the seller is smaller or equal to the posted price and the valuation of the buyer is greater or equal to the posted price and both agree to trade at that price.

For the case where a central planner is involved in the design process, and he/she may want to maximize the expected proceeds from the trade but is unsure about either the probability mass of the buyer or of the seller, we specify an uncertainty set for the unknown probability mass which constrains the mean valuation to a fixed value. Then we obtain a mixed-integer programming formulation for the maximization of worst-case expected proceeds from trade. The problem is solved by a simple procedure using the extreme point characterization and graphical linear programming on the plane. The extreme point solutions are still posted price mechanisms. However, it is shown, as expected, that the specification of the mean valuation affects the optimal posted price in the foreseeable direction, e.g., for increasing mean valuation of the seller, the posted price advocated by the optimal mechanism also increases.

In a continuous type space setting, Hagerty and Rogerson (1987) studied the problem of designing a robust trade mechanism for a seller and a buyer where each trader is entitled to a dominant strategy implementation. Their approach has the benefit of rendering the optimal mechanism independent of any common prior assumption on the valuations of other parties involved, hence the term "robust". Formally, this class of robust mechanisms is obtained by enforcing the so-called dominant strategy incentive compatibility (DIC), ex-post individual rationality (EIR) and budget balance requirements on both parties. They establish that posted price mechanisms are essentially the only mechanisms such that the traders have a dominant strategy implementation. Their proofs use the classical tools of calculus as introduced by Myerson and Satterthwaite (1983).

Our contribution, with respect to previous work and in particular (Hagerty and Rogerson 1987), is threefold. First, we shall place the trade design problem into the realm of mixed-integer (non-linear and non-convex) feasibility (or, optimization) problems in the spirit of Vohra (2011, 2012) where a study of mechanism design problems was undertaken using the tools of finite-dimensional optimization, and especially linear programming. The problem is then equivalently posed as a linear integer optimization problem, and analyzed using the tools of finite dimensional optimization. Second, we introduce an added robustness concept, and give a simple polynomial procedure to solve the associated design problem. Third, we extend our result to the case where budget balance is relaxed to feasibility, i.e., an intermediary maximizing the expected surplus from trade acts as the designer of the mechanism under the constraints of dominant strategy incentive compatibility and ex-post individual rationality. Schwartz and Wen (2012) showed that this relaxation may increase the expected gains from trade, and posed the question of characterization of such robust trade mechanisms. We give a simple linear programming characterization of such mechanisms. In a yet unpublished reference by Kos and Manea (2009) bilateral trade with discrete values is studied to characterize the necessary and sufficient conditions on the distribution of valuations for the existence of a budget balanced, efficient mechanism. A VCG-like mechanism (VCG') is also proposed as a solution to the problem, and conditions guaranteeing its validity are established. In connection to Sect. 5 of the present paper where an intermediary is present, it is shown in Kos and Manea (2009) that efficient, budget balanced, incentive compatible and individually rational trade is possible if and only if the broker running the VCG' mechanism makes a non-negative expected profit. Our result in Sect. 5 also yields optimal mechanisms which are of VCG' type.

A readable summary of bilateral trade mechanism design can be found in the monograph by Spulber (1999). A more general reference is Laffont and Mortimer (2002). More recent and relevant literature includes Čopič and Ponsati (2016) where it is proved that optimal robust bilateral trade mechanisms are payoff equivalent to non-wasteful randomized posted prices, and Drexl and Kleiner (2015) where the focus is on strategy proof mechanisms maximizing the agents' residual surplus in an independent private value auction environment. It is shown that optimal mechanisms are of a posted price nature under a monotonicity condition of the hazard rate of type distributions. In a bilateral trade environment, optimality of posted price is shown without any monotonicity assumption. It appears that posted price mechanisms are a recurrent theme in bilateral trade. In this context, the present paper may be seen as yet another justification for the pervasive nature of posted price mechanisms.

In Sect. 4 of the present paper we investigate the impact of imprecision in the type distribution (referred to as ambiguity) of the seller (or, the buyer) on the optimization problem of a central planner who is averse to ambiguity. This section was inspired by both the progress in robust optimization, a research effort initiated by Ben-Tal and

Nemirovski (1998, 1999), and robust mechanism design research that started with a paper by Bergemann and Morris (2005). The research of Ben-Tal and Nemirovski spawned a new field of research in optimization. The idea is to treat imprecise parameters of an optimization problem by means of a worst-case optimization with respect to an uncertainty set for the imprecise parameters for which the robust solution assumes full responsability. On the other hand, Bergemann and Morris relaxed the common knowledge assumptions pervasive in mechanism design in a similar worst-case framework in the spirit of Knightian uncertainty (Gilboa and Schmeidler 1989). An in-depth review can be found in Bergemann and Morris (2013). Relevant references in mechanism design include Bergemann and Schlag (2008, 2011) where the problem of a max–min utility seller with imperfect information about the valuation distribution of the buyer is studied, and optimal pricing schemes are investigated as well as Auster (2013) where a privately informed seller is assumed, Bandi and Bertsimas (2014) where a robust optimization approach is applied in the context of auction design, and Wolitzky (2016) where the buyer and the seller know only the mean of each other's valuations. We cite Bodoh-Creed (2012), Boze et al. (2006), Dong (2004) and Lopomo et al. (2009) for other related research where more complex settings were used. Lopomo et al. (2009) consider mechanism design problems with Knightian uncertainty formalized using incomplete preferences. In a related study Bodoh-Creed (2012) develops a payoff equivalence theorem for mechanisms with ambiguity averse participants in the sense of Gilboa and Schmeidler (1989). Dong studies the impact of ambiguity aversion of bidders on their bidding strategies in four classical auction mechanisms. Boze et al. (2006) reports on the optimal auction problem allowing for ambiguity about the distribution of valuations

The rest of the paper is organized as follows. In Sect. 2 we introduce the notation and the initial formulation. In Sect. 3, we transform the problem and study the structure of the resulting transformed problem. We apply the results of Sect. 3 to a further robust trade design problem in Sect. 4 in the presence of a central planner averse to ambiguity in type distribution. In Sect. 5, the budget balance requirement is relaxed into feasibility in the presence of a surplus maximizing intermediary, and the robust trades are characterized as the solution of a linear programming problem.

2 Initial formulation

We follow Hagerty and Rogerson (1987) for the notation. Individual 1 owns a good that he/she wishes to sell while individual 2 is interested in acquiring the good. Both parties attach a value to the good as one of the integers $T \equiv \{1, \ldots, m\}$,[1] independently. We shall refer to the valuation of the seller by i, and the valuation of the buyer as j. An *allocation rule* is a pair of discrete functions (p, x) mapping the pair of integers from $T \times T$ to appropriate values. Namely, $p_{ij} \in [0, 1]$ is the probability that trade occurs if valuation of seller is equal to i, and the valuation of the buyer is equal to j. Similarly,

[1] This is only for simplicity; one can define different (for each participant) and more general finite type sets. However, this only complicates the notation, and offers no added insight.

x_{ij} is the expected payment (which can be zero) from the buyer to the seller if valuation of seller is equal to i, and the valuation of the seller is equal to j.

We shall seek allocation rules where each trader has a dominant strategy implementation as detailed further below. By the Revelation Principle (see e.g., Myerson and Satterthwaite 1983) it is sufficient to consider mechanisms where traders truthfully report their type or choose not to participate.

A *trading mechanism* is defined as a triplet of functions (p, g, x) giving the probability of trade and the price at which trade occurs if it occurs at all. Let r denote the price at which trade occurs if it occurs and $g(r; i, j)$ denote the probability of r conditional on the valuations equal to i and j, respectively. The expected payment x_{ij} conditional on valuations i and j is determined by the relation:

$$x_{ij} = p_{ij} \sum_{r \in \{i,\ldots,j\}} rg(r; i, j), \quad \forall i, j \in T. \tag{1}$$

Since for given p and g, x can be inferred from the above we henceforth refer to as a mechanism using the couple (p, g). The utility to individual 1 (the seller) conditional on valuations i and j is denoted $U_1(i, j)$ and given by

$$U_1(i, j) = x_{ij} - ip_{ij}, \tag{2}$$

whereas the utility for individual 2 (the buyer), $U_2(i, j)$ is defined as

$$U_2(i, j) = jp_{ij} - x_{ij}. \tag{3}$$

Now, we can define the DIC and EIR requirements. A mechanism satisfies DIC if for every $i, j \in T$ and $k, \ell \in T$ one has

$$U_1(i, j) \geq x_{kj} - ip_{kj}, \tag{4}$$

and

$$U_2(i, j) \geq jp_{i\ell} - x_{i\ell}. \tag{5}$$

These inequalities say that reporting his/her type truthfully is a dominant strategy for both parties. A mechanism is said to satisfy EIR if

$$\text{support}\, g(.; , i, j) \subseteq [i, j] \tag{6}$$

for i, j such that $p_{ij} > 0$. An *allocation rule* (p, x) is said to be *DIC–EIR implementable* if there exists a mechanism (p, g) such that (i) (p, g) satisfies DIC and EIR, and (ii) (p, g) induces an expected payment rule x according to (1).

A *posted price mechanism* is defined as one where $p_{ij} = 1$ and $x_{ij} = a$ for i, j such that $i \leq a$ and $j \geq a$, and both seller and buyer agree to trade at a price equal to a.

Summarizing the above development we are interested in computing (p, g, x) satisfying the following constraints (budget balance is trivially satisfied since we deal with a single payment function x):

$$x_{ij} = p_{ij} \sum_{r \in \{i,\ldots,j\}} r g(r; i, j), \quad \forall i, j \in T, \tag{7}$$

$$x_{ij} - i p_{ij} \geq x_{kj} - i p_{kj}, \quad \forall i, j, k, \tag{8}$$

$$j p_{ij} - x_{ij} \geq j p_{i\ell} - x_{i\ell}, \quad \forall i, j, \ell, \tag{9}$$

$$g(r; i, j) = 0, \quad \text{if } r < i \text{ or } r > j, \tag{10}$$

$$p_{ij} \in \{0, 1\} \quad \text{and} \quad g(r; i, j) \in [0, 1] \quad \forall i, j, r \in T, \tag{11}$$

$$\sum_{r \in \{i,\ldots,j\}} g(r; i, j) = 1, \quad \forall i, j \in T. \tag{12}$$

We refer to the mixed-integer non-linear and non-convex feasibility problem (7)–(12) as **TR1**.

3 The transformed problem and its solution

The following result, which is a rephrasing of Theorem 1 of Hagerty and Rogerson (1987) in discrete type space, gives an equivalent formulation as a linear integer optimization problem. We shall give a proof partially based on network duality different from Hagerty and Rogerson (1987) since the proof of Hagerty and Rogerson (1987) relies on calculus arguments which do not apply in our setting.

Theorem 1 *The allocation rule is DIC–EIR implementable if and only if the following conditions are satisfied:*

$$x_{ij} = j p_{ij} - \sum_{k=i}^{j-1} p_{ik}, \quad \forall i \leq j, \tag{13}$$

$$x_{ij} = i p_{ij} + \sum_{\ell=i+1}^{j} p_{\ell j}, \quad \forall i \leq j, \tag{14}$$

$$p \text{ is weakly monotone increasing in } j \text{ for fixed } i \tag{15}$$

i.e., for each $i \in T$

$$p_{i1} \leq p_{i2} \leq \cdots \leq p_{im}, \ldots$$

$$p \text{ is weakly monotone decreasing in } i \text{ for fixed } j \tag{16}$$

i.e., for each $j \in T$

$$p_{1j} \geq p_{2j} \geq \cdots \geq p_{mj}, \ldots$$

$$p_{ij} = 0, \quad x_{ij} = 0 \quad \forall i > j, \quad \textit{and} \quad p_{ij} \in \{0, 1\}, \quad \forall i, j \in T. \tag{17}$$

Proof We shall first deal with necessity, i.e., we shall assume an allocation rule which is DIC–EIR implementable, and establish that the conditions given in the body of the theorem are implied.

The conditions (17) are immediate from (10). Now, let us consider inequalities (8) first. We shall follow a reasoning similar to Vohra (2011, 2012), namely we shall interpret these inequalities as dual constraints corresponding to a (multiple) longest path problem on an appropriate layered graph as follows (there is no connection between layers): let us define a node for each pair (i, j) such that $i \leq j$. For fixed j, we define an arc from node (i, j) to node (k, j) for $k < i$ with length $k[p_{kj} - p_{ij}]$ and an arc from (k, j) to (i, j) with length $i[p_{ij} - p_{kj}]$. We also define a dummy node $(0, j)$ and connect it with a forward arc to every node (i, j) at the same level j with length ip_{ij}. The variables x_{0j} and p_{0j} are set to zero. That is, we act as if we are looking for the longest path from $(0, j)$ to every other node of the same layer j. By the duality theorem of linear programming applied to the longest path problem from node $(0, j)$ to node (i, j) at the same level j (see e.g., Ahuja et al. 1993), the inequalities

$$x_{\ell j} - x_{ij} \leq i(p_{\ell j} - p_{ij})$$

are feasible if and only if no positive cycle is present in the graph, which is equivalent to the fact p is weakly monotone decreasing in i for fixed j. This proves the monotonicity property (14). Now, since the above inequalities constitute necessary and sufficient conditions for x_{ij} to be equal to the length of the longest path from $(0, j)$ to (i, j) and the longest path length in our graph is easily found to be equal to $ip_{ij} + \sum_{k=i+1}^{j} p_{kj}$ we have

$$x_{ij} = ip_{ij} + \sum_{k=i+1}^{j} p_{kj}, \quad \forall i \leq j,$$

which gives (14).

On the other hand, the inequalities (9) are viewed as the dual constraints of a multiple shortest path problem on a layered graph: define a node (i, j) for each pair (i, j) such that $i \leq j$, and a dummy node $(i, 0)$ for each layer $i = 1, \ldots, m - 1$. There is an arc from (i, j) to (i, k) for $k > j$ with length $k[p_{ik} - p_{ij}]$ and an arc from (i, k) to (i, j) with length $j[p_{ij} - p_{ik}]$. For the inequalities to be consistent no negative cycle should be allowed in any layer, hence the weakly monotone increasing property of p in j for fixed i. From $(i, 0)$ we make an arc to every (i, j) in the same layer with length jp_{ij}. Now we are looking for the shortest path from $(i, 0)$ to every other node (i, j) in the same layer i. Assuming x_{ij} to represent the length of a path from $(i, 0)$ to (i, j), (the variables x_{i0} and p_{i0} are again set to zero) we arrive by a reasoning similar to that for the longest path above at

$$x_{ij} = jp_{ij} - \sum_{k=i}^{j-1} p_{ik},$$

i.e., x_{ij} equals the length of the shortest path from $(i, 0)$ to (i, j), which gives (13).

Now, we turn to sufficiency of the conditions stated in the theorem for implying DIC–EIR implementability. We shall do the proof for the buyer. The proof for the seller is similar and thus omitted. If (13) is satisfied, then using the definition of $U_2(i, j)$ and (13) we have

$$U_2(i, j) = \sum_{k=i}^{j-1} p_{ik}. \tag{18}$$

Now, we want to show that (9) holds, i.e.,

$$U_2(i, j) \geq j p_{i\ell} - x_{i\ell}.$$

Now, substituting for $U_2(i, j)$ from (18) we have to show

$$\sum_{k=i}^{j-1} p_{ik} \geq j p_{i\ell} - x_{i\ell}.$$

We can re-write $x_{i\ell}$ as

$$x_{i\ell} = \ell p_{i\ell} - \sum_{k=1}^{\ell-1} p_{ik}$$

using (17). Hence we have to show

$$\sum_{k=i}^{j-1} p_{ik} \geq j p_{i\ell} - \ell p_{i\ell} + \sum_{k=1}^{\ell-1} p_{ik}.$$

However, this last inequality holds by weakly monotone increasing property of p in j for fixed i. To complete the proof, under the conditions stated in the theorem, one can create $g(.; i, j)$ to have a single mass point on x_{ij}/p_{ij} since g need to be defined only for i, j such that $p_{ij} > 0$. □

We remark that the above proof does not make use of the binary nature of the variables p_{ij}. Therefore, one can replace conditions (11) by

$$p_{ij} \in [0, 1] \quad \text{and} \quad g(r; i, j) \in [0, 1] \quad \forall i, j, r \in T, \tag{19}$$

and thus also replace (17) by

$$p_{ij} = 0, \quad x_{ij} = 0 \quad \forall\, i > j, \quad \text{and} \quad p_{ij} \in [0, 1], \quad \forall i, j \in T, \tag{20}$$

i.e., one can allow a probability of trade strictly between 0 and 1 without affecting the validity of Theorem 1.

We refer to the system of constraints (13)–(17) as **TR2**. The problem **TR2** can be equivalently cast as a binary feasibility problem **TRB** with constraints (15), (16), (17) and the equations obtained by equating (13) and (14) thereby eliminating the variables x_{ij}:

$$(j-i)p_{ij} = \sum_{k=i}^{j-1} p_{ik} + \sum_{\ell=i+1}^{j} p_{\ell j}, \quad \forall i \leq j. \tag{21}$$

Let **TRB-LP** denote the LP relaxation of **TRB**, i.e., the problem obtained by relaxing the requirements $p_{ij} \in \{0, 1\}$ to $p_{ij} \in [0, 1]$ for all i, j. Then we have the following theorem which establishes that every extreme point of **TRB-LP** is a binary vector corresponding to a posted price mechanism and therefore the polytope associated with the problem is an integral polytope with a restricted number of extreme points.

Theorem 2 1. *The polytope associated with* **TRB-LP** *with type space* $T = \{1, \ldots, m\}$ *is an* m*-simplex with* $m+1$ *integer extreme points, enumerated as* $k = 0, 1, 2, \ldots, m$ *where extreme point indexed* k *corresponds to a posted price mechanism with price equal to* k.
2. *The convex hull* X_m *of binary solutions to* **TRB-LP** *with type space* $T = \{1, \ldots, m\}$ *is given as*

$$X_m = \left\{ \begin{pmatrix} \frac{\alpha_2}{\alpha_2+\alpha_3} \\ \frac{\alpha_3}{\alpha_2+\alpha_3+\alpha_4} \\ \frac{\alpha_3+\alpha_4}{\vdots} \\ \sum_{j=2}^{m+1} \alpha_j \\ \sum_{j=3}^{m+1} \alpha_j \\ \vdots \\ \alpha_{m+1} \end{pmatrix} : \sum_{j=2}^{m+1} \alpha_j \leq 1, \alpha_j \geq 0, j = 2, \ldots, m+1 \right\}. \tag{22}$$

Proof We first give a simple procedure to construct recursively all binary solutions to **TRB-LP**. Start from $m = 2$. It is easy to verify that the following table gives the three binary solutions to **TRB-LP**:

k	0	1	2
p_{11}	0	1	0
p_{12}	0	1	1
p_{22}	0	0	1

Now, passing to $m = 3$, one simply adds the variables p_{13}, p_{23}, p_{33} in this order vertically to the leftmost column, fills the now empty portion of the second column with zeros (the origin is always a feasible binary solution), and adds an $m \times m$ upper triangular matrix filled with ones on the diagonal and above the diagonal. The remaining entries of the upper now empty portion of the right most column is filled with zeros. This operation results in the table:

k	0	1	2	3
p_{11}	0	1	0	0
p_{12}	0	1	1	0
p_{22}	0	0	1	0
p_{13}	0	1	1	1
p_{23}	0	0	1	1
p_{33}	0	0	0	1

The fact that these four points are the only feasible binary solutions is verified by direct calculation. Let us do one more step of the above procedure for $m = 4$, which should now be obvious. We add $p_{14}, p_{24}, p_{34}, p_{44}$ to the leftmost column, fill the second column with zeros, and next to it add a 4×4 upper triangular matrix with ones on and above diagonal. The upper portion of the rightmost newly added column is filled with zeros.

k	0	1	2	3	4
p_{11}	0	1	0	0	0
p_{12}	0	1	1	0	0
p_{22}	0	0	1	0	0
p_{13}	0	1	1	1	0
p_{23}	0	0	1	1	0
p_{33}	0	0	0	1	0
p_{14}	0	1	1	1	1
p_{24}	0	0	1	1	1
p_{34}	0	0	0	1	1
p_{44}	0	0	0	0	1

Using this procedure we can generate all binary feasible solutions to **TRB-LP** for any positive integer m recursively. The point $p^{(k)}$ indexed k for $k = 1, \ldots, m$ has the general form

$$p_{ij}^{(k)} = \begin{cases} 1 & i = 1, \ldots, k,\ j = k, \ldots, m \\ 0 & \text{otherwise,} \end{cases}$$

and $p_{ij}^{(0)} = 0$ for all i, j. It is verified by simple algebra using (13) or (14) that the posted price assertion holds, i.e., that for the extreme point indexed k, a posted price mechanism with payment equal to k is optimal.

The second part of the theorem is verified by induction starting with $m = 2$. Let $P(m)$ represent the polytope corresponding to **TRB-LP**. Then it is easy to show that $P(2) = X_2$ where X_2 is given as

$$X_2 = \left\{ \begin{pmatrix} \alpha_2 \\ \alpha_2 + \alpha_3 \\ \alpha_3 \end{pmatrix} : \alpha_2 + \alpha_3 \leq 1, \alpha_j \geq 0, \ j = 2, 3 \right\}.$$

Then, assuming the result true for n one easily verifies for $n + 1$. □

The above result confirms the belief by Hagerty and Rogerson (1987) that essentially the only trade mechanisms satisfying the constraints of their setting are posted price mechanisms. We note that solving the linear relaxation by the simplex method will result in a binary solution among the binary extreme points which are all posted price mechanisms. Therefore, at least in theory, the linear relaxation could have multiple optimal extreme point solutions, which implies that any convex combination is also optimal, and hence a mechanism allowing probabilistic trade (p_{ij} strictly between 0 and 1) could also be optimal.

4 Ambiguity averse planner

Now, we consider the problem of a central planner who wishes to maximize the expected proceeds from the trade, e.g., by maximizing the objective function

$$\sum_{i=1}^{m} \sum_{j=1}^{m} f_i h_j x_{ij} \tag{23}$$

where f_i is the probability that the seller is of type i, and h_j is the probability that the buyer is of type j over the constraints of **TR1**. By virtue of Theorem 2, this problem is solved by simply enumerating all integer extreme points (we have $m + 1$ of them) and choosing the best. This is an $\mathcal{O}(m^3)$ procedure.

Now, we shall assume that the central planner is uncertain about the probabilities f_i or h_j and wishes to have some protection against fluctuations in expected proceeds from trade. Therefore he/she takes an ambiguity-averse attitude by maximizing the worst-case expected proceeds. For illustration, let us consider the case where the central planner specifies an ambiguity set for f_i's of the form

$$U_f = \left\{ f \in \mathbb{R}^m : \sum_{i=1}^{m} f_i = 1, \sum_{i=1}^{m} i f_i = \alpha, \ f_i \geq 0, i = 1, \ldots, m \right\},$$

where $\alpha \in [1, m]$ is a given number. We refer to the specification above as a mean-constrained ambiguity specification. Here, the central planner is assumed to have an idea of the mean valuation of the buyer, and includes this information into the set of probability distributions U_f. A similar set of distributions was used in Neeman (2003) to describe the average magnitudes of the buyer valuations in an English auction setting. Neeman (2003) performed a worst-case analysis of the performance of the English auction from the viewpoint of a seller who is uncertain about which environment he is facing within a certain class of environments. Specifying a mean value and seeking a robust solution in a worst-case sense over all distributions resulting in the given mean value is also practised in portfolio optimization; see e.g., Chen et al. (2011) and El Ghaoui et al. (2003). The main point of the present section is that incorporating ambiguity aversion into the decision problem of a planner unsure about type distributions does not result in a more difficult problem since it can be solved efficiently using the posted price characterization and a very simple linear program.

Now, the central planner wishes to devise a trade mechanism maximizing the worst-case expected proceeds: i.e.,

$$\max_{p,g,x} \min_{f \in U_f} \sum_{i=1}^{m} \sum_{j=1}^{m} f_i h_j x_{ij} \tag{24}$$

over the constraints defining **TR1**. Using Theorem 1, the above problem is equivalent to

$$\max_{p} \min_{f \in U_f} \sum_{i=1}^{m} f_i \sum_{j=i}^{m} p_{ij} \left(j h_j - \sum_{\ell=j+1}^{m} h_\ell \right) \tag{25}$$

over the constraints defining **TRB**. Now, by linear programming duality the previous problem is equivalent to

$$\max_{p,y,z} y + \alpha z \tag{26}$$

subject to the constraints defining **TRB** and the inequalities

$$y + iz \leq \gamma_i, \quad \forall i = 1, \ldots, m,$$

where we defined for readability $\gamma_i = \sum_{j=i}^{m} p_{ij}(j h_j - \sum_{\ell=j+1}^{m} h_\ell)$. We refer to this problem as **AASPf**. By virtue of Theorem 2 we have the following result which describes a simple solution procedure for problem **AASPf**.

Proposition 1 *Problem* **AASPf** *is solved by enumerating each of* $m + 1$ *vertices of the* **TRB-LP** *polytope, and solving a two-variable LP over* (y, z) *for each vertex.*

It is expected that increasing α will result in a larger posted price mechanism. This is indeed observed. Consider, for example, $m = 10$ with uniform probabilities $h_j = 1/m$ for all $j = 1, \ldots, m$. With $\alpha = 2$, there is trade for seller valuations i in $I^* = \{1, \ldots, 6\}$ and $j \in J^* = \{6, \ldots, 10\}$ for pairs $(i, j) \in I^* \times J^*$ with $i \leq j$. The posted price is equal to 6. For $\alpha = 5$, the posted price is equal to 7, and there is trade for $I^* = \{1, \ldots, 7\}$ and $j \in J^* = \{7, \ldots, 10\}$. For $\alpha = 8$, the posted price is equal to 10 with $I^* = \{1, \ldots, 10\}$ and $j \in J^* = \{10\}$.

We remark that one could equally treat the probability mass h of the buyer using a similar ambiguity set, say U_h for a fixed (known) seller probability mass f. The ensuing development is parallel to the above, and is therefore omitted.

5 Robust mediated trade

Consider now the trade problem in the presence of a third party who acts an intermediary by extracting some surplus for his/herself from the trade. The presence of the intermediary serves to relax the budget balance requirement of the previous sections in that the payment made by the buyer is larger than the payment received by the seller where the difference goes to the intermediary. The problem of robust trade in the presence of an intermediary can be easily cast in the formulation of the previous sections. In this case, we are dealing with the expected payment x_{ij} by the buyer and the expected payment y_{ij} received by the seller when the seller valuation is equal to i and the buyer valuation is equal to j. Let p_{ij} represent the probability of trade at the pair (i, j) as usual, and $g(k; i, j)$ denote the probability of trade with surplus equal to k conditional on the respective seller and buyer valuations i and j.

Kos and Manea (2009) defines a VCG' mechanism for this setting as follows: the buyer receives the object when his/her reported value is larger than the seller's reported value, and in such instances he/she pays the lowest value he/she may have reported to win the object given the seller valuation. The seller trades the object when his/her reported value is smaller than the buyer valuation, and in such instances he/she gets the maximum she/she could have reported to let go of the object. Formally, a VCG' mechanism in our context is defined as $p_{ij} = 1$ if $i \leq j$ and $p_{ij} = 0$ if $i > j$ with payment rules:

$$x_{ij} = \begin{cases} \min\{j : j \geq i\} & \text{if } j \geq i \\ 0 & \text{otherwise,} \end{cases}$$

$$y_{ij} = \begin{cases} \max\{i : j \geq i\} & \text{if } j \geq i \\ 0 & \text{otherwise.} \end{cases}$$

We shall establish an important connection to VCG' mechanisms below.

Maximizing the expected surplus, we are dealing with the following design problem referred to as **ITR1**:

$$\max \sum_{i=1}^{m} \sum_{j=1}^{m} f_i h_j (x_{ij} - y_{ij}) \tag{27}$$

subject to

$$x_{ij} - y_{ij} = p_{ij} \sum_{k \in \{0,\ldots,j-i\}} k g(k; i, j), \quad \forall i, j \in T. \tag{28}$$

$$y_{ij} - i p_{ij} \geq y_{\ell j} - i p_{\ell j}, \quad \forall i, j, \ell. \tag{29}$$

$$j p_{ij} - x_{ij} \geq j p_{ik} - x_{ik}, \quad \forall i, j, k \tag{30}$$

$$g(k; i, j) = 0, \text{ if } k \notin \{0, \ldots, j-i\}. \tag{31}$$

$$p_{ij} \in \{0, 1\} \quad \text{and} \quad g(k; i, j) \in [0, 1] \quad \forall i, j \in T \quad \text{and} \quad k \in \{i, \ldots, j\}. \tag{32}$$

$$\sum_{k \in \{0,\ldots,j-i\}} g(k; i, j) = 1, \quad \forall i, j \in T. \tag{33}$$

Using the proof techniques of Theorem 1 we have the following result which gives an integer linear programming description of (ITR1) and a structural property of optimal mechanisms. An implication of the result is that the integer requirement on p_{ij} can be ignored.

Theorem 3 1. *Problem ITR1 is equivalent to the following optimization problem (ITR2):*

$$\max \sum_{i=1}^{m} \sum_{j=1}^{m} f_i h_j (x_{ij} - y_{ij}) \tag{34}$$

subject to

$$x_{ij} = j p_{ij} - \sum_{k=i}^{j-1} p_{ik}, \quad \forall i \leq j, \tag{35}$$

$$y_{ij} = i p_{ij} + \sum_{\ell=i+1}^{j} p_{\ell j}, \quad \forall i \leq j, \tag{36}$$

$$p \text{ is weakly monotone increasing in } j \text{ for fixed } i \tag{37}$$

$$p \text{ is weakly monotone decreasing in } i \text{ for fixed } j \tag{38}$$

$$p_{ij} = 0, \quad x_{ij} = 0, \quad y_{ij} = 0, \quad \forall\, i > j, \quad \text{and} \quad p_{ij} \in \{0, 1\}, \quad \forall i, j \in T. \tag{39}$$

2. *ITR2 admits an optimal VCG' mechanism.*

Proof The equivalent reformulation ITR2 is obtained exactly as in Theorem 1. By eliminating the variables x_{ij} and y_{ij} in ITR2 and by substituting in the objective function, we get a linear optimization problem over the simple 0/1 polytope defined by constraints (37), (38) (i.e., we can relax the binary requirements). For part 2, observe that an optimal (extreme point) solution p_{ij} corresponds to an upper triangular $m \times m$ matrix **P** with all non-zero entries equal to one. For such feasible **P**, computing x_{ij} and y_{ij} shows by simple algebra that one deals with a VCG' mechanism. □

Thus, Theorem 3 gives a simple linear programming characterization for DIC, EIR and feasible (not budget balanced) optimal robust trades, a problem also studied by Schwartz and Wen (2012) in the continuous type case (they show, among other things, that relaxing the budget balance to feasibility increases the expected gains from trade) and by Kos and Manea (2009) in the discrete type case.

For illustration, let us consider the case $m = 10$ with $f_i = h_j = 1/m$, i.e., uniform probabilities. Then the optimal expected payments x and y are given in the tables below.

	6	7	8	9	10
1	6	6	6	6	6
2		7	7	7	7
3			8	8	8
4				9	9
5					10

for x (payment by buyer), and

	6	7	8	9	10
1	1	2	3	4	5
2		2	3	4	5
3			3	4	5
4				4	5
5					5

for y (seller). The empty entries (as well as the other i, j pairs not present in the tables) correspond to points of no-trade. For example, when the seller's value is equal to 1, for all buyer values from 6 to 10, the buyer makes a fixed payment equal to 6, of which the seller gets only 1, or 2, or 3, or 4 or 5 units, respectively for increasing valuations of the buyer. In other words, the optimal trade mechanism advocates charging a fixed fee to the buyer (the smallest valuation he can declare to get the object given the seller valuation) for a given fixed seller valuation, while reserving an increasing fee to the seller for increasing valuation of the buyer, i.e., for a fixed seller valuation, the intermediary retains less surplus for increasing buyer values. Likewise, for a fixed buyer valuation, the seller gets the same payment for his/her increasing valuations (the largest he can declare to relinquish the object for fixed buyer valuation), while the intermediary retains an increasing amount since the buyer is charged a higher price for increasing seller valuations. The above mechanism is exactly a VCG' mechanism as defined in Kos and Manea (2009) and revisited above.

The main contribution of the present section was to show a linear programming representation of DIC, EIR and feasible (not budget balanced) robust trades maximizing the expected revenue of the intermediary, and show optimality of VCG' mechanisms.

6 Concluding remarks

In a departure from common practice in the economics literature we studied bilateral trade problems in discrete type spaces instead of continuous type spaces, which opens the way to finite-dimensional optimization formulations and combinatorial problems. We derived formulations amenable to solution by linear programming after transforming the associated optimization problems into integer programming problems by means of duality arguments. We completely characterized the convex hull of binary solutions to the problem of robust trade design, and explored a further layer of robustness when probabilities governing buyer or seller types are uncertain as well as the characterization of robust trade when budget balance is relaxed. The main points to retain from the present study is 1. that posted price mechanism which is a common thread of the results (at least in Sects. 3, 4) is "robust" to variations (e.g., discrete type space, probability of trade continuous between 0 and 1, worst-case expected proceeds maximization under mean-constrained ambiguity specification) in the classical setup, and 2. that the presence of an surplus maximizing intermediary leads to a VCG-like optimal mechanism.

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