Arch. Math. 109 (2017), 323–334

0003-889X/17/040323-12
published online August 3, 2017
DOI 10.1007/s00013-017-1071-y

Archiv der Mathematik

Arithmetical rank of binomial ideals

Anargyros Katsabekis

Abstract. In this paper, we investigate the arithmetical rank of a binomial ideal J. We provide lower bounds for the binomial arithmetical rank and the J-complete arithmetical rank of J. Special attention is paid to the case where J is the binomial edge ideal of a graph. We compute the arithmetical rank of such an ideal in various cases.

Mathematics Subject Classification. 13F20, 14M12, 05C25.

Keywords. Arithmetical rank, Binomial ideals, Graphs, Indispensable monomials.

1. Introduction. Consider the polynomial ring $K[x_1,\dots,x_m]$ in the variables $x_1,\dots,x_m$ over a field K. For the sake of simplicity, we will denote by $\mathbf{x}^{\mathbf{u}}$ the monomial $x_1^{u_1}\cdots x_m^{u_m}$ of $K[x_1,\dots,x_m]$, with $\mathbf{u}=(u_1,\dots,u_m)\in\mathbb{N}^m$, where $\mathbb{N}$ stands for the set of non-negative integers. A *binomial* in the sense of [12, Chapter 8] is a difference of two monomials, i.e. it is of the form $\mathbf{x}^{\mathbf{u}}-\mathbf{x}^{\mathbf{v}}$. A *binomial ideal* is an ideal generated by binomials.

Toric ideals serve as important examples of binomial ideals. Let $\mathcal{A}=\{\mathbf{a}_1,\dots,\mathbf{a}_m\}$ be a subset of $\mathbb{Z}^n$. The *toric ideal* $I_{\mathcal{A}}$ is the kernel of the K-algebra homomorphism $\phi: K[x_1,\dots,x_m]\to K[t_1^{\pm1},\dots,t_n^{\pm1}]$ given by

$$\phi(x_i)=\mathbf{t}^{\mathbf{a}_i}=t_1^{a_{i,1}}\cdots t_n^{a_{i,n}}\quad\text{for all}\ \ i=1,\dots,m,$$

where $\mathbf{a}_i=(a_{i,1},\dots,a_{i,n})$.

We grade $K[x_1,\dots,x_m]$ by the semigroup $\mathbb{N}\mathcal{A}:=\{l_1\mathbf{a}_1+\cdots+l_m\mathbf{a}_m|l_i\in\mathbb{N}\}$ setting $\deg_{\mathcal{A}}(x_i)=\mathbf{a}_i$ for $i=1,\dots,m$. The $\mathcal{A}$-degree of a monomial $\mathbf{x}^{\mathbf{u}}$ is defined by

$$\deg_{\mathcal{A}}(\mathbf{x}^{\mathbf{u}})=u_1\mathbf{a}_1+\cdots+u_m\mathbf{a}_m\in\mathbb{N}\mathcal{A}.$$

A polynomial $F\in K[x_1,\dots,x_m]$ is $\mathcal{A}$-homogeneous if the $\mathcal{A}$-degrees of all the monomials that occur in F are the same. An ideal is $\mathcal{A}$-homogeneous if it is generated by $\mathcal{A}$-homogeneous polynomials. The ideal $I_{\mathcal{A}}$ is generated by all the binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ such that $\deg_{\mathcal{A}}(\mathbf{x}^{\mathbf{u}}) = \deg_{\mathcal{A}}(\mathbf{x}^{\mathbf{v}})$ (see [11, Lemma 4.1]), thus $I_{\mathcal{A}}$ is $\mathcal{A}$-homogeneous.

Let $J \subset K[x_1, \ldots, x_m]$ be a binomial ideal. There exist a positive integer n and a vector configuration $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subset \mathbb{Z}^n$ such that $J \subset I_{\mathcal{A}}$, see for instance [7, Theorem 1.1]. We say that a polynomial $F = c_1M_1 + \cdots + c_sM_s \in J$, where $c_i \in K$ and $M_1, \ldots, M_s$ are monomials, is J-*complete* if $M_i - M_l \in J$ for every $1 \leq i < l \leq s$. Clearly every J-complete polynomial F is also $\mathcal{A}$-homogeneous.

Computing the least number of polynomial equations defining an algebraic set is a classical problem in Algebraic Geometry which goes back to Kronecker [9]. This problem is equivalent, over an algebraically closed field, with the corresponding problem in Commutative Algebra of the determination of the smallest integer s for which there exist polynomials $F_1, \ldots, F_s$ in J such that $rad(J) = rad(F_1, \ldots, F_s)$. The number s is commonly known as the *arithmetical rank* of J and will be denoted by $\text{ara}(J)$. Since J is generated by binomials, it is natural to define the *binomial arithmetical rank* of J, denoted by $\text{bar}(J)$, as the smallest integer s for which there exist binomials $B_1, \ldots, B_s$ in J such that $rad(J) = rad(B_1, \ldots, B_s)$. Furthermore we can define the J-*complete arithmetical rank* of J, denoted by $\text{ara}_c(J)$, as the smallest integer s for which there exist J-complete polynomials $F_1, \ldots, F_s$ in J such that $rad(J) = rad(F_1, \ldots, F_s)$. Finally we define the $\mathcal{A}$-*homogeneous arithmetical rank* of J, denoted by $\text{ara}_{\mathcal{A}}(J)$, as the smallest integer s for which there exist $\mathcal{A}$-homogeneous polynomials $F_1, \ldots, F_s$ in J such that $rad(J) = rad(F_1, \ldots, F_s)$. From the definitions and [2, Corollary 3.3.3] we deduce the following inequalities:

$$\text{cd}(J) \leq \text{ara}(J) \leq \text{ara}_{\mathcal{A}}(J) \leq \text{ara}_c(J) \leq \text{bar}(J)$$

where $\text{cd}(J)$ is the cohomological dimension of J.

In Sect. 2 we introduce the simplicial complex Δ_J and use combinatorial invariants of the aforementioned complex to provide lower bounds for the binomial arithmetical rank and the J-complete arithmetical rank of J. In particular we prove that $\text{bar}(J) \geq \delta(\Delta_J)_{\{0,1\}}$ and $\text{ara}_c(J) \geq \delta(\Delta_J)_{\Omega}$, see Theorem 2.6.

In Sect. 3 we study the arithmetical rank of the binomial edge ideal J_G of a graph G. This class of ideals generalizes naturally the determinantal ideal generated by the 2-minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ x_{n+1} & x_{n+2} & \ldots & x_{2n} \end{pmatrix}.$$

We prove (see Theorem 3.3) that, for a binomial edge ideal J_G, both the binomial arithmetical rank and the J_G-complete arithmetical rank coincide with the number of edges of G. If G is the complete graph on the vertex set $\{1, \ldots, n\}$, then, from [3, Theorem 2], the arithmetical rank of J_G equals $2n-3$. It is still an open problem to compute $\text{ara}(J_G)$ when G is not the complete graph. We show that $\text{ara}(J_G) \geq n + l - 2$, where n is the number of vertices of G and l is the vertex connectivity of G. Furthermore we prove that in several cases $\text{ara}(J_G) = \text{cd}(J_G) = n + l - 2$, see Theorems 3.7, 3.9, Corollary 3.10, and Theorem 3.13.

2. Lower bounds. First we will use the notion of indispensability to introduce the simplicial complex Δ_J. Let $J \subset K[x_1,\ldots,x_m]$ be a binomial ideal containing no binomials of the form $\mathbf{x}^{\mathbf{u}}-1$, where $\mathbf{u} \neq \mathbf{0}$. A *binomial* $B = M - N \in J$ is called *indispensable* of J if every system of binomial generators of J contains B or $-B$, while a *monomial* M is called *indispensable* of J if every system of binomial generators of J contains a binomial B such that M is a monomial of B. Let $\mathcal{M}_J$ be the ideal generated by all monomials M for which there exists a nonzero $M - N \in J$. By [7, Proposition 1.5] the set $G(\mathcal{M}_J)$ of indispensable monomials of J is the unique minimal generating set of $\mathcal{M}_J$.

The support of a monomial $\mathbf{x}^{\mathbf{u}}$ of $K[x_1,\ldots,x_m]$ is $\mathrm{supp}(\mathbf{x}^{\mathbf{u}}) := \{i | x_i \text{ divides } \mathbf{x}^{\mathbf{u}}\}$. Let $\mathcal{T}$ be the set of all $E \subset \{1,\ldots,m\}$ for which there exists an indispensable monomial M of J such that $E = \mathrm{supp}(M)$. Let $\mathcal{T}_{\min}$ denote the set of minimal elements of $\mathcal{T}$.

Definition 2.1. We associate to J a simplicial complex Δ_J with vertices the elements of $\mathcal{T}_{\min}$. Let $T = \{E_1,\ldots,E_k\}$ be a subset of $\mathcal{T}_{\min}$, then $T \in \Delta_J$ if there exist M_i, $1 \le i \le k$, such that $\mathrm{supp}(M_i) = E_i$ and $M_i - M_l \in J$ for every $1 \le i < l \le k$.

Next we will study the connection between the radical of J and Δ_J. The *induced subcomplex* Δ' of Δ_J by certain vertices $\mathcal{V} \subset \mathcal{T}_{\min}$ is the subcomplex of Δ_J with vertices $\mathcal{V}$ and $T \subset \mathcal{V}$ is a simplex of the subcomplex Δ' if T is a simplex of Δ_J. A subcomplex H of Δ_J is called a *spanning subcomplex* if both have exactly the same set of vertices.

Let F be a polynomial in $K[x_1,\ldots,x_m]$. We associate to F the induced subcomplex $\Delta_J(F)$ of Δ_J consisting of those vertices $E_i \in \mathcal{T}_{\min}$ with the property: there exists a monomial M_i in F such that $E_i = \mathrm{supp}(M_i)$. The next theorem provides a necessary condition under which a set of polynomials in the binomial ideal J generates the radical of J up to radical.

Proposition 2.2. *Let K be any field. If $rad(J) = rad(F_1,\ldots,F_s)$ for some polynomials $F_1,\ldots,F_s$ in J, then $\cup_{i=1}^{s}\Delta_J(F_i)$ is a spanning subcomplex of Δ_J.*

Proof. Let $E = \mathrm{supp}(\mathbf{x}^{\mathbf{u}}) \in \mathcal{T}_{\min}$, where $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in J$ and $\mathbf{x}^{\mathbf{u}}$ is an indispensable monomial of J. We will show that there exists a monomial M in some F_l, $1 \le l \le s$, such that $E = \mathrm{supp}(M)$. Since $rad(J) = rad(F_1,\ldots,F_s)$, there is a power B^r, $r \ge 1$, which belongs to the ideal generated by $F_1,\ldots,F_s$. Thus there is a monomial M in some F_l dividing the monomial $(\mathbf{x}^{\mathbf{u}})^r$ and therefore $\mathrm{supp}(M) \subseteq \mathrm{supp}(\mathbf{x}^{\mathbf{u}})$. But $F_l \in J$ and J is generated by binomials, so there exists $\mathbf{x}^{\mathbf{z}} - \mathbf{x}^{\mathbf{w}} \in J$ such that $\mathbf{x}^{\mathbf{z}}$ divides M. Since $\mathbf{x}^{\mathbf{z}} \in \mathcal{M}_J$ and $G(\mathcal{M}_J)$ generates $\mathcal{M}_J$, there is an indispensable monomial N dividing $\mathbf{x}^{\mathbf{z}}$, thus

$$\mathrm{supp}(N) \subseteq \mathrm{supp}(\mathbf{x}^{\mathbf{z}}) \subseteq \mathrm{supp}(M) \subseteq E.$$

Since $E \in \mathcal{T}_{\min}$, we deduce that $E = \mathrm{supp}(N)$, and therefore $E = \mathrm{supp}(M)$.

$\square$

Remark 2.3. (1) If F is a J-complete polynomial of J, then $\Delta_J(F)$ is a simplex. To see that $\Delta_J(F)$ is a simplex, suppose that $\Delta_J(F) \neq \emptyset$ and let $T = \{E_1, \dots, E_k\}$ be the set of vertices of $\Delta_J(F)$. For every $1 \leq i \leq k$ there exists a monomial M_i, $1 \leq i \leq k$, in F such that $E_i = \mathrm{supp}(M_i)$. Since F is J-complete, we have that $M_i - M_l \in J$ for every $1 \leq i < l \leq k$. Thus $\Delta_J(F)$ is a simplex.

(2) If B is a binomial of J, then $\Delta_J(B)$ is either a vertex, an edge, or the empty set.

Remark 2.4. If the equality $rad(J) = rad(F_1, \dots, F_s)$ holds for some J-complete polynomials $F_1, \dots, F_s$ in J, then $\cup_{i=1}^{s} \Delta_J(F_i)$ is a spanning subcomplex of Δ_J and each $\Delta_J(F_i)$ is a simplex.

For a simplicial complex Δ we denote by r_Δ the smallest number s of simplices T_i of Δ, such that the subcomplex $\cup_{i=1}^{s} T_i$ is spanning and by b_Δ the smallest number s of simplices T_i of Δ, such that the subcomplex $\cup_{i=1}^{s} T_i$ is spanning and each T_i is either an edge, a vertex, or the empty set.

Theorem 2.5. *Let K be any field, then $b_{\Delta_J} \leq \mathrm{bar}(J)$ and $r_{\Delta_J} \leq \mathrm{ara}_c(J)$.*

It turns out that both b_{Δ_J} and r_{Δ_J} have a combinatorial interpretation in terms of matchings in Δ_J.

Let Δ be a simplicial complex on the vertex set $\mathcal{T}_{\min}$ and Q be a subset of $\Omega := \{0, 1, \dots, \dim(\Delta)\}$. A set $\mathcal{N} = \{T_1, \dots, T_s\}$ of simplices of Δ is called a *Q-matching* in Δ if $T_k \cap T_l = \emptyset$ for every $1 \leq k, l \leq s$ and $\dim(T_k) \in Q$ for every $1 \leq k \leq s$; see also [8, Definition 2.1]. Let $\mathrm{supp}(\mathcal{N}) = \cup_{i=1}^{s} T_i$, which is a subset of the vertices $\mathcal{T}_{\min}$. We denote by $\mathrm{card}(\mathcal{N})$ the cardinality s of the set $\mathcal{N}$. A Q-matching $\mathcal{N}$ in Δ is called a *maximal Q-matching* if $\mathrm{supp}(\mathcal{N})$ has the maximum possible cardinality among all Q-matchings. By $\delta(\Delta)_Q$, we denote the minimum of the set

$$\{\mathrm{card}(\mathcal{N}) | \mathcal{N} \text{ is a maximal } Q - \text{matching in } \Delta\}.$$

Theorem 2.6. *Let K be any field, then $\mathrm{bar}(J) \geq \delta(\Delta_J)_{\{0,1\}}$ and $\mathrm{ara}_c(J) \geq \delta(\Delta_J)_\Omega$.*

Proof. By [8, Proposition 3.3], $b_{\Delta_J} = \delta(\Delta_J)_{\{0,1\}}$ and $r_{\Delta_J} = \delta(\Delta_J)_\Omega$. Now the result follows from Theorem 2.5. □

Proposition 2.7. *Let J be a binomial ideal. Suppose that there exists a minimal generating set $\mathcal{S}$ of J such that every element of $\mathcal{S}$ is a difference of two squarefree monomials. Assume that J is generated by the indispensable binomials, namely $\mathcal{S}$ consists precisely of the indispensable binomials (up to sign). Then $\mathrm{bar}(J) = \mathrm{card}(\mathcal{S})$.*

Proof. Let $\mathrm{card}(\mathcal{S}) = t$. Since $\mathcal{S}$ is a generating set of J, we have that $\mathrm{bar}(J) \leq t$. It is enough to prove that $t \leq \mathrm{bar}(J)$. Let $|\mathcal{T}_{\min}| = g$. By [4, Corollary 3.6] it holds that $\mathrm{card}(G(\mathcal{M}_J)) = 2t$, so $g = 2t$. For every maximal $\{0,1\}$-matching $\mathcal{M}$ in Δ_J, we have that $\mathrm{supp}(\mathcal{M}) = \mathcal{T}_{min}$, so $\delta(\Delta_J)_{\{0,1\}} \geq \left\lfloor \frac{g}{2} \right\rfloor$ and therefore $\delta(\Delta_J)_{\{0,1\}} \geq t$. Thus, from Theorem 2.6, $\mathrm{bar}(J) \geq t$. □

Example 2.8. Let J be the binomial ideal generated by $f_1 = x_1x_6 - x_2x_5$, $f_2 = x_2x_7 - x_3x_6$, $f_3 = x_1x_8 - x_4x_5$, $f_4 = x_3x_8 - x_4x_7$, and $f_5 = x_1x_7 - x_3x_5$. Actually J is the binomial edge ideal of the graph G with edges $\{1, 2\}$, $\{2, 3\}$, $\{1, 4\}$, $\{3, 4\}$, and $\{1, 3\}$, see Sect. 3 for the definition of such an ideal. Note that J is $\mathcal{A}$-homogeneous where $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_8\}$ is the set of columns of the matrix

$$D = \begin{pmatrix} 1&0&0&0&1&0&0&0 \\ 0&1&0&0&0&1&0&0 \\ 0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1 \end{pmatrix}.$$

By [4, Theorem 3.3] every binomial f_i is indispensable of J. Thus

$$\begin{aligned}\mathcal{T}_{\min} = \{&E_1 = \{1, 6\}, E_2 = \{2, 5\}, E_3 = \{2, 7\}, E_4 = \{3, 6\}, E_5 = \{1, 8\},\\ &E_6 = \{4, 5\}, E_7 = \{3, 8\}, E_8 = \{4, 7\}, E_9 = \{1, 7\}, E_{10} = \{3, 5\}\}.\end{aligned}$$

By Proposition 2.7 the binomial arithmetical rank of J equals 5. The simplicial complex Δ_J has 5 connected components and all of them are 1-simplices, namely $\Delta_1 = \{E_1, E_2\}$, $\Delta_2 = \{E_3, E_4\}$, $\Delta_3 = \{E_5, E_6\}$, $\Delta_4 = \{E_7, E_8\}$, and $\Delta_5 = \{E_9, E_{10}\}$. Consequently

$$\delta(\Delta_J)_\Omega = \sum_{i=1}^{5} \delta(\Delta_i)_\Omega = 1 + 1 + 1 + 1 + 1 = 5,$$

and therefore $5 \leq \mathrm{ara}_c(J)$. Since $\mathrm{ara}_c(J) \leq \mathrm{bar}(J)$, we get that $\mathrm{ara}_c(J) = 5$. We will show that $\mathrm{ara}_\mathcal{A}(J) = 5$. Suppose that $\mathrm{ara}_\mathcal{A}(J) = s < 5$, and let $F_1, \dots, F_s$ be $\mathcal{A}$-homogeneous polynomials in J such that $rad(J) = rad(F_1, \dots, F_s)$. For every vertex $E_i \in \mathcal{T}_{\min}$ there exists, from Proposition 2.2, a monomial M_i in F_k such that $E_i = \mathrm{supp}(M_i)$. But $s < 5$, so there exist $E_i \in \mathcal{T}_{\min}$ and $E_j \in \mathcal{T}_{\min}$ such that

(1) $\{E_i, E_j\}$ is not a 1-simplex of Δ_J,
(2) $E_i = \mathrm{supp}(M_i)$, $E_j = \mathrm{supp}(M_j)$, and
(3) M_i and M_j are monomials of some F_k.

Since F_k is $\mathcal{A}$-homogeneous, it holds that $\deg_\mathcal{A}(M_i) = \deg_\mathcal{A}(M_j)$. Considering all possible combinations of E_i and E_j, we finally arrive at a contradiction. Thus $\mathrm{ara}_\mathcal{A}(J) = 5$. Note that J is $\mathcal{B}$-homogeneous where $\mathcal{B}$ is the set of columns of the matrix

$$N = \begin{pmatrix} 1&1&1&1&0&0&0&0 \\ 1&0&0&0&1&0&0&0 \\ 0&1&0&0&0&1&0&0 \\ 0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1 \end{pmatrix}.$$

Since every row of D is a row of N, we deduce that every $\mathcal{B}$-homogeneous polynomial in J is also $\mathcal{A}$-homogeneous. So $\mathrm{ara}_\mathcal{B}(J)$ is an upper bound for $\mathrm{ara}_\mathcal{A}(J)$, therefore $\mathrm{ara}_\mathcal{B}(J) = 5$. We have that $rad(J) = rad(f_1, f_2 + f_3, f_4, f_5)$, since the second power of both binomials f_2 and f_3 belongs to the ideal generated by the polynomials f_1, f_2+f_3, f_4, f_5. Remark that the polynomials f_1, f_2+f_3, f_4, and f_5 are $\mathcal{C}$-homogeneous, where $\mathcal{C}$ is the set of columns of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 5 & 6 & 7 & 8 \end{pmatrix}.$$

Thus $\operatorname{ara}_{\mathcal{C}}(J) \leq 4$, so $\operatorname{ara}(J) \leq 4$. A primary decomposition of J is

$$J = (f_1, f_2, f_3, f_4, f_5, x_2x_8 - x_4x_6) \cap (x_1, x_3, x_5, x_7).$$

Hence, by [2, Proposition 19.2.7], it follows that $\operatorname{ara}(J) \geq 4$. Thus

$$\operatorname{ara}(J) = \operatorname{ara}_{\mathcal{C}}(J) = 4 < 5 = \operatorname{ara}_{\mathcal{A}}(J) = \operatorname{ara}_{\mathcal{B}}(J) = \operatorname{ara}_c(J) = \operatorname{bar}(J).$$

3. Binomial edge ideals of graphs. In this section we consider a special class of binomial ideals, namely binomial edge ideals of graphs. This ideal was introduced in [6] and independently at the same time in [10].

Let G be an undirected connected simple graph on the vertex set $[n] := \{1, \ldots, n\}$ and with edge set $E(G)$. Consider the polynomial ring

$$R := K[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}]$$

in $2n$ variables, $x_1, \ldots, x_n$, $x_{n+1}, \ldots, x_{2n}$, over K.

Definition 3.1. The binomial edge ideal $J_G \subset R$ associated to the graph G is the ideal generated by the binomials $f_{ij} = x_i x_{n+j} - x_j x_{n+i}$, with $i < j$, such that $\{i, j\}$ is an edge of G.

Remark 3.2. From [7, Corollary 1.13] every binomial f_{ij}, where $\{i, j\}$ is an edge of G, is indispensable of J_G. Thus

$$\mathcal{T}_{min} = \left\{E_{ij}^1 = \{i, n+j\}, E_{ij}^2 = \{j, n+i\} | \{i, j\} \in E(G)\right\}.$$

We recall some fundamental material from [6]. Let G be a connected graph on $[n]$ and let $S \subset [n]$. By $G \backslash S$, we denote the graph that results from deleting all vertices in S and their incident edges from G. Let $c(S)$ be the number of connected components of $G \backslash S$, and let $G_1, \ldots, G_{c(S)}$ denote the connected components of $G \backslash S$. Also let $\widetilde{G}_i$ denote the complete graph on the vertices of G_i. We set

$$P_S(G) = \left(\cup_{i \in S}\{x_i, x_{n+i}\}, J_{\widetilde{G}_1}, \ldots, J_{\widetilde{G}_{c(S)}}\right) R.$$

Then $P_S(G)$ is a prime ideal for every $S \subset [n]$. The ring $R/P_\emptyset(G)$ has Krull dimension $n+1$. For $S \neq \emptyset$ the ring $R/P_S(G)$ has Krull dimension $n - \operatorname{card}(S) + c(S)$. The ideal $P_S(G)$ is a minimal prime of J_G if and only if $S = \emptyset$ or $S \neq \emptyset$, and for each $i \in S$ one has $c(S \backslash \{i\}) < c(S)$. Moreover J_G is a radical ideal and it admits the minimal primary decomposition $J_G = \cap_{S \in \mathcal{M}(G)} P_S(G)$, where $\mathcal{M}(G) = \{S \subset [n] : P_S(G) \text{ is a minimal prime of } J_G\}$.

Theorem 3.3. *Let G be a connected graph on the vertex set $[n]$ with m edges. Then* $\operatorname{bar}(J_G) = \operatorname{ara}_c(J_G) = m$.

Proof. Every binomial f_{ij}, where $\{i,j\}$ is an edge of G, is indispensable of J_G, thus, from Proposition 2.7, $\mathrm{bar}(J_G) = m$. Note that, for every edge $\{i,j\}$ of G, $\{E^1_{ij}, E^2_{ij}\}$ is a 1-simplex of Δ_{J_G}. Furthermore Δ_{J_G} has exactly m connected components and all of them are 1-simplices. Thus $\delta(\Delta_{J_G})_\Omega = m$ and therefore, from Theorem 2.6, $\mathrm{ara}_c(J_G) \geq m$. Consequently $\mathrm{ara}_c(J_G) = m$. □

Theorem 3.4. *Let G be a connected graph on the vertex set $[n]$ with m edges. Consider the canonical basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{Z}^n$ and the canonical basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_{n+1}\}$ of $\mathbb{Z}^{n+1}$. Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_{2n}\} \subset \mathbb{N}^n$ be the set of vectors where $\mathbf{a}_i = \mathbf{e}_i$, $1 \leq i \leq n$, and $\mathbf{a}_{n+i} = \mathbf{e}_i$ for $1 \leq i \leq n$. Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_{2n}\} \subset \mathbb{N}^{n+1}$ be the set of vectors where $\mathbf{b}_i = \mathbf{w}_1 + \mathbf{w}_{i+1}$, $1 \leq i \leq n$, and $\mathbf{b}_{n+i} = \mathbf{w}_{i+1}$ for $1 \leq i \leq n$. Then $\mathrm{ara}_{\mathcal{A}}(J_G) = \mathrm{ara}_{\mathcal{B}}(J_G) = m$.*

Proof. Suppose that $\mathrm{ara}_{\mathcal{A}}(J_G) = t < m$, and let $F_1, \ldots, F_t$ be $\mathcal{A}$-homogeneous polynomials in J_G such that $J_G = rad(F_1, \ldots, F_t)$. For every edge $\{i,j\}$ of G with $i < j$ there exist, from Proposition 2.2, monomials M^k_{ij} and N^l_{ij} in F_k and F_l, respectively, such that $E^1_{ij} = \mathrm{supp}(M^k_{ij})$ and $E^2_{ij} = \mathrm{supp}(N^l_{ij})$. But $t < m$, so there exists $E^1_{rs} \in \mathcal{T}_{\min}$, where $\{r,s\}$ is an edge of G with $r < s$, such that

(1) $\{E^1_{ij}, E^1_{rs}\}$ is not a 1-simplex of Δ_{J_G},
(2) $E^1_{ij} = \mathrm{supp}(M^k_{ij})$, $E^1_{rs} = \mathrm{supp}(M^k_{rs})$, and
(3) M^k_{ij} and M^k_{rs} are monomials of some F_k.

Let $M^k_{ij} = x_i^{g_i} x_{n+j}^{g_j}$ and $M^k_{rs} = x_r^{g_r} x_{n+s}^{g_s}$. Since F_k is $\mathcal{A}$-homogeneous, we deduce that $\deg_{\mathcal{A}}(M^k_{ij}) = \deg_{\mathcal{A}}(M^k_{rs})$, and therefore $g_i\mathbf{e}_i + g_j\mathbf{e}_j = g_r\mathbf{e}_r + g_s\mathbf{e}_s$. Consequently $i = r$, $j = s$, and also $M^k_{ij} = M^k_{rs}$ is a contradiction. Let D and Q be the matrices with columns $\mathcal{A}$ and $\mathcal{B}$, respectively. Since every row of D is a row of Q, we deduce that every $\mathcal{B}$-homogeneous polynomial in J_G is also $\mathcal{A}$-homogeneous. Thus $\mathrm{ara}_{\mathcal{B}}(J_G)$ is an upper bound for $\mathrm{ara}_{\mathcal{A}}(J_G)$, so $m \leq \mathrm{ara}_{\mathcal{B}}(J_G)$ and therefore $\mathrm{ara}_{\mathcal{B}}(J_G) = m$. □

The graph G is called *l-vertex-connected* if $l < n$ and $G \backslash S$ is connected for every subset S of $[n]$ with $\mathrm{card}(S) < l$. The *vertex connectivity* of G is defined as the maximum integer l such that G is l-vertex-connected.

In [1] the authors study the relationship between algebraic properties of a binomial edge ideal J_G, such as the dimension and the depth of R/J_G, and the vertex connectivity of the graph. It turns out that this notion is also useful for the computation of the arithmetical rank of a binomial edge ideal.

Theorem 3.5. *Let K be a field of any characteristic and G be a connected graph on the vertex set $[n]$. Suppose that the vertex connectivity of G is l. Then $\mathrm{ara}(J_G) \geq n + l - 2$.*

Proof. If G is the complete graph on the vertex set $[n]$, its vertex connectivity is $n-1$, then $\mathrm{ara}(J_G) = 2n-3 = n+l-2$ by [3, Theorem 2]. Assume now that G is not the complete graph. Let $P_\emptyset(G), W_1, \ldots, W_t$ be the minimal primes of J_G. It holds that $J_G = P_\emptyset(G) \cap L$ where $L = \cap_{i=1}^t W_i$. First we will prove that $\dim(R/(P_\emptyset(G) + L)) \leq n-l+1$. For every prime ideal Q such that $P_\emptyset(G)+L \subseteq Q$, we have that $L \subseteq Q$, so there is $1 \leq i \leq t$ such that $W_i \subseteq Q$. Thus

$P_\emptyset(G)+W_i \subseteq Q$ and therefore $\dim(R/(P_\emptyset(G)+L)) \leq \dim(R/(P_\emptyset(G)+W_i))$. It is enough to show that $\dim(R/(P_\emptyset(G)+W_i)) \leq n-l+1$. Let $W_i = P_S(G)$ for $\emptyset \neq S \subset [n]$. We have that $P_\emptyset(G)+P_S(G)$ is generated by

$$\{x_i x_{n+j} - x_j x_{n+i} : i,j \in [n]\backslash S\} \cup \{x_i, x_{n+i} : i \in S\}.$$

Then $\dim(R/(P_\emptyset(G)+P_S(G))) = n - \mathrm{card}(S)+1$. If $l=1$, then $\mathrm{card}(S) \geq 1$ since $S \neq \emptyset$, and therefore $\dim(R/(P_\emptyset(G)+W_i)) \leq n$. Suppose that $l \geq 2$ and also that $\mathrm{card}(S) < l$. Since $P_S(G)$ is a minimal prime, for every $i \in S$ we have that $c(S\backslash\{i\}) < c(S)$. But G is l-vertex-connected, namely $G\backslash S$ is connected, so $P_\emptyset(G) \subset P_S(G)$, a contradiction to the fact that $P_S(G)$ is a minimal prime. Thus $\dim(R/(P_\emptyset(G)+W_i)) \leq n-l+1$ and therefore $\dim(R/(P_\emptyset(G)+L)) \leq n-l+1$. Next we will show that $\min\{\dim(R/P_\emptyset(G)), \dim(R/L)\} > \dim(R/(P_\emptyset(G)+L))$. Recall that $\dim(R/P_\emptyset(G)) = n+1$, so $\dim(R/(P_\emptyset(G)+L)) < \dim(R/P_\emptyset(G))$. Since $L \subset P_\emptyset(G)+L$, we deduce that $\dim(R/(P_\emptyset(G)+L)) \leq \dim(R/L)$. Suppose that $\dim(R/(P_\emptyset(G)+L)) = \dim(R/L)$, say equal to s, and let $Q_1 \subsetneqq Q_2 \subsetneqq \cdots \subsetneqq Q_s$ be a chain of prime ideals containing $P_\emptyset(G)+L$. Then there is $1 \leq j \leq t$ such that $Q_1 = W_j$. So $P_\emptyset(G) \subset W_j$, a contradiction. By [2, Proposition 19.2.7] it holds that

$$\begin{aligned}\mathrm{cd}(J_G) &\geq \dim(R) - \dim(R/(P_\emptyset(G)+L)) - 1 = 2n - \dim(R/(P_\emptyset(G)+L)) \\ &\quad -1 \geq 2n-(n-l+1)-1 = n+l-2.\end{aligned}$$

Consequently $\mathrm{ara}(J_G) \geq n+l-2$. □

Example 3.6. Let G be the graph on the vertex set [5] with edges $\{1,2\}$, $\{2,3\}$, $\{1,3\}$, $\{2,4\}$, $\{4,5\}$, and $\{3,5\}$. Here the vertex connectivity is $l=2$. By Theorem 3.5, $\mathrm{ara}(J_G) \geq 5$. The ideal J_G is generated up to radical by the polynomials $f_{12}, f_{23}, f_{13}+f_{24}, f_{35}$, and f_{45}, since both f_{13}^2 and f_{24}^2 belong to the ideal generated by $f_{12}, f_{23}, f_{13}+f_{24}, f_{35}$, and f_{45}. Thus $\mathrm{ara}(J_G) = 5 < 6 = \mathrm{bar}(J_G)$.

Theorem 3.7. *If G is a cycle of length $n \geq 3$, then* $\mathrm{ara}(J_G) = \mathrm{bar}(J_G) = n$.

Proof. The vertex connectivity of G is 2, so, from Theorem 3.5, the inequality $n \leq \mathrm{ara}(J_G)$ holds. Since G has n edges, we have that $\mathrm{ara}(J_G) \leq \mathrm{bar}(J_G) = n$ and therefore $\mathrm{ara}(J_G) = n$. □

Proposition 3.8. *Let G be a connected graph on $[n]$, with m edges and $n \geq 4$. If G contains an odd cycle of length 3, then* $\mathrm{ara}(J_G) \leq m-1$.

Proof. Let C be an odd cycle of G of length 3, with edge set $\{\{1,2\},\{2,3\},\{1,3\}\}$. Since G is connected, without loss of generality, there is a vertex $4 \leq i \leq n$ such that $\{1,i\}$ is an edge of G. We will show that $(x_1 x_{n+i} - x_i x_{n+1})^2$ belongs to the ideal L generated by the polynomials $f_{12}, f_{13}, f_{1i}+f_{23}$. We have that

$$\begin{aligned}x_1^2 x_{n+i}^2 &\equiv x_1 x_{n+i} x_i x_{n+1} - x_1 x_2 x_{n+i} x_{n+3} + x_1 x_3 x_{n+i} x_{n+2} \equiv x_1 x_i x_{n+i} x_{n+1} \\ &\quad - x_2 x_{n+i} x_3 x_{n+1} + x_2 x_3 x_{n+1} x_{n+i} \equiv x_1 x_i x_{n+i} x_{n+1} \bmod L.\end{aligned}$$

Similarly we have that $x_i^2 x_{n+1}^2 \equiv x_1 x_i x_{n+i} x_{n+1} \bmod L$. Thus $x_1^2 x_{n+i}^2 + x_i^2 x_{n+1}^2 \equiv 2x_1 x_i x_{n+i} x_{n+1} \bmod L$, so $(x_1 x_{n+i} - x_i x_{n+1})^2$ belongs to L. Next we prove that $(x_2x_{n+3} - x_3x_{n+2})^2$ belongs to L. We have that

$$\begin{aligned} x_2^2x_{n+3}^2 &\equiv x_2x_{n+3}x_3x_{n+2} - x_2x_{n+3}x_1x_{n+i} + x_2x_{n+3}x_ix_{n+1} \\ &\equiv x_2x_{n+3}x_3x_{n+2} - x_2x_{n+i}x_3x_{n+1} + x_{n+3}x_ix_1x_{n+2} \\ &\equiv x_2x_{n+3}x_3x_{n+2} - x_1x_{n+2}x_{n+i}x_3 + x_ix_{n+2}x_3x_{n+1} \bmod L. \end{aligned}$$

Furthermore

$$x_3^2x_{n+2}^2 \equiv x_2x_{n+3}x_3x_{n+2} - x_3x_{n+2}x_ix_{n+1} + x_3x_{n+2}x_1x_{n+i} \bmod L.$$

Thus $x_2^2x_{n+3}^2 + x_3^2x_{n+2}^2 \equiv 2x_2x_{n+3}x_3x_{n+2} \bmod L$, so $(x_2x_{n+3} - x_3x_{n+2})^2 \in L$. Let H be the subgraph of G consisting of the cycle C and the edge $\{1, i\}$. Then J_G is generated up to radical by the following set of $m - 1$ binomials:

$$\{f_{kl} | \{k, l\} \in E(G) \backslash E(H)\} \cup \{f_{12}, f_{13}, f_{1i} + f_{23}\}.$$

Therefore $\text{ara}(J_G) \leq m - 1$. □

Let $G_1 = (V(G_1), E(G_1))$, $G_2 = (V(G_2), E(G_2))$ be graphs such that $G_1 \cap G_2$ is a complete graph. The new graph $G = G_1 \bigoplus G_2$ with the vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$ is called the *clique sum* of G_1 and G_2 in $G_1 \cap G_2$. If the cardinality of $V(G_1) \cap V(G_2)$ is $k + 1$, then this operation is called a k-clique sum of the graphs G_1 and G_2. We write $G = G_1 \bigoplus_{\widehat{v}} G_2$ to indicate that G is the clique sum of G_1 and G_2 and that $V(G_1) \cap V(G_2) = \widehat{v}$.

Theorem 3.9. *Let G be a connected graph on the vertex set $[n]$. Suppose that G has exactly one cycle C. If $n \geq 4$ and C is odd of length 3, then* $\text{ara}(J_G) = n-1$.

Proof. The graph G can be written as the 0-clique sum of the cycle C and some trees. More precisely,

$$G = C \bigoplus_{v_1} T_1 \bigoplus_{v_2} \cdots \bigoplus_{v_s} T_s$$

for some vertices $v_1, \ldots, v_s$ of C. The vertex connectivity of G is 1. By Theorem 3.5, the inequality $n - 1 \leq \text{ara}(J_G)$ holds. Since G has exactly one cycle, we have that $\text{card}(E(G)) = n$. From Proposition 3.8, $\text{ara}(J_G) \leq n - 1$, and therefore $\text{ara}(J_G) = n - 1$. □

Let $\text{ht}(J_G)$ be the height of J_G, then we have, from the generalized Krull's principal ideal theorem, that $\text{ht}(J_G) \leq \text{ara}(J_G)$. We say that J_G is a *set-theoretic complete intersection* if $\text{ara}(J_G) = \text{ht}(J_G)$.

Corollary 3.10. *Let G be a connected graph on the vertex set $[n]$ with $n \geq 4$. Suppose that G has exactly one cycle C and its length is 3. Then the following properties are equivalent:*

(a) *J_G is unmixed,*
(b) *J_G is Cohen–Macaulay,*
(c) *J_G is a set-theoretic complete intersection,*
(d) *$G = C \bigoplus_{v_1} T_1 \bigoplus_{v_2} \cdots \bigoplus_{v_s} T_s$, where $\{v_1, \ldots, v_s\} \subset V(C)$, $s \geq 1$, v_h are pairwise distinct and T_h are paths.*

In particular, if one of the above conditions is true, then $\mathrm{ara}(J_G) = \mathrm{ht}(J_G) = n-1$.

Proof. The implication (b)⇒(a) is well known. If J_G is a set-theoretic complete intersection, then, from Theorem 3.9, $\mathrm{ht}(J_G) = n-1$ and $\dim(R/J_G) = n+1$. Also $\mathrm{depth}(R/J_G) = n+1$ by [5, Theorem 1.1], so J_G is Cohen–Macaulay, whence (c)⇒(b). Recall that $\mathcal{M}(G) = \{S \subset [n] : P_S(G)$ is a minimal prime of $J_G\}$. If J_G is unmixed, then every vertex v of T_h, $v \neq v_h$, has degree at most 2. In fact, $\{v\} \in \mathcal{M}(G)$ and, if $\deg_G(v) \geq 3$, then by [6, Lemma 3.1], one has $\mathrm{ht}(P_{\{v\}}(G)) = n + \mathrm{card}(\{v\}) - c(\{v\}) = n+1-\deg_G(v) \leq n-2 < n-1 = \mathrm{ht}(P_{\emptyset}(G))$, a contradiction. Moreover, v_h has degree at most 3 for every h. In fact, $\{v_h\} \in \mathcal{M}(G)$ and, if $\deg_G(v_h) \geq 4$, then by [6, Lemma 3.1], one has $\mathrm{ht}(P_{\{v_h\}}(G)) = n + \mathrm{card}(\{v_h\}) - c(\{v_h\}) = n+1-(\deg_G(v_h)-1) \leq n-2 < n-1 = \mathrm{ht}(P_{\emptyset}(G))$, a contradiction. Thus, (d) follows. Finally, assuming (d), J_G is unmixed by [5, Theorem 1.1] and $\mathrm{ht}(J_G) = n-1$. By Theorem 3.9, it follows that

$$\mathrm{ara}(J_G) = n-1 = \mathrm{ht}(J_G).$$

□

If C_1 and C_2 are cycles of G having no common vertex, then a *bridge* between C_1 and C_2 is an edge $\{i,j\}$ of G with $i \in V(C_1)$ and $j \in V(C_2)$.

Proposition 3.11. *Let G be a connected graph on the vertex set $[n]$ with m edges. Suppose that G contains a subgraph H consisting of two vertex disjoint odd cycles of length 3, namely C_1 and C_2, and also two bridges between the cycles C_1 and C_2. Then* $\mathrm{ara}(J_G) \leq m-2$.

Proof. Let $E(C_1) = \{\{1,2\},\{2,3\},\{3,1\}\}$ and $E(C_2) = \{\{4,5\},\{5,6\},\{4,6\}\}$. Suppose first that the bridges have no common vertex. Let $e_1 = \{1,4\}$ and $e_2 = \{3,6\}$ be the bridges of the two cycles. Then f_{14}^2 belongs to the ideal generated by the polynomials $f_{12}, f_{13}, f_{14}+f_{23}$. Furthermore f_{36}^2 belongs to the ideal generated by the polynomials $f_{46}, f_{56}, f_{36}+f_{45}$. Thus J_G is generated up to radical by the union of $\{f_{12}, f_{13}, f_{14}+f_{23}, f_{46}, f_{56}, f_{36}+f_{45}\}$ and $\{f_{ij} | \{i,j\} \in E(G)$ and $\{i,j\} \notin E(H)\}$. If the bridges have a common vertex, then without loss of generality, we can assume that $e_1 = \{1,4\}$ and $e_2 = \{3,4\}$ are the bridges of the two cycles. Applying similar arguments as before, we deduce that $\mathrm{ara}(J_G) \leq m-2$. □

Example 3.12. Suppose that G is a graph with 6 vertices and 8 edges consisting of two vertex disjoint odd cycles of length 3, namely C_1 and C_2, and also two vertex disjoint bridges between the cycles C_1 and C_2. Here the vertex connectivity is $l = 2$. Thus $\mathrm{ara}(J_G) \geq 6$. By Proposition 3.11, $\mathrm{ara}(J_G) \leq 6$ and therefore $\mathrm{ara}(J_G) = 6$.

Theorem 3.13. *Let G_k be a graph containing k odd cycles $C_1, \ldots, C_k$ of length 3 such that the cycles C_i and C_j have disjoint vertex sets, for every $1 \leq i < j \leq k$. Suppose that there exists exactly one path $P_{i,i+1}$ of length $r_i \geq 2$ connecting a vertex of C_i with a vertex of C_{i+1}, $1 \leq i \leq k-1$. If G_k has no more vertices* *or edges, then* $\operatorname{ara}(J_{G_k}) = \operatorname{ht}(J_{G_k}) = 2k + \sum_{i=1}^{r-1} r_i$. *In particular,* J_{G_k} *is a set-theoretic complete intersection.*

Proof. The graph G_k has $3k + \sum_{i=1}^{k-1}(r_i - 1)$ vertices. Here the vertex connectivity is $l = 1$, so

$$2k + \sum_{i=1}^{k-1} r_i = 3k + \sum_{i=1}^{k-1}(r_i - 1) + 1 - 2 \leq \operatorname{ara}(J_{G_k}).$$

We will prove that $\operatorname{ara}(J_{G_k}) \leq 2k + \sum_{i=1}^{k-1} r_i$ by induction on $k \geq 2$. Suppose that $k = 2$ and let $E(C_1) = \{\{1,2\},\{2,3\},\{1,3\}\}$, $P_{1,2} = \{\{3,4\},\{4,5\},\ldots,\{r+2,r+3\}\}$, and $C_2 = \{\{r+3,r+4\},\{r+4,r+5\},\{r+3,r+5\}\}$. Then J_{G_2} is generated up to radical by the union of

$$\{f_{12} + f_{34}, x_{r+2}x_{n+r+3} - x_{r+3}x_{n+r+2} + x_{r+4}x_{n+r+5} - x_{r+5}x_{n+r+4}\}$$

and

$$\{f_{ij} | \{i,j\} \in E(G_2) \backslash \{\{1,2\},\{3,4\},\{r+2,r+3\},\{r+4,r+5\}\}\}.$$

Thus $\operatorname{ara}(J_{G_2}) \leq 4 + r$. Assume that the inequality $\operatorname{ara}(J_{G_k}) \leq 2k + \sum_{i=1}^{k-1} r_i$ holds for k, and we will prove that $\operatorname{ara}(J_{G_{k+1}}) \leq 2(k+1) + \sum_{i=1}^{k} r_i$. We have that $J_{G_{k+1}} = J_{G_k} + J_H$ where H is the graph consisting of the path $P_{k,k+1}$ and the cycle C_{k+1}. By Theorem 3.9, $\operatorname{ara}(J_H) = r_k + 2$. Then, from the induction hypothesis,

$$\operatorname{ara}(J_{G_{k+1}}) \leq \operatorname{ara}(J_{G_k}) + \operatorname{ara}(J_H) \leq 2k + \sum_{i=1}^{k-1} r_i + r_k + 2 = 2(k+1) + \sum_{i=1}^{k} r_i.$$

Since J_{G_k} is unmixed by [5, Theorem 1.1], we have that

$$\operatorname{ht}(J_{G_k}) = \operatorname{card}(V(G_k)) - 1 = 2k + \sum_{i=1}^{r-1} r_i.$$

□

Remark 3.14. All the results presented are independent of the field K.

Acknowledgements. The author is grateful to an anonymous referee for useful suggestions and comments that helped improve an earlier version of the manuscript. This work was supported by the Scientific and Technological Research Council of Turkey (TÜBITAK) through BIDEB 2221 Grant.

References

[1] A. Banerjee and L. Núnez-Betancourt, Graph connectivity and binomial edge ideals, Proc. Amer. Math. Soc. **145** (2017), 487–499.

[2] M. P. Brodmann and R. Y. Sharp, Local Cohomology: An Algebraic Introduction with Geometric Applications, Cambridge Studies in Advanced Mathematics, 60, Cambridge University Press, Cambridge, 1998.

[3] W. Bruns and R. Schwänzl, The number of equations defining a determinantal variety, Bull. London Math. Soc. **22** (1990), 439–445.

[4] H. Charalambous, A. Thoma, and M. Vladoiu, Binomial fibers and indispensable binomials, J. Symbolic Comput. **74** (2016), 578–591.

[5] V. Ene, J. Herzog, and T. Hibi, Cohen-Macaulay binomial edge ideals, Nagoya Math. J. **204** (2011), 57–68.

[6] J. Herzog, T. Hibi, F. Hreinsdóttir, T. Kahle, and J. Rauh, Binomial edge ideals and conditional independence statements, Adv. in Appl. Math. **45** (2010), 317–333.

[7] A. Katsabekis and I. Ojeda, An indispensable classification of monomial curves in $\mathbb{A}^4(K)$, Pacific J. Math. **268** (2014), 95–116.

[8] A. Katsabekis and A. Thoma, Matchings in simplicial complexes, circuits and toric varieties, J. Comb. Theory Ser. A **114** (2007), 300–310.

[9] L. Kronecker, Grundzüge einer arithmetischen Theorie der algebraischen Grössen, J. Reine Angew. Math. **92** (1882), 1–122.

[10] M. Ohtani, Graphs and ideals generated by some 2-minors, Comm. Algebra **39** (2011), 905–917.

[11] B. Sturmfels, Gröbner Bases and Convex Polytopes, University Lecture Series, 8, American Mathematical Society, Providence, RI, 1995.

[12] R. Villarreal, Monomial Algebras, Second Edition, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2015.

Anargyros Katsabekis
Department of Mathematics
Bilkent University
06800 Ankara
Turkey
e-mail: katsampekis@bilkent.edu.tr

Received: 20 March 2017